

## PREDICTING AND ESTIMATING PROBABILITY DENSITY FUNCTIONS OF CHAOTIC SYSTEMS

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(Communicated by Stephen Pankavich)

**ABSTRACT.** In the present work, for the first time, we employ Ulam's method to estimate and to predict the existence of the probability density functions of single species populations with chaotic dynamics. In particular, given a chaotic map, we show that Ulam's method generates a sequence of density functions in  $L^1$ -space that may converge weakly to a function in  $L^1$ -space. In such a case we show that the limiting function generates an absolutely continuous (w.r.t. the Lebesgue measure) invariant measure (w.r.t. the given chaotic map) and therefore the limiting function is the probability density function of the chaotic map. This fact can be used to determine the existence and estimate the probability density functions of chaotic biological systems.

**1. Introduction.** Several ecologists have studied the implications of chaotic dynamics both in field experiments [11, 12, 13, 16, 25, 39, 40] and experimental population research [7, 14, 20, 22, 36, 45]. In wildlife, there are only a few proofs of chaos that seem convincing. This includes chaotic population dynamics of boreal rodents due to mustelid predators [15], dynamics of vole populations in northern Fennoscandia [17, 32, 41] and the epidemics of childhood diseases [34, 33]. Although laboratory experiments are far different than the wildlife situations, they can be used for testing some ecological concepts. There are more convincing examples of chaotic behaviors in laboratory experiments. In particular, ecologists have observed experimental evidence of chaos in cultures of the flour beetle *Tribolium* [8, 10], Nicholson's sheep blowflies [29, 35, 43] and the microbial food web with constant experimental conditions [3].

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2010 *Mathematics Subject Classification.* \*\*\*.

*Key words and phrases.* \*\*\*.

The research of Prof. Paweł Góra was supported by NSERC grant.

There are several mathematical and statistical models that can be used to explore chaotic behaviors. In the present work we consider discrete-time models of the form

$$x_{n+1} = f(x_n), \quad (1)$$

where  $f(x)$  is the density-dependent recruitment function.

In biology, the recruitment functions are often classified into overcompensating and compensating density dependent functions [2]. The former occurs when  $f$  has a local maximum and decreases for large population sizes, whereas the latter corresponds to a recruitment function that is bounded and increases monotonically. Compensating density dependence relates to contest competition, in which only a portion of the population obtains enough resources for survival and the rest will gradually die [5]. Whereas overcompensating density relates to scramble competition, in which resources are divided between all members and large population sizes will result in decreased survival rates [2, 5].

It can be shown that there is no possibility of period doubling or chaotic behavior for populations with compensating density dependence. In particular, when the recruitment function  $f(x)$  is a bounded monotone increasing function, the slope of  $f(x)$  at the positive equilibrium is between zero and one. Therefore, the positive equilibrium remains stable at all times. On the contrary, populations that are ruled by overcompensating density dependence may exhibit period doubling and chaotic behaviors (see for example chapter 2 of [5]). When the model is chaotic, the solution  $x_n$  behaves randomly and it is highly sensitive to the initial condition. Studies of discrete chaotic dynamical systems often lead to bifurcation diagrams, frequency plots and many other computational aspects of the chaotic systems [1, 7, 10, 38]. In the present work, for the first time, we employ a mathematical method known as Ulam's method [42] to estimate the probability density function (PDF) of single species populations that exhibit chaotic behaviors. We will also establish the conditions, under which the PDF related to a chaotic system exists. Using the estimated PDF the ecologist will be able to further investigate the behavior of the chaotic populations and compute the estimated probability of any interval of population sizes.

Mathematically, we study the existence of invariant measures of chaotic maps. The existence of a continuous invariant measure shows that the system in question is chaotic. The existence of an absolutely continuous invariant measure (acim) proves even more chaotic behavior and gives tools to statistically predict the future of the system.

The most popular method to practically approximate an acim is the so-called Ulam's method of approximating the Frobenius-Perron operator of the system by a sequence of finite-dimensional operators (matrices), [24]. We prove some facts about Ulam's approximations. In particular, we show that any \*-weak limit of Ulam's approximations is an invariant measure, and give sufficient conditions for this limit measure to be absolutely continuous.

As an example, we show that the Smith-Slatkin model for some parameters has an acim. We use the Schwarzian derivative method and sketch the methodology which can give the existence of an acim for a large set of parameter values.

**2. Preliminaries.** Throughout the paper we assume that  $[a, b]$  is a closed and bounded interval in the real number line,  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $[a, b]$ , and  $\mu$

is a measure on  $\mathcal{B}$ . We recall some basic definitions. More details can be found in [4].

**Definition 1.** The map  $\tau : [a, b] \rightarrow [a, b]$  is called measurable if  $\tau^{-1}(\mathcal{B}) \subseteq \mathcal{B}$ , that is,  $B \in \mathcal{B}$  implies  $\tau^{-1}(B) \in \mathcal{B}$ , where  $\tau^{-1}(B) = \{x \in [a, b] : \tau(x) \in B\}$ .

**Definition 2.** Let  $\tau : [a, b] \rightarrow [a, b]$  be a measurable map. A set  $B \in \mathcal{B}$  is said to be an invariant set of  $\tau$  if  $\tau^{-1}(B) = B$ . The map  $\tau$  is said to be ergodic w.r.t. the measure  $\mu$  if  $B \in \mathcal{B}$  is an invariant set of  $\tau$  then  $\mu(B) = 0$  or  $\mu(B^c) = 0$ .

**Definition 3.** We say that the measurable map  $\tau : [a, b] \rightarrow [a, b]$  preserves measure  $\mu$  or that  $\mu$  is  $\tau$ -invariant if  $\mu(\tau^{-1}(B)) = \mu(B)$  for all  $B \in \mathcal{B}$ , that is,  $\mu \circ \tau^{-1} = \mu$  on  $\mathcal{B}$ .

**Definition 4.** The measure  $\mu$  on  $\mathcal{B}$  is said to be an absolutely continuous measure w.r.t. the Lebesgue measure  $m$  if  $B \in \mathcal{B}$  and  $m(B) = 0$ , then  $\mu(B) = 0$ .

Now we are ready to recall the Birkhoff Ergodic Theorem (see [4] for the proof).

**The Birkhoff Ergodic Theorem** Suppose  $\tau : [a, b] \rightarrow [a, b]$  is measurable and the measure  $\mu$  is  $\tau$ -invariant. Then for any  $f \in L^1[a, b]$ , there exists a function  $\hat{f} \in L^1[a, b]$  such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\tau^n(x)) = \hat{f}(x), \quad \mu - \text{a.e.} \quad (2)$$

Furthermore, we have

$$\hat{f} \circ \tau = \hat{f}, \quad \mu - \text{a.e.} \quad (3)$$

and

$$\int_a^b \hat{f} d\mu = \int_a^b f d\mu. \quad (4)$$

In addition, if  $\tau$  is ergodic w.r.t. the measure  $\mu$ , then (3) implies that  $\hat{f}$  is constant  $\mu$ -a.e., so from (4) we obtain

$$\hat{f} = \frac{1}{\mu([a, b])} \int_a^b f d\mu, \quad \mu - \text{a.e.}$$

So if  $\mu([a, b]) = 1$ , that is, the measure  $\mu$  is a probability measure, (2) becomes

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\tau^n(x)) = \int_a^b f d\mu, \quad \mu - \text{a.e.} \quad (5)$$

Thus, by letting  $f = \chi_B$  in (5) we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_B(\tau^n(x)) = \mu(B), \quad \mu - \text{a.e.} \quad (6)$$

Here  $\chi_B$  denotes the characteristic function over the set  $B$ , that is,

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases},$$

**Remark 1.** Let's assume the measure  $\mu$  defined by

$$\mu(B) = \int_B f^* dm, \quad \forall B \in \mathcal{B}, \quad (7)$$

where  $f^* \in L^1[a, b]$ . Then we say that the measure  $\mu$  has density  $f^*$ . Note that the measure  $\mu$  defined by (7) is absolute continuous w.r.t. the Lebesgue measure  $m$ . So if  $\tau$  is ergodic w.r.t. the Lebesgue measure  $m$ , then it is also ergodic w.r.t. the measure  $\mu$ . Now we further assume that the measure  $\mu$  defined by (7) is  $\tau$ -invariant probability measure. If  $\tau$  is also ergodic w.r.t. the Lebesgue measure  $\mu$ , from (6) and (7) we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_B(\tau^n(x)) = \int_B f^* dm, \quad \forall B \in \mathcal{B}, \quad (8)$$

Note that if we regard  $(x, \tau(x), \tau^2(x), \dots)$  as the occurrences of a random variable  $X$  at each discrete step, then (8) can be written

$$\text{Prob}(X \in B) = \int_B f^* dm, \quad \forall B \in \mathcal{B}.$$

So  $f^*$  is the probability density function of the random variable  $X$ . In section 5, we will find conditions under which there exists a density function  $f^* \in L^1[a, b]$  such that the probability measure  $\mu$  with density  $f^*$  is  $\tau$ -invariant using the concept of weak precompactness.

