# HARMONIC AVERAGE OF SLOPES AND THE STABILITY OF ABSOLUTELY CONTINUOUS INVARIANT MEASURE

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Abstract. For families of piecewise expanding maps which converge to a map with a fixed or periodic turning point touching a branch with slope of modulus equal or less than 2, the standard Lasota-Yorke argument fails to prove stability. It is the goal of this paper to use instead the harmonic average of slopes condition for a large class of maps satisfying the summable oscillation condition for the reciprocal of the derivative. Using Rychlik's Theorem for a family of perturbations we prove weak compactness in $L^1$ of the density functions associated with them. From this it follows that we have stability of absolutely continuous invariant measures of the limit map.

## 1. Introduction

The main motivation for this investigation is to prove stability of the absolutely continuous invariant measure (acim) for some maps with fixed or periodic turning points. The difficulty caused by periodic turning points was first noticed by Keller [7] who introduced the so called W-maps to display a wide range of limiting behaviour. We will study classes of maps more general than the W-maps.

For almost forty years the Lasota-Yorke inequality [9, 1] has played a crucial role in establishing existence of absolutely continuous invariant measures and in studying properties of these measures. More precisely, in the setting where we have a single piecewise expanding map $\tau: I \to I$, the Lasota-Yorke method requires that we use an iterate $\tau^n$ for which we have $\inf |(\tau^n)'| > 2$. Then, the partition $\mathcal{P}^{(n)}$ of $\tau^n$ is used in an argument where the magnitude of the minimum length of $\mathcal{P}^{(n)}$ appears in the denominator of a term. This works if we are dealing with a single map or with a family of maps for which the $n$ th iterate of all

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members of the family has slopes uniformly bounded away from 2 in modulus. Stability of acim in this situation was considered in [7, 8]. However, in some important situations this does not happen. Consider, for example, Keller's W maps [7], where the limit map has $|\text{slope}| = 2$ at a turning fixed point $1/2$. In this situation the standard Lasota-Yorke inequality cannot be applied to a family of approximating maps since taking an iterate of these maps creates partition elements which go to 0 length. The papers [4, 11] show instability of acim for this map. In the paper [10] stability of a more general W shaped map has been considered. The results of this paper inspired the introduction of the harmonic average of slopes condition.

Recently the Lasota-Yorke inequality has been strengthened [3] by using the harmonic average of the slopes on each side of the partition points rather than the doubled reciprocal of the minimal slope. This allows us to show stability of the acim of the limit map for a larger class of maps. The smoothness assumption in [3] is piecewise $C^{1+1}$.

In this note we generalize the use of the harmonic average of slopes condition to maps with much weaker smoothness properties, namely we assume only the summable oscillation condition for the reciprocal of the derivative. Unlike [3], we do not use the bounded variation technique. Our main tool is Rychlik's Theorem (see, e.g., [1]). We show that the invariant densities of families of perturbed maps form a uniformly bounded set in $L^\infty$ which implies that it is weakly compact in $L^1$. From this compactness property it follows that we have stability of the acim associated with the limit map.

In section 2 we will define the class of maps we will consider and introduce the harmonic average slope condition. Then, we will recall Rychlik's Theorem [1, Theorem 6.2.1]. In section 3 we rewrite Rychlik's proof and show that the harmonic average condition is enough for the result to hold. In section 4 we prove the main result of this paper, which establishes weak compactness in $L^1$ of the densities associated with the perturbing family of maps. This is turn proves stability of acim of the limit map. This results stays true in many situation not covered by previous works. Examples are presented in section 5.

## 2. NOTATION AND PRELIMINARY RESULTS

Let $I = [0, 1]$ and let $m$ be Lebesgue measure on $I$. We present the usual definition of a piecewise expanding map.