We also need to review the Frobenius-Perron operator [4] associated with a map  $\tau$ .

**Definition 5.** A measurable map  $\tau : [a, b] \rightarrow [a, b]$  is nonsingular w.r.t. the Lebesgue measure  $m$  if  $m(B) = 0$  implies  $m(\tau^{-1}(B)) = 0$ , where  $B \in \mathcal{B}$ .

**Definition 6.** If  $\tau : [a, b] \rightarrow [a, b]$  is a nonsingular map w.r.t. the Lebesgue measure  $m$ , the operator  $P_\tau : L^1[a, b] \rightarrow L^1[a, b]$  defined (implicitly) by

$$\int_B P_\tau f dm = \int_{\tau^{-1}(B)} f dm \quad \forall B \in \mathcal{B}, \quad \forall f \in L^1[a, b]$$

is called the Frobenius-Perron operator associated with  $\tau$ .

**Remark 2.** By choosing  $B = [a, x]$  we see that

$$\int_a^x P_\tau f dm = \int_{\tau^{-1}([a, x])} f dm \quad \forall f \in L^1[a, b].$$

So by differentiating both sides w.r.t.  $x$  we have

$$(P_\tau f)(x) = \frac{d}{dx} \int_{\tau^{-1}([a, x])} f dm \quad \forall f \in L^1[a, b],$$

which is the explicit formula of the Frobenius-Perron operator associated with  $\tau$ .

Now we have the following theorem that tells us under what condition on the density function  $f^* \in L^1[a, b]$ , the probability measure  $\mu$  with density  $f^*$  is  $\tau$ -invariant.

**Theorem 1.** *Let  $\tau : [a, b] \rightarrow [a, b]$  be a nonsingular map, and let  $P_\tau$  be the Frobenius-Perron operator associated with  $\tau$ . Let  $f^* \in L^1[a, b]$  be a density function. Then the probability measure  $\mu$  with density  $f^*$  is  $\tau$ -invariant if and only if  $f^*$  is a fixed point of  $P_\tau$ , that is,  $P_\tau(f^*) = f^*$ .*

*Proof.* See, for example, [4, Proposition 4.2.7].  $\square$

**Remark 3.** In general, for a given nonsingular map  $\tau : [a, b] \rightarrow [a, b]$ , there may not exist a density function  $f^* \in L^1[a, b]$  such that  $P_\tau(f^*) = f^*$ .

**3. Partially invariant measures.** In this section we show that Ulam's method gives an almost  $\tau$ -invariant measure. We briefly explain Ulam's method. We let  $\mathcal{P}_n = \{I_i\}_{i=1}^n$  be a partition of  $[a, b]$  into  $n$  equal length subintervals. We define an  $n \times n$  nonnegative matrix  $A = (a_{ij})$ , which is called Ulam's matrix associated with  $\tau$  based on the partition  $\mathcal{P}_n$  of  $[a, b]$ , as follows:

$$a_{ij} = \frac{m(\tau^{-1}(I_i) \cap I_j)}{m(I_j)}, \quad 1 \leq i, j \leq n.$$

Note that Ulam's matrix is a column stochastic matrix. So  $A$  can be viewed as the transition matrix of a finite state Markov chain so that the  $a_{ij}$  can be interpreted as the probability that a point in  $I_j$  is mapped into  $I_i$  under  $\tau$ .

Since Ulam's matrix is a column stochastic matrix, 1 is its eigenvalue. Since  $A$  is nonnegative, there exists a nonnegative eigenvector  $y$  of  $A$  corresponding to the eigenvalue 1, that is,  $Ay = y$  and  $y \geq 0$ . We may assume that  $\sum_{j=1}^n y_j = n$ . We let  $f_n = \sum_{j=1}^n y_j \chi_{I_j}$ . Since  $y$  is a nonnegative vector such that  $\sum_{j=1}^n y_j = n$ , it follows that  $f_n$  is a density function. We call  $f_n$  Ulam's density function associated with  $\tau$  based on the partition  $\mathcal{P}_n$  of  $[a, b]$ .

Now we prove that if  $\mu_n$  is the measure with density  $f_n$ , then  $\mu_n$  is  $\tau$ -invariant on  $\mathcal{P}_n$ .

**Lemma 1.** *If we let  $\mu_n$  be the measure with density  $f_n$ , where  $f_n$  is Ulam's density function associated with  $\tau$  based on the partition  $\mathcal{P}_n = \{I_i\}_{i=1}^n$ , then we have*

$$\mu_n(\tau^{-1}(I_i)) = \mu_n(I_i), \quad 1 \leq i \leq n.$$

*Proof.* Let  $A = (a_{ij})$  be Ulam's matrix associated with  $\tau$  based on the partition  $\mathcal{P}_n$ , so that

$$a_{ij} = \frac{m(\tau^{-1}(I_i) \cap I_j)}{m(I_j)}, \quad 1 \leq i, j \leq n.$$

So we have

$$m(I_j)a_{ij} = m(\tau^{-1}(I_i) \cap I_j), \quad 1 \leq i, j \leq n$$

or

$$\frac{b-a}{n} a_{ij} = m(\tau^{-1}(I_i) \cap I_j), \quad 1 \leq i, j \leq n.$$

Recall that  $Ay = y$  and  $f_n = \sum_{j=1}^n y_j \chi_{I_j}$ .

We first compute  $\mu_n(\tau^{-1}(I_i))$ . Observe that

$$\begin{aligned} \mu_n(\tau^{-1}(I_i)) &= \sum_{j=1}^n \mu_n(\tau^{-1}(I_i) \cap I_j) \\ &= \sum_{j=1}^n \int_{\tau^{-1}(I_i) \cap I_j} f_n \, dm \\ &= \sum_{j=1}^n y_j m(\tau^{-1}(I_i) \cap I_j) \end{aligned}$$

$$= \frac{b-a}{n} \sum_{j=1}^n a_{ij} y_j = \frac{b-a}{n} [Ay]_i = \frac{b-a}{n} y_i.$$

On the other hand, note that

$$\mu_n(I_i) = \int_{I_i} f_n dm = y_i m(I_i) = \frac{b-a}{n} y_i.$$

So we have proved that

$$\mu_n(\tau^{-1}(I_i)) = \mu_n(I_i), \quad 1 \leq i \leq n.$$

□

**Remark 4.** Let  $\mathcal{P}_{2^k} = \{I_i\}_{i=1}^{2^k}$  be a partition of  $[a, b]$  into  $2^k$  equal length subintervals for  $k = 1, 2, \dots$ . Then by Lemma 1 the measure  $\mu_{2^n}$  is  $\tau$ -invariant on  $\mathcal{P}_{2^n}$ . But since  $\mu_{2^n}$  is a measure, it is finitely additive. So it follows that  $\mu_{2^n}$  is  $\tau$ -invariant on  $\cup_{k=1}^n \mathcal{P}_{2^k}$ .

**4. Invariant measures.** In this section we will prove that if a measure  $\mu$  is a limit point of the measures  $\{\mu_{2^n}\}_{n \geq 1}$  in  $*$ -weak topology, then  $\mu$  is  $\tau$ -invariant. We recall that the convergence of  $\mu_{2^n}$  to  $\mu$  in  $*$ -weak topology means that

$$\lim_{n \rightarrow \infty} \int_a^b g d\mu_{2^n} = \int_a^b g d\mu, \quad \forall g \in C[a, b].$$

(See, for example, [4, Chapter 2, Definition 2.2.8]).

We first prove the following lemma.

**Lemma 2.** *Suppose  $g$  is a step function on  $\mathcal{P}_{2^n} = \{I_i\}_{i=1}^{2^n}$ . Then*

$$\int_a^b g d\mu_{2^n} = \int_a^b g \circ \tau d\mu_{2^n}.$$

*Proof.* Write  $g = \sum_{i=1}^{2^n} a_i \chi_{I_i}$ . Then clearly

$$\int_a^b g d\mu_{2^n} = \sum_{i=1}^{2^n} a_i \mu_{2^n}(I_i).$$

On the other hand, using Lemma 1 we also have

$$\begin{aligned} \int_a^b g \circ \tau d\mu_{2^n} &= \sum_{i=1}^{2^n} a_i \int_a^b \chi_{I_i} \circ \tau d\mu_{2^n} \\ &= \sum_{i=1}^{2^n} a_i \int_a^b \chi_{\tau^{-1}(I_i)} d\mu_{2^n} \\ &= \sum_{i=1}^{2^n} a_i \mu_{2^n}(\tau^{-1}(I_i)) \\ &= \sum_{i=1}^{2^n} a_i \mu_{2^n}(I_i). \end{aligned}$$

So, we have shown that  $\int_a^b g d\mu_{2^n} = \int_a^b g \circ \tau d\mu_{2^n}$ , provided that  $g$  is a step function on  $\mathcal{P}_{2^n} = \{I_i\}_{i=1}^{2^n}$ . □

We will need the following result in proving Proposition 1.

**Theorem 2.** *Let  $\{\nu_n\}_{n \geq 1}$  be a sequence of probability measures. Then there exists a subsequence  $\{\nu_{n_k}\}_{k \geq 1}$  such that  $\nu_{n_k} \rightarrow \nu$  in  $*$ -weak topology as  $k \rightarrow \infty$  and  $\nu$  is a probability measure, that is, the set of probability measures is compact in  $*$ -weak topology.*

*Proof.* See, for example [4, Chapter 2, Corollary 2.2.3]. □

We also need the following result.

**Theorem 3.** *Let  $\tau : [a, b] \rightarrow [a, b]$  be a measurable map. Then a measure  $\mu$  is  $\tau$ -invariant if and only if*

$$\int_a^b g \, d\mu = \int_a^b g \circ \tau \, d\mu \quad \text{for any } g \in C[a, b].$$

*Proof.* See, for example, [4, Chapter 3, Theorem 3.1.2]. □

Now we prove the following.

**Proposition 1.** *Suppose  $\tau : [a, b] \rightarrow [a, b]$  is in  $C[a, b]$ . Let us consider a sequence of Ulam's densities  $\{f_{2^n}\}_{n \geq 1}$  associated with  $\tau$  based on the sequence of partitions  $\{\mathcal{P}_{2^n}\}_{n \geq 1}$ . Let  $\mu_{2^n}$  be probability measure with density  $f_{2^n}$ . Since  $\{\mu_{2^n}\}_{n \geq 1}$  is a family of probability measures, by Theorem 2 there exists a subsequence  $\{\mu_{2^{n_k}}\}_{k \geq 1}$  such that  $\mu_{2^{n_k}} \rightarrow \mu$  as  $k \rightarrow \infty$  in  $*$ -weak topology, where  $\mu$  is a probability measure. Then  $\mu$  is  $\tau$ -invariant.*

*Proof.* For notational convenience, we will write  $\{\mu_{2^{n_k}}\}_{k \geq 1}$  as  $\{\mu_{2^n}\}_{n \geq 1}$ . By Theorem 3, it is enough to show that

$$\int_a^b g \, d\mu = \int_a^b g \circ \tau \, d\mu \quad \text{for any } g \in C[a, b].$$

Suppose  $g \in C[a, b]$ . Let  $\epsilon > 0$ . For each positive integer  $n$ , let  $g_{2^n} = \sum_{i=1}^{2^n} c_i \chi_{I_i}$ , where each  $I_i \in \mathcal{P}_{2^n}$  and  $c_i$  is the infimum value of  $g$  on  $I_i$ . Since  $g \in C[a, b]$ ,  $g_{2^n} \rightarrow g$  uniformly on the interval  $[a, b]$ . In particular, we can find an  $N_1 \geq 1$  such that

$$n \geq N_1 \implies \sup_{[a, b]} |g - g_{2^n}| < \epsilon/4.$$

This implies also

$$n \geq N_1 \implies \sup_{[a, b]} |g \circ \tau - g_{2^n} \circ \tau| < \epsilon/4.$$

Since  $\mu_{2^n} \rightarrow \mu$  in  $*$ -weak topology and  $g, \tau \in C[a, b]$ , we can find  $N_2$  and  $N_3$  such that

$$n \geq N_2 \implies \left| \int_a^b g \, d\mu - \int_a^b g \, d\mu_{2^n} \right| < \epsilon/4,$$

and

$$n \geq N_3 \implies \left| \int_a^b g \circ \tau \, d\mu - \int_a^b g \circ \tau \, d\mu_{2^n} \right| < \epsilon/4.$$

Now let  $N = \max\{N_1, N_2, N_3\}$ . Let  $n \geq N$ . Then using Lemma 2 and the fact that  $\mu_{2^n}$  is a probability measure, we have

$$\begin{aligned}
& \left| \int_a^b g \, d\mu - \int_a^b g \circ \tau \, d\mu \right| \\
& \leq \left| \int_a^b g \, d\mu - \int_a^b g \, d\mu_{2^n} \right| + \left| \int_a^b g \, d\mu_{2^n} - \int_a^b g_{2^n} \, d\mu_{2^n} \right| \\
& \quad + \left| \int_a^b g_{2^n} \, d\mu_{2^n} - \int_a^b g_{2^n} \circ \tau \, d\mu_{2^n} \right| + \left| \int_a^b g_{2^n} \circ \tau \, d\mu_{2^n} - \int_a^b g \circ \tau \, d\mu_{2^n} \right| \\
& \quad + \left| \int_a^b g \circ \tau \, d\mu_{2^n} - \int_a^b g \circ \tau \, d\mu \right| \\
& < \frac{\epsilon}{4} + \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

Since  $\epsilon > 0$  was arbitrary, we have  $\int_a^b g \, d\mu = \int_a^b g \circ \tau \, d\mu$ . Since  $g$  was an arbitrary function in  $C[a, b]$ ,  $\mu$  is  $\tau$ -invariant.  $\square$

The following example shows that even though the measure  $\mu$  in Proposition 1 is  $\tau$ -invariant, it may not be absolutely continuous measure w.r.t. the Lebesgue measure  $m$ .

**Example 1.** Let  $\tau : [0, 1] \rightarrow [0, 1]$  be defined by  $\tau(x) = x/2$  for  $\forall x \in [0, 1]$ . It is easy to show that  $f_{2^n} = 2^n \chi_{[0, 1/2^n]}$ . So for any  $g \in C[0, 1]$ , we have

$$\int_0^1 g \, d\mu_{2^n} = \int_0^1 g \cdot f_{2^n} \, dm = \int_0^1 g \cdot 2^n \chi_{[0, 1/2^n]} \, dm = 2^n \int_0^{\frac{1}{2^n}} g \, dm = g(x_n),$$

where  $x_n \in [0, 1/2^n]$ . So

$$\lim_{n \rightarrow \infty} \int_0^1 g \, d\mu_{2^n} = \lim_{n \rightarrow \infty} g(x_n) = g(0).$$

If we define  $\mu$  as

$$\mu(B) = \begin{cases} 1 & \text{if } 0 \in B \\ 0 & \text{if } 0 \notin B \end{cases},$$

then clearly  $\int_0^1 g \, d\mu = g(0)$ . So we see that  $\mu_n \rightarrow \mu$  as  $n \rightarrow \infty$  in \*-weak topology. So by Proposition 1  $\mu$  is  $\tau$ -invariant, but clearly  $\mu$  is not absolutely continuous with respect to the Lebesgue measure  $m$ .

**5. Absolutely continuous invariant measures.** In this section we consider constructing  $\tau$ -invariant measures which are also absolutely continuous measures w.r.t. the Lebesgue measure  $m$ .

**Definition 7.** The set of functions  $\mathcal{F} \subseteq L^1[a, b]$  is called weakly precompact if every sequence of functions  $\{g_n\} \subseteq \mathcal{F}$  contains a subsequence  $\{g_{n_k}\}$  that converges weakly to a function  $g \in L^1[a, b]$ .

Recall that weak convergence of  $\{g_{n_k}\}$  to  $g$  in  $L^1[a, b]$  means that for any  $h \in L^\infty[a, b]$  we have

$$\int_a^b g_{n_k} \cdot h \, dm \rightarrow \int_a^b g \cdot h \, dm \quad \text{as } k \rightarrow \infty.$$



Now the following sufficient condition of the family of sets in  $L^1[a, b]$  being weakly precompact is useful.

**Theorem 4.** *Let  $g \in L^1[a, b]$  be a nonnegative function. Then the set of all functions  $f \in L^1[a, b]$  such that*

$$|f(x)| \leq g(x) \quad m - \text{a.e.}$$

*is weakly precompact in  $L^1[a, b]$ .*

*Proof.* See, for example [23, page 87]. □

We now prove the following:

**Proposition 2.** *Let  $\tau : [a, b] \rightarrow [a, b]$  be nonsingular. Let us consider a sequence of Ulam's densities  $\{f_{2^n}\}_{n \geq 1}$  associated with  $\tau$  base on the sequence of partitions  $\{\mathcal{P}_{2^n}\}_{n \geq 1}$ . Let  $\mu_{2^n}$  be the measure with density  $f_{2^n}$  for each  $n$ . If the family  $\{f_{2^n}\}_{n \geq 1}$  is weakly precompact in  $L^1[a, b]$  and hence  $f_{2^{n_k}} \rightarrow f^*$  weakly in  $L^1[a, b]$  as  $k \rightarrow \infty$ , then the measure  $\mu$  with density  $f^*$  is an absolutely continuous  $\tau$ -invariant probability measure.*

*Proof.* For notational convenience we may assume that  $f_{2^n} \rightarrow f^*$  weakly in  $L^1[a, b]$  as  $n \rightarrow \infty$ . We first show that  $\mu$  is  $\tau$ -invariant on  $\cup_{n \geq 1} \mathcal{P}_{2^n}$ . Suppose  $I \in \cup_{n \geq 1} \mathcal{P}_{2^n}$ . Then  $I \in \mathcal{P}_{2^{N_1}}$  for some  $N_1$ . By Remark 4 we have  $\mu_{2^n}(I) = \mu_{2^n}(\tau^{-1}(I))$  for all  $n \geq N_1$ , that is,

$$\int_I f_{2^n} dm = \int_{\tau^{-1}(I)} f_{2^n} dm, \quad n \geq N_1$$

or

$$\int_a^b f_{2^n} \chi_I dm = \int_a^b f_{2^n} \chi_{\tau^{-1}(I)} dm, \quad n \geq N_1.$$

Let  $\epsilon > 0$  be given. Since  $f_{2^n}$  converges to  $f^*$  weakly and  $\chi_I \in L^\infty[a, b]$ , there exists  $N_2$  such that

$$\left| \int_a^b f_{2^n} \chi_I dm - \int_a^b f^* \chi_I dm \right| < \frac{\epsilon}{2}, \quad n \geq N_2.$$

Similarly, there exists  $N_3$  such that

$$\left| \int_a^b f_{2^n} \chi_{\tau^{-1}(I)} dm - \int_a^b f^* \chi_{\tau^{-1}(I)} dm \right| < \frac{\epsilon}{2}, \quad n \geq N_3.$$

Let  $N = \max\{N_1, N_2, N_3\}$ . Observe that if  $n \geq N$ , then we have

$$\begin{aligned} |\mu(I) - \mu(\tau^{-1}(I))| &= \left| \int_I f^* dm - \int_{\tau^{-1}(I)} f^* dm \right| \\ &= \left| \int_a^b f^* \chi_I dm - \int_a^b f^* \chi_{\tau^{-1}(I)} dm \right| \\ &\leq \left| \int_a^b f^* \chi_I dm - \int_a^b f_{2^n} \chi_I dm \right| \\ &\quad + \left| \int_a^b f_{2^n} \chi_I dm - \int_a^b f_{2^n} \chi_{\tau^{-1}(I)} dm \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_a^b f_{2^n} \chi_{\tau^{-1}(I)} dm - \int_a^b f^* \chi_{\tau^{-1}(I)} dm \right| \\
& < \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since  $\epsilon > 0$  was arbitrary, we have  $\mu(I) = \mu(\tau^{-1}(I))$ . Since  $I$  was an arbitrary element in  $\cup_{n \geq 1} \mathcal{P}_{2^n}$ ,  $\mu$  is  $\tau$ -invariant on  $\cup_{n \geq 1} \mathcal{P}_{2^n}$ .

Now we show that  $\mu$  is  $\tau$ -invariant on  $\mathcal{B}$ . Since  $\tau$  is nonsingular, both  $\mu$  and  $\mu \circ \tau^{-1}$  are absolutely continuous measures w.r.t. the Lebesgue measure  $m$ . So for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$m(A) < \delta \implies \mu(A) < \frac{\epsilon}{6} \text{ and } (\mu \circ \tau^{-1})(A) < \frac{\epsilon}{6}. \quad (9)$$

Now suppose  $B \in \mathcal{B}$ . Then there is a finite disjoint collection of finite intervals  $\{I_k\}_{k=1}^m$  such that

$$m(B - I) + m(I - B) < \delta, \quad (10)$$

where  $I = \bigcup_{k=1}^m I_k$ . Since the collection of the end points of the intervals in  $\{\mathcal{P}_{2^n}\}_{n \geq 1}$  is dense in the interval  $[a, b]$ , for each  $k$  we can choose an interval  $J_k$  such that

- (a)  $J_k \subseteq I_k$ ;
- (b)  $J_k$  is a finite disjoint union of the intervals in  $\mathcal{P}_{2^{n_k}}$  for some  $n_k$ ;
- (c)  $m(I_k - J_k) < \delta/m$ .

So if we let  $J = \bigcup_{k=1}^m J_k$ , we have

$$\begin{aligned}
m(I - J) &= m\left(\bigcup_{k=1}^m I_k - \bigcup_{k=1}^m J_k\right) = m\left(\bigcup_{k=1}^m (I_k - J_k)\right) \\
&= \sum_{k=1}^m m(I_k - J_k) < \sum_{k=1}^m \frac{\delta}{m} = \delta.
\end{aligned} \quad (11)$$

Since  $\mu = \mu \circ \tau^{-1}$  on  $\{\mathcal{P}_{2^n}\}_{n \geq 1}$ , it follows that  $\mu(J_k) = (\mu \circ \tau^{-1})(J_k)$  for  $1 \leq k \leq m$ , and hence it follows that

$$\mu(J) = (\mu \circ \tau^{-1})(J). \quad (12)$$

So using (9), (11), and (12), we have

$$\begin{aligned}
|\mu(I) - (\mu \circ \tau^{-1})(I)| &= |\mu(J \cup (I - J)) - (\mu \circ \tau^{-1})(J \cup (I - J))| \\
&= |\mu(J) + \mu(I - J) - (\mu \circ \tau^{-1})(J) - (\mu \circ \tau^{-1})(I - J)| \\
&= |\mu(I - J) - (\mu \circ \tau^{-1})(I - J)| \\
&\leq |\mu(I - J)| + |(\mu \circ \tau^{-1})(I - J)| \\
&< \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\end{aligned} \quad (13)$$

Now we let

$$C = B - I, \quad D = B \cap I, \quad \text{and} \quad E = I - B$$

and note that by (10) we have

$$m(C) < \delta \text{ and } m(E) < \delta. \quad (14)$$

So, using (9), (13), and (14), we have

$$\begin{aligned}
|\mu(D) - (\mu \circ \tau^{-1})(D)| &= |\mu(I - E) - (\mu \circ \tau^{-1})(I - E)| \\
&= |\mu(I) - \mu(E) - (\mu \circ \tau^{-1})(I) + (\mu \circ \tau^{-1})(E)| \\
&\leq |\mu(I) - (\mu \circ \tau^{-1})(I)| + \mu(E) + (\mu \circ \tau^{-1})(E) \quad (15) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{2\epsilon}{3}.
\end{aligned}$$

Finally, using (9), (14), and (15), we have

$$\begin{aligned}
|\mu(B) - (\mu \circ \tau^{-1})(B)| &= |\mu(C \cup D) - (\mu \circ \tau^{-1})(C \cup D)| \\
&= |\mu(C) + \mu(D) - (\mu \circ \tau^{-1})(C) - (\mu \circ \tau^{-1})(D)| \\
&\leq |\mu(D) - (\mu \circ \tau^{-1})(D)| + \mu(C) + (\mu \circ \tau^{-1})(C) \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\end{aligned}$$

Since  $\epsilon$  was arbitrary, we have  $\mu(B) = (\mu \circ \tau^{-1})(B)$ . Since  $B$  was an arbitrary element of  $\mathcal{B}$ ,  $\mu$  is  $\tau$ -invariant on  $\mathcal{B}$ . Since  $\mu$  has its density function  $f^* \in L^1[a, b]$ , it is also an absolutely continuous probability measure w.r.t. the Lebesgue measure.  $\square$

**Example 2.** Let  $\tau : [0, 1] \rightarrow [0, 1]$  be defined by

$$\tau(x) = \begin{cases} \frac{2x}{1-x}, & 0 \leq x \leq \frac{1}{3} \\ \frac{1-x}{2x}, & \frac{1}{3} \leq x \leq 1. \end{cases}$$

Using Ulam's method, we found  $\{f_{2^n}\}_{n=1}^{10}$  and obtained the data displayed in Table 1. It shows strong evidence that  $\{f_{2^n}\}_{n \geq 1}$  is bounded by 2. So, by Theorem 4,  $\{f_{2^n}\}_{n \geq 1}$  is weakly pre-compact, and hence there exists a sub-sequence  $\{f_{2^{n_k}}\}_{n \geq 1}$  of  $\{f_{2^n}\}_{n \geq 1}$  which converges weakly to  $f^* \in L^1[a, b]$ . Thus by Proposition 2 the measure  $\mu$  with density  $f^*$  is an absolutely continuous  $\tau$ -invariant probability measure  $\mu$ . Then by Theorem 1, we have  $P_\tau(f^*) = f^*$ . Since  $\tau$  is ergodic,  $f^*$  is the unique fixed density of  $P_\tau$ . Hence again by Theorem 1 we see that  $\mu$  is the unique absolutely continuous  $\tau$ -invariant probability measure on  $\mathcal{B}$ . One can verify that

$$f^*(x) = \frac{2}{(1+x)^2}.$$

Since  $f^*$  is the unique fixed density of  $P_\tau$ , we may expect that  $\{f_{2^n}\}_{n \geq 1}$  converges strongly to  $f^*$  (see [24]). See Figure 1 for the plot of  $f_{2^5}$  and  $f^*$ .

**Example 3.** Let  $\tau : [0, 1] \rightarrow [0, 1]$  be defined by  $\tau(x) = 4x(1-x)$ . Using Ulam's method, we found  $\{f_{2^n}\}_{n=5}^{10}$ . Each  $f_n$  has two peak values. Those peak values happened near 0 and near 1 for each  $n$ . We compiled those peak values and displayed them in Table 2. We also presented their ratios of two consecutive  $f_n$ 's, and displayed them in Table 3, which show that the limit behavior of  $\{f_{2^n}\}_{n \geq 1}$  is like  $O(1/\sqrt{x})$  near 0 and  $O(1/\sqrt{1-x})$  near 1. Since both  $1/\sqrt{x}$  and  $1/\sqrt{1-x}$  are in  $L^1[0, 1]$ , the collection  $\{f_{2^n}\}_{n \geq 1}$  is bounded by some  $g \in L^1[0, 1]$ . So, by Theorem 4,  $\{f_{2^n}\}_{n \geq 1}$  is weakly precompact, and hence by Proposition 2 there is a density  $f^* \in L^1[a, b]$  such that if the measure  $\mu$  has  $f^*$  as its density, then the measure  $\mu$  is the absolutely continuous  $\tau$ -invariant probability measure. Since  $\tau$  is ergodic,  $f^*$  the only density function which makes  $\mu$   $\tau$ -invariant. In fact, it is well known that

$$f^*(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

TABLE 1. Maximum values of  $f_{2^n}$ 

$n$	$2^n$	maximum values of $f_{2^n}$
1	2	1.25
2	4	1.5075
3	8	1.7232
4	16	1.8157
5	32	1.9126
6	64	1.9465
7	128	1.9748
8	256	1.9871
9	512	1.9930
10	1024	1.9964

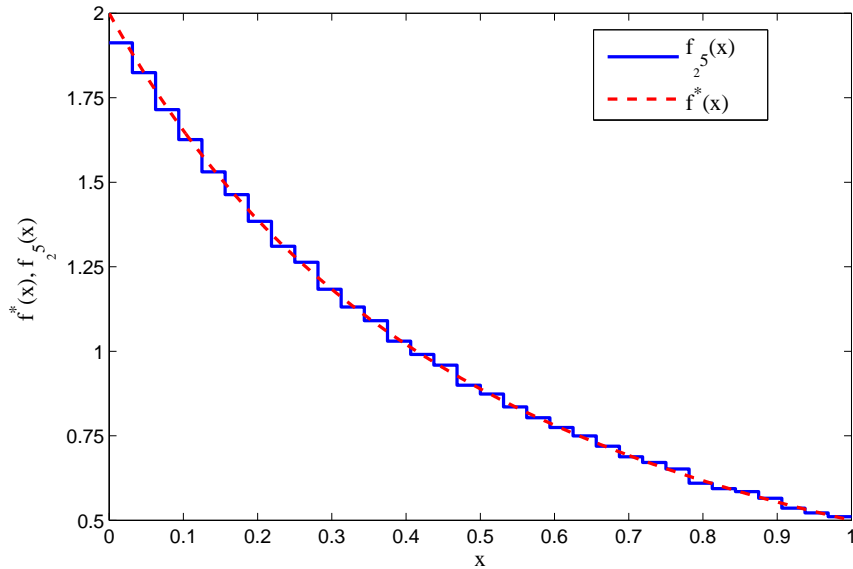


FIGURE 1. Graphs of the probability density function  $f^*$  and its estimation  $f_{2^5}$  for Example 2. Since  $\{f_{2^n}\}_{n \geq 1}$  is bounded by 2, by Theorem 4  $\{f_{2^n}\}_{n \geq 1}$  is weakly pre-compact, and hence there exists a subsequence  $\{f_{2^{n_k}}\}_{n \geq 1}$  of  $\{f_{2^n}\}_{n \geq 1}$  which converges weakly to  $f^* \in L^1[a, b]$ . In fact, we may expect that  $\{f_{2^n}\}_{n \geq 1}$  converges strongly to  $f^*$ .

Again we may expect that  $\{f_{2^n}\}_{n \geq 1}$  converges strongly to  $f^*$ . See Figure 2 for the plot of  $f_{2^9}$  and  $f^*$ .

**Example 4.** Let  $\tau : [0, 1] \rightarrow [0, 1]$  be defined by  $\tau(x) = x/2$ . As was discussed in Example 1, Ulam's method gives  $\{f_{2^n}\}_{n \geq 1} = \{2^n \chi_{[0, 1/2^n]}\}_{n \geq 1}$ , and hence the peak value of  $f_{2^n}$  is  $2^n$ , and the ratio of the peak values of two consecutive  $f_n$ 's is 2. This suggests that the limit behavior of  $\{f_{2^n}\}_{n \geq 1}$  is like  $O(1/x)$  near 0. Since  $1/x$

TABLE 2. Peak values of  $f_{2^n}$  in Example 3

$n$	$2^n$	1st peak value of $f_{2^n}$	2nd peak value of $f_{2^n}$
5	32	1.2899	3.8291
6	64	1.8337	5.4724
7	128	2.4475	7.3235
8	256	3.5149	10.5309
9	512	4.8498	14.5401
10	1024	6.9200	20.7534

TABLE 3. The consecutive ratio of peak values of  $f_{2^n}$  in Example 3

$n$	$2^n$	$f_{2^n}/f_{2^{n-1}}$ at 1st peak value	$f_{2^n}/f_{2^{n-1}}$ at 2nd peak value
6	64	1.4216	1.4292
7	128	1.3347	1.3382
8	256	1.4361	1.4380
9	512	1.3798	1.3807
10	1024	1.4269	1.4273

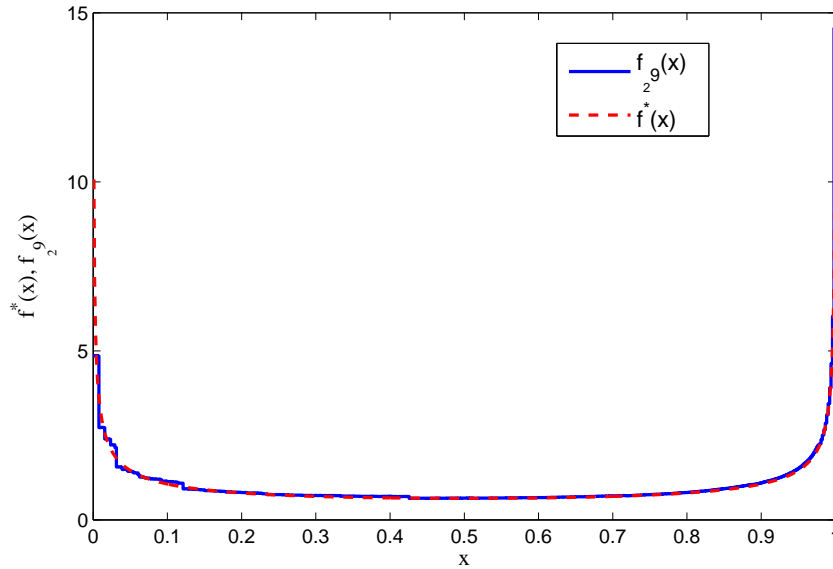


FIGURE 2. Graphs of the probability density function  $f^*$  and its estimation  $f_{2^9}$  for Example 3. Again we may expect that  $\{f_{2^n}\}_{n \geq 1}$  converges strongly to  $f^*$ .

is not in  $L^1[0, 1]$ , the collection  $\{f_{2^n}\}_{n \geq 1}$  is not bounded by an integrable function, and hence Theorem 4 and Proposition 2 do not apply. In fact, in Example 1 we showed that there is no absolutely continuous invariant measure in this case.

**6. Applications to biological systems.** The recursive equation

$$x_{n+1} = f(x_n) + \gamma x_n,$$

can be used for nonlinear discrete-time epidemic models. Here  $\gamma$  ( $0 \leq \gamma < 1$ ) denotes the constant probability of surviving per generation;  $f$  models the typically nonlinear birth or recruitment process;  $x(n)$  denotes the population size at time  $n$ ; and time is measured in discrete units.

If

$$f(x_n) = \frac{\alpha x_n}{1 + (\beta x_n)^p},$$

the above recursive equation is called the Smith-Slatkin model. Hence the Smith-Slatkin model gives

$$x_{n+1} = \frac{\alpha x_n}{1 + (\beta x_n)^p} + \gamma x_n.$$

Here  $\alpha$  is the maximal per-capita intrinsic growth rate of the population;  $p$  reflects the type and strength of intra-specific competition; and  $\beta$  scales the carrying capacity of the population. Details of this model and its applications in ecology and epidemiology have been mentioned in [6].

We write the above recursive equation as

$$x_{n+1} = \tau(x_n),$$

where

$$\tau(x) = \frac{\alpha x}{1 + (\beta x)^p} + \gamma x. \quad (16)$$

If we let  $x^*$  denote the fixed population of the Smith-Slatkin model under  $\tau$ , from the equation

$$x^* = \tau(x^*),$$

we obtain

$$x^* = \frac{1}{\beta} \left( \frac{\alpha + \gamma - 1}{1 - \gamma} \right)^{\frac{1}{p}}.$$

The following lemma establishes the condition under which  $|\tau'(x^*)| > 1$ .

**Lemma 3.** *If  $p > \frac{2\alpha}{(1-\gamma)(\alpha+\gamma-1)}$ , then  $|\tau'(x^*)| > 1$ .*

*Proof.* Since

$$\tau'(x) = \frac{\alpha[1 + (\beta x)^p] - \alpha p(\beta x)^p}{[1 + (\beta x)^p]^2} + \gamma,$$

by a direct computation we obtain

$$\tau'(x^*) = 1 - \frac{(1-\gamma)(\alpha+\gamma-1)}{\alpha}.$$

It follows that if  $p > \frac{2\alpha}{(1-\gamma)(\alpha+\gamma-1)}$ , then  $|\tau'(x^*)| > 1$ .  $\square$

Whenever  $0 < p < \frac{2\alpha}{(1-\gamma)(\alpha+\gamma-1)}$ ,  $\{x_n\}_{n \geq 1}$  approaches to a simple demographic attractor  $x^*$ . But, as  $p$  increases past  $\frac{2\alpha}{(1-\gamma)(\alpha+\gamma-1)}$ , the equilibrium point undergoes period-doubling bifurcation and eventually to chaos, see Figure 3. See also [6].

The critical point of  $\tau$  to the left of  $x^*$  is

$$x_c = \frac{1}{\beta} \left( -\frac{\sqrt{\alpha^2 p^2 - 2\alpha^2 p - 4\alpha\gamma p + \alpha^2} - (\alpha p - \alpha - 2\gamma)}{2\gamma} \right)^{1/p}.$$

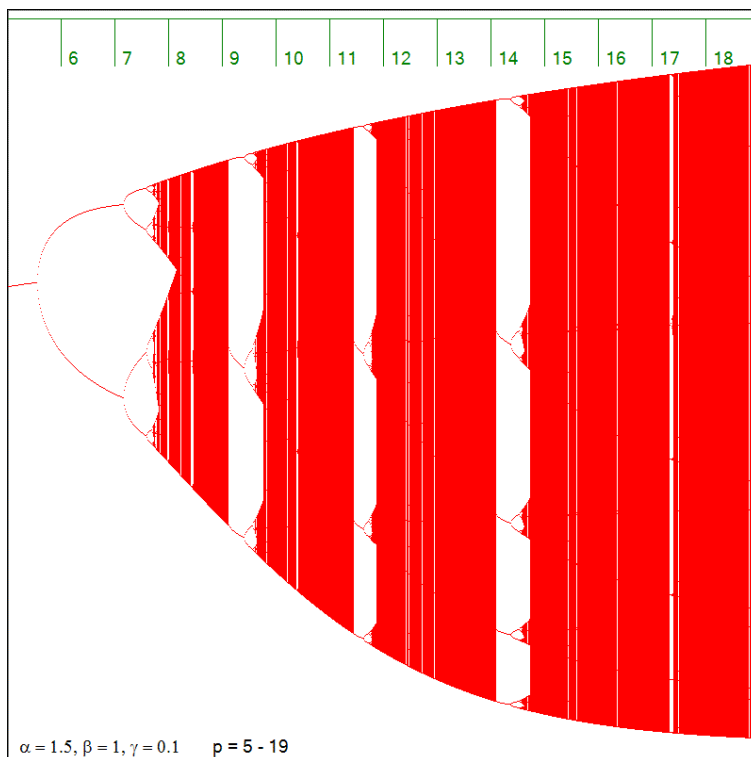


FIGURE 3. Bifurcation diagram of Smith-Slatkin map for  $\alpha = 1.5, \beta = 1, \gamma = 0.1$ , as  $p$  changes from 5 to 19.

We will study the dynamics of  $\tau$  on the invariant interval  $[a = \tau^2(x_c), b = \tau(x_c)]$ , see Figure 4.

We will use the Schwarzian derivative of  $\tau$  defined by

$$S\tau = \frac{\tau'''}{\tau'} - \frac{3}{2} \left( \frac{\tau''}{\tau'} \right)^2.$$

Its negativity can be used to prove the existence of absolutely continuous invariant measure [27, 28, 31, 21, 30]. First, we will prove that  $S\tau < 0$  for a wide range of parameters.

**Proposition 3.** For  $p > 2$  and  $\frac{\gamma}{\alpha} < \frac{p^3 - 3p^2 + 5p - 3}{6(p^2 + 1)}$  we have  $S\tau < 0$  for  $x > 0$ .

*Proof.* Let  $p > 2$ . Let  $f_1(x) = x/(1 + x^p) + kx$ , where  $k = \frac{\gamma}{\alpha}$ . It can be proved by elementary but tedious calculations that

$$Sf_1(x) = \frac{-F(x)}{2x^2(1 + x^p)^2(x^{2p}k - x^p(p - 1 - 2k) + k + 1)^2},$$

where

$$\begin{aligned} F(x) = & x^{4p}2k(p^2 - 1) + x^{3p}(p^3 - p^2 + 5p - 2 - 4k(p^2 + 2)) \\ & + x^{2p}(2p^3 - 6p^2 + 10p - 6 - 12k(p^2 + 1)) \\ & + x^p(p^3 + 5p - 6 - k(4p^2 + 8)) + p^2(k + 2) - 2(k + 1). \end{aligned}$$

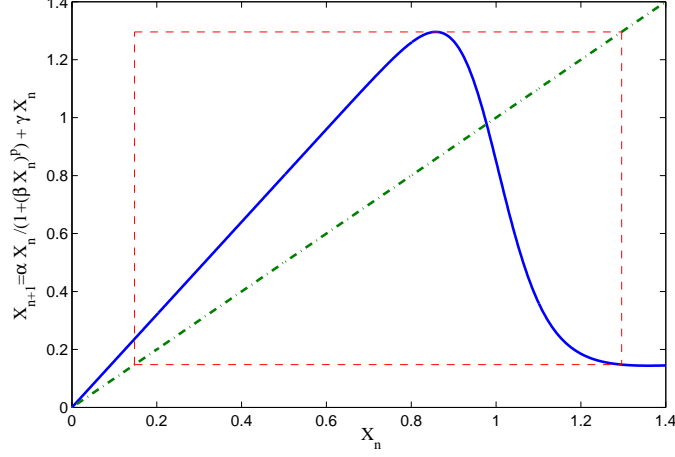


FIGURE 4. Graph of Smith-Slatkin map for  $\alpha = 1.5, \beta = 1, \gamma = 0.1$  and  $p = 18$  with  $x^* = 0.977725973073732$ ,  $x_c = 0.857817481013884$ ,  $b = \tau(x_c) = 1.295956066342802$ , and  $a = \tau^2(x_c) = 0.147708628629491$ . Note that  $\tau : [a, b] \rightarrow [a, b]$ .

Thus, a sufficient condition for  $Sf_1 < 0$  is

$$k < \min \left\{ \frac{p^3 - 3p^2 + 5p - 3}{6(p^2 + 1)}, \frac{p^3 - p^2 + 5p - 2}{4(p^2 + 2)}, \frac{p^3 + 5p - 6}{4(p^2 + 2)} \right\},$$

which for  $p > 2$  is equivalent to

$$k < \frac{p^3 - 3p^2 + 5p - 3}{6(p^2 + 1)}.$$

Let  $g(x) = \beta x$ . Since in general  $S(f \circ g) = (Sf \circ g)(g')^2 + S(g)$  and in our case  $S(g) = 0$  we get  $S(f_1 \circ g) < 0$  or  $Sf_2 < 0$  for  $f_2(x) = \beta x / (1 + (\beta x)^p) + \frac{\gamma}{\alpha} x$ . Since for any function  $g$  we have  $S(\text{const} \cdot g) = S(g)$  we also have  $S\tau < 0$  since  $\tau(x) = \frac{\alpha}{\beta} f_2(x)$ .  $\square$

In Figure 5 we plot the graph of  $\frac{p^3 - 3p^2 + 5p - 3}{6(p^2 + 1)}$  on the  $p$ -interval  $[2, 20]$ . For example, if  $\alpha = 1.5$  and  $\gamma = 0.1$ , the condition  $\frac{\gamma}{\alpha} < \frac{p^3 - 3p^2 + 5p - 3}{6(p^2 + 1)}$  in Proposition 3 is fulfilled for all  $p > 2$ .

A strong result giving sufficient condition for the existence of acim for such maps was given in [31]. We give a simplified version applicable to our map  $\tau$ :

**Theorem 5.** *Suppose that  $\tau$  is unimodal,  $C^3$  and has a negative Schwarzian derivative. Moreover assume that*

$$\sum_{n=0}^{\infty} |(\tau^{n+1})'(x_c)|^{-1/2} < +\infty.$$

*Then,  $\tau$  has a unique absolutely continuous invariant probability measure  $\mu$  which is ergodic and of positive entropy. Furthermore there exists a positive constant  $K$  such that  $\mu(A) < K\sqrt{m(A)}$  for any  $A \in \mathcal{B}$ .*



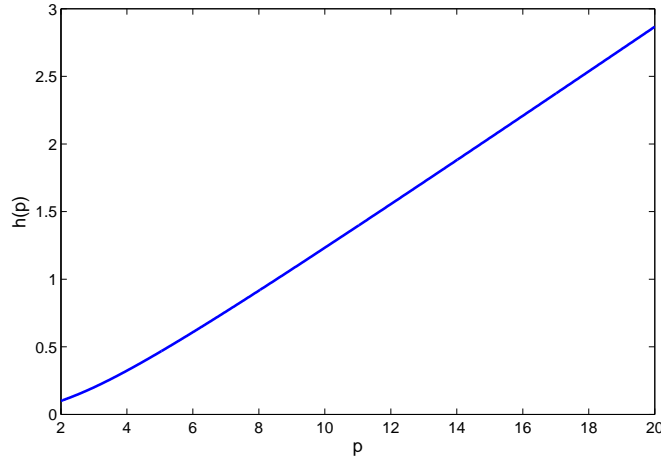


FIGURE 5. Graph of  $\frac{p^3-3p^2+5p-3}{6(p^2+1)}$ . For  $\alpha = 1.5$  and  $\gamma = 0.1$ , the condition  $\frac{\gamma}{\alpha} < \frac{p^3-3p^2+5p-3}{6(p^2+1)}$  in Proposition 3 is fulfilled for all  $p > 2$ .

As it is shown in Figures 6 and 7, when  $\alpha = 1.5$ ,  $\beta = 1$ , and  $\gamma = 0.1$ , for  $p = 8$  the third image of the critical point  $\tau^3(x_c)$  is above the fixed point  $x^*$  and for  $p = 8.5$  we have  $\tau^3(x_c) < x^*$ . Thus, for some value  $p = \hat{p}$  in between we have  $\tau^3(x_c) = x^*$ . We found that  $\hat{p} \sim 8.151184748169387$ . Note that

$$\hat{p} > \frac{2\alpha}{(1-\gamma)(\alpha+\gamma-1)} = \frac{2(1.5)}{(1-0.1)(1.5+0.1-1)} = \frac{50}{9}.$$

So by Lemma 3 we have  $|\tau'(x^*)| > 1$ . Note that the mapping  $\tau$  defined (16) on the interval  $[a, b]$  is unimodal and  $C^3$ . We already observed that if  $\alpha = 1.5$  and  $\gamma = 0.1$ , then the condition in Proposition 3 is fulfilled for all  $p > 2$ . Hence  $\tau$  with  $\alpha = 1.5$ ,  $\beta = 1$ ,  $\gamma = 0.1$ , and  $p = \hat{p}$ , has negative Schwarzian derivative on  $[a, b]$ .

**Corollary 1.** *Suppose  $\alpha = 1.5$ ,  $\beta = 1$ , and  $\gamma = 0.1$ . If  $p = \hat{p}$  the Smith-Slatkin map  $\tau$  admits a unique ergodic absolutely continuous invariant measure  $\mu$ . It is of positive entropy and  $\mu(A) < K\sqrt{m(A)}$  for any  $A \in \mathcal{B}$ .*

*Proof.* Since  $\tau^3(x_c) = x^*$ , for  $n \geq 3$ , the series in Theorem 5 becomes

$$|\tau'(x_c)|^{-1/2} + |(\tau^2)'(x_c)|^{-1/2} + |(\tau^3)'(x_c)|^{-1/2} \sum_{n=0}^{\infty} \left( |\tau'(x^*)|^{-1/2} \right)^n.$$

Since  $|\tau'(x^*)| > 1$ , this series converges. So the Corollary follows from Theorem 5.  $\square$

The famous theorem of Jakobson [27, 18, 19, 26, 37, 44] states that for a one parameter family of unimodal maps  $\tau_\alpha$  with negative Schwarzian derivative, similar to the family of parabolas  $x \mapsto \alpha x(1-x)$ , the set of parameters for which  $\tau_\alpha$  has an acim is “large” (e.g., it is of positive Lebesgue measure) and in particular it is of Lebesgue density 1 around each “Misiurewicz parameter” (a parameter for which some iterate of critical point hits a periodic point).

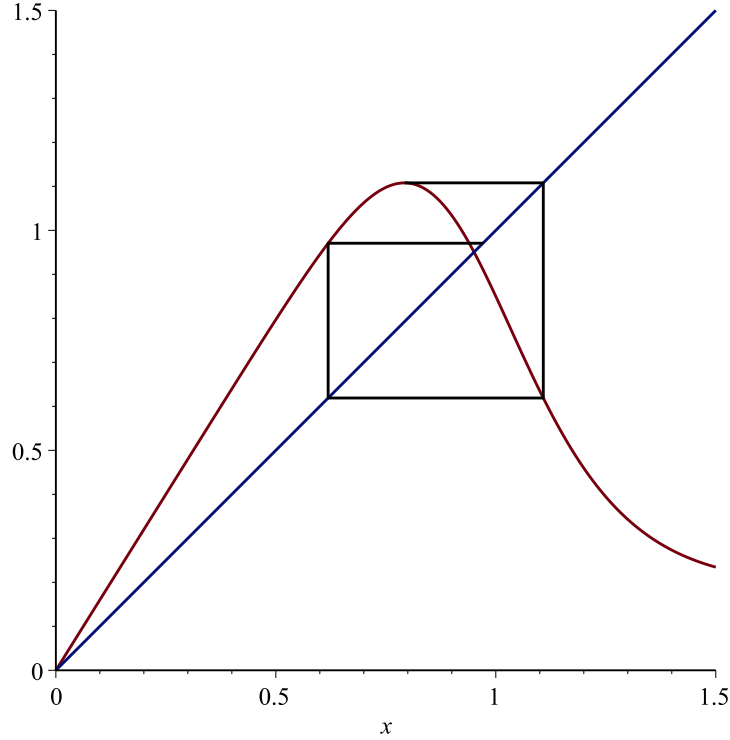


FIGURE 6. Graph of Smith-Slatkin map for  $\alpha = 1.5, \beta = 1, \gamma = 0.1$  and  $p = 8$ . Three first iterates of the critical point are shown.

Unfortunately, Smith-Slatkin family (say, with  $\alpha, \beta, \gamma$  fixed and  $p$  as a parameter) does not satisfy the assumptions of any version of Jakobson's theorem we know. On the other hand the bifurcation diagram of Figure 3 suggests that map  $\tau$  has an acim for a large set of  $p$ 's. The following Example reinforces this evidence.

**Example 5.** We let  $\alpha = 1.5, \beta = 1$ , and  $\gamma = 0.1$ . If we let  $p = 18$ , then the orbit  $\{\tau^n(x)\}_{n=0}^{\infty}$  exhibits a chaotic behavior. We have  $x_c = 0.857817481013884$ ,  $b = \tau(x_c) = 1.295956066342802$ , and  $a = \tau^2(x_c) = 0.147708628629491$ . Note that  $\tau : [a, b] \rightarrow [a, b]$ .

Now we want to study whether there is a density  $f^* \in L^1[a, b]$  such that if the measure  $\mu$  has  $f^*$  as its density function, then  $\mu$  is a  $\tau$ -invariant absolutely continuous probability measure on  $\mathcal{B}$ . If we can find such a density  $f^* \in L^1[a, b]$ , then by Remark 1  $f^*$  is the p.d.f. of the population random variable  $X$  whose occurrences are according to  $(x, \tau(x), \tau^2(x), \dots)$ . So even though the population at discrete times may exhibit a chaotic behavior, we can do statistical analysis of the population random variable  $X$ .

Using Ulam's method we computed  $\{f_{2^n}\}_{n=8}^{11}$ , and we found out that there are six peak values. We compiled those peak values and displayed them in Tables 4-6. We also present their ratios of two consecutive  $f_{2^n}$  and displayed them in Tables 7-9. Note that at each singular point  $\hat{x}$  the limit behavior is like  $O(1/\sqrt{\hat{x} - x})$  when  $x < \hat{x}$  and  $x \approx \hat{x}$  and  $O(1/\sqrt{x - \hat{x}})$  when  $x > \hat{x}$  and  $x \approx \hat{x}$ . Since  $1/\sqrt{\hat{x} - x}$  and

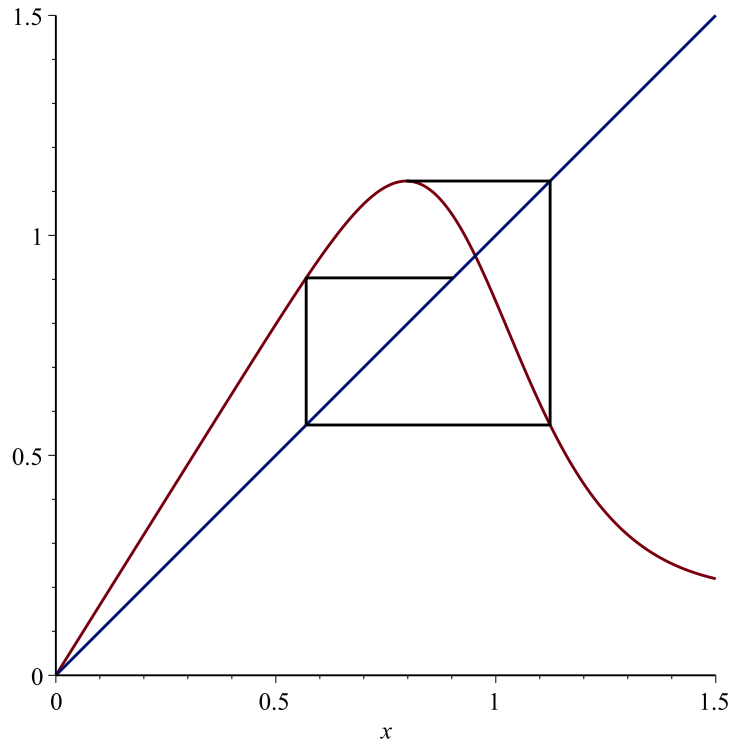


FIGURE 7. Graph of Smith-Slatkin map for  $\alpha = 1.5, \beta = 1, \gamma = 0.1$  and  $p = 8.5$ . Three first iterates of the critical point are shown.

TABLE 4. Peak values of  $f_{2^n}$  for  $\tau$  in Example 5

$n$	$2^n$	The 1st peak values	The 2nd peak values
8	256	9.9001	6.5190
9	512	13.987	9.0862
10	1024	20.609	12.770
11	2048	28.397	16.860

$1/\sqrt{\hat{x} - x_*}$  are in  $L^1[a, b]$ , the collection  $\{f_{2^n}\}_{n \geq 1}$  is bounded by some  $g \in L^1[a, b]$ . So by Theorem 5,  $\{f_{2^n}\}_{n \geq 1}$  is weakly precompact and hence by proposition 2 there is a density  $f^* \in L^1[a, b]$  such that if the measure  $\mu$  has  $f^*$  as its density function, then the measure  $\mu$  is the absolutely continuous  $\tau$ -invariant probability measure. Since  $\tau$  is ergodic,  $\mu$  is the only  $\tau$ -invariant measure. So we may expect that  $\{f_{2^n}\}_{n \geq 1}$  converges to  $f^*$ . Figure 8 is the graph of  $f_{2^8}$ , in which we can recognize six peak values. It is clear that the first peak value occurs at  $x = a$  and the last peak value at  $x = b$ .

Note that  $f^*$  is the p.d.f. of the population random variable  $X$  whose occurrences are according to  $(x, \tau(x), \tau^2(x), \dots)$ . Using (5) with  $f(x) = x$ , we have

$$\int_a^b x d\mu(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tau^n(x).$$

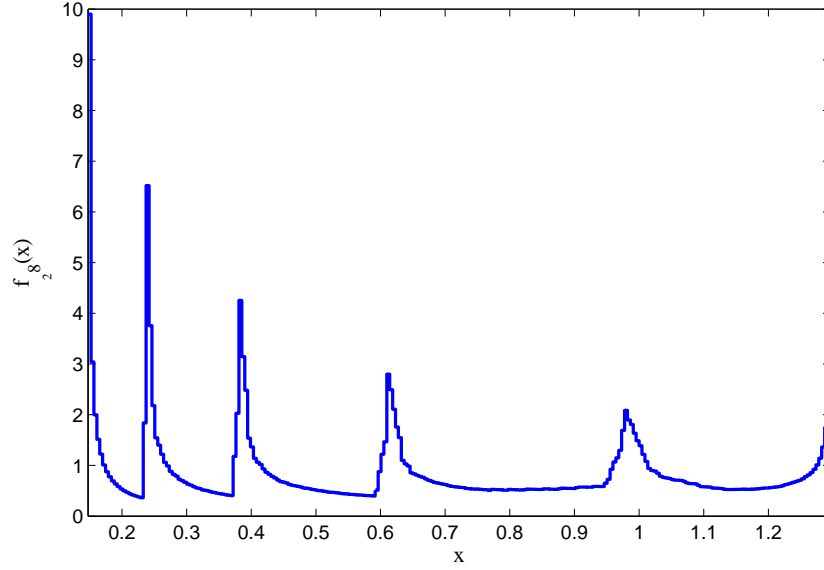


FIGURE 8. Graph of  $f_{2^8}$  for Example 5. There are six peak values, where the first peak value occurs at  $x = a$  and the last peak value at  $x = b$ . Although the closed form of the probability density function  $f^*$  is unknown, the numerical simulations strongly show that  $\{f_{2^n}\}_{n \geq 1}$  is bounded by an integrable function. So, by Theorem 4,  $\{f_{2^n}\}_{n \geq 1}$  is weakly pre-compact, and hence there exists a sub-sequence  $\{f_{2^{n_k}}\}_{n \geq 1}$  of  $\{f_{2^n}\}_{n \geq 1}$  which converges weakly to  $f^* \in L^1[a, b]$ . By Proposition 2 the measure  $\mu$  with density function  $f^*$  is an absolutely continuous  $\tau$ -invariant measure. So  $f^*$  is the PDF of the chaotic map  $\tau$ . In fact, we may expect that  $\{f_{2^n}\}_{n \geq 1}$  converges strongly to  $f^*$ .

TABLE 5. Peak values of  $f_{2^n}$  for  $\tau$  in Example 5

$n$	$2^n$	The 3rd peak values	The 4th peak values
8	256	4.2530	2.7992
9	512	5.8493	3.8154
10	1024	8.1477	5.2147
11	2048	10.711	6.8142

Since the measure  $\mu$  has  $f^*$  as its density, the above equation becomes

$$\bar{X} = \int_a^b x f^*(x) dm(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tau^n(x),$$

where  $\bar{X}$  denotes the mean of the population random variable  $X$ . So we can estimate the mean  $\bar{X}$  by  $\bar{x}$ , where

TABLE 6. Peak values of  $f_{2^n}$  for  $\tau$  in Example 5

$n$	$2^n$	The 5th peak values	The 6th peak values
8	256	2.0871	4.1312
9	512	2.8524	5.5220
10	1024	3.5875	7.8931
11	2048	4.5694	10.607

TABLE 7. The consecutive ratio of the first two peak values of  $f_{2^n}$  for  $\tau$  in Example 5

$n$	$2^n$	$f_{2^n}/f_{2^{n-1}}$ at 1st peak value	$f_{2^n}/f_{2^{n-1}}$ at 2nd peak value
9	512	1.4128	1.3938
10	1024	1.4735	1.4055
11	2048	1.3779	1.3203

TABLE 8. The consecutive ratio of the middle two peak values of  $f_{2^n}$  for  $\tau$  in Example 5

$n$	$2^n$	$f_{2^n}/f_{2^{n-1}}$ at 3rd peak value	$f_{2^n}/f_{2^{n-1}}$ at 4th peak value
9	512	1.3753	1.3630
10	1024	1.3929	1.3668
11	2048	1.3146	1.3067

TABLE 9. The consecutive ratio of the last two peak values of  $f_{2^n}$  for  $\tau$  in Example 5

$n$	$2^n$	$f_{2^n}/f_{2^{n-1}}$ at 5th peak value	$f_{2^n}/f_{2^{n-1}}$ at 6th peak value
9	512	1.3667	1.3367
10	1024	1.2577	1.4294
11	2048	1.2737	1.3439

$$\bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} \tau^n(x),$$

using a large  $N$ . Similarly, using (5) with  $f(x) = (x - \bar{x})^2$ , the variance  $\sigma^2$  can be approximated by  $s^2$ , where

$$s^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} (\tau^n(x) - \bar{x})^2,$$

So we can also estimate the standard deviation  $\sigma$  of the population random variable  $X$  by  $s$ , where

$$s = \sqrt{\frac{\sum_{n=0}^{N-1} (\tau^n(x) - \bar{x})^2}{N}}.$$

Using these two formulas with  $N = 10^5, 10^6$ , and  $10^7$ , we compiled the estimates of  $\bar{X}$  and  $\sigma$  in Table 10.

TABLE 10. The estimates of  $\bar{X}$  and  $\sigma$  for Example 5

$N$	$\bar{x}$	s
$10^5$	0.6658	0.3570
$10^6$	0.6657	0.3574
$10^7$	0.6657	0.3575

**Acknowledgments.** Prof. Paweł Góra thanks the Applied Mathematics Group at the University of Missouri-Kansas City, for hospitality during the visit while this research was initialized.

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Received November 2016; revised March 2017.

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