**Definition 1.** *Suppose there exists a partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, \ldots, q\}$ of $I$ such that $\tau : I \to I$ satisfies the following conditions:*

(1) *$\tau$ is monotonic on each interval $I_i$;*
(2) *$\tau_i := \tau|_{I_i}$ is $C^1$ and $\lim_{x\to a_{i-1}^+} \tau'(x)$, $\lim_{x\to a_i^-} \tau'(x)$ exist; Let $M = \max_{x\in I} |\tau'(x)|$ ;*
(3) *$|\tau_i'(x)| \geq s_i > 1$ for any $i$ and for all $x \in (a_{i-1}, a_i)$.*

*Then, we say $\tau \in \mathcal{T}(I)$, the class of piecewise expanding transformations.*

Let

$$s := \min_{1\leq i\leq q} s_i \ . \tag{1}$$

Suppose $\tau \in \mathcal{T}(I)$ satisfies the following condition.

$$s_H = \max_{i=1,\dots,q-1} \left\{ \frac{1}{s_i} + \frac{1}{s_{i+1}} \right\} < 1 \ . \tag{2}$$

The number $H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}$ is called the harmonic average of $a$ and $b$. Condition $H(a,b) > 2$ is equivalent to condition $\frac{1}{a}+\frac{1}{b} < 1$. If $\tau$ satisfies $s_H < 1$ we say that $\tau$ satisfies the harmonic average condition. Let

$$\delta := \min_{2\leq i\leq q-1} m(I_i) \ . \tag{3}$$

Note, that to calculate the $\delta$ we do not use the first and the last subintervals of the partition.

Let

$$g_n = \frac{1}{|(\tau^n)'|},$$

wherever $(\tau^n)'$ is defined. Let $\mathcal{P}^{(n)} = \bigvee_{i=0}^{n-1} \tau^{-i}(\mathcal{P})$. Note that $\mathcal{P} = \mathcal{P}^{(1)}$. For any measurable subset $A$ of $[a,b]$, let

$$\mathcal{P}(A) = \{J \in \mathcal{P} : \lambda(J \cap A) > 0\} \ .$$

Let $\gamma_n = \sum_{J\in\mathcal{P}^{(n)}} \sup_J \ g_n$.

For $J \in \mathcal{P}^{(n)}$, we define $\operatorname{osc}_J \frac{1}{|\tau'|} = \max_J \frac{1}{|\tau'|} - \min_J \frac{1}{|\tau'|}$ and

$$d_n = \max_{J\in\mathcal{P}^{(n)}} \operatorname{osc}_J \frac{1}{|\tau'|} \ .$$

**Definition 2.** *We say that a map $\tau \in \mathcal{T}(I)$ satisfies the summable oscillation condition, or $\tau \in \mathcal{T}_\Sigma(I)$, if*

$$\sum_{n\geq 1} d_n \leq D < +\infty \ .$$

Note that usually the summable oscillation condition means a similar condition for $|\tau'|$ and not $\frac{1}{|\tau'|}$ as here.

This condition is satisfied for example for the following maps:

(i) piecewise in $C^{1+\varepsilon}$, i.e., with bounded derivative satisfying a Hölder condition;

(ii) piecewise satisfying Collet's condition [2], i.e, the modulus of continuity of $\tau'$ satisfies

$$\omega(t) \leq \frac{K}{(1+\log|t|)^{1+\gamma}} ,$$

as $t \to 0$, for some $K, \gamma > 0$ ($\gamma = 0$ is not enough);

(iii) satisfying Schmitt's condition [13, 5], i.e, summable oscillation condition for $|\tau'|$.

## 3. MAIN RESULT

We now recall Rychlik's Theorem. The proof can be found in [12] or [1] (Theorem 6.2.1).

**Theorem 1.** *Let $\tau$ be a piecewise monotonic transformation of an interval $[a,b]$ satisfying the following three conditions:*

(1) *There exists $d > 0$ such that for any $n \geq 1$ and any $J \in \mathcal{P}^{(n)}$,*

$$\sup_J \ g_n \leq d \cdot \inf_J \ g_n;$$

(2) *There exist $\varepsilon > 0$ and $r \in (0,1)$ such that for any $n \geq 1$ and any $J \in \mathcal{P}^{(n)}$,*

$$m(\tau^n(J)) < \varepsilon \Rightarrow \sum_{J' \in \mathcal{P}(\tau^n(J))} \sup_{J'} g \leq r;$$

(3) $\gamma_1 = \sum_{J \in \mathcal{P}} \ \sup_J \ g < +\infty$.

*Then $\tau$ admits an absolutely continuous invariant measure. Moreover, if $f$ is a $\tau$-invariant density then*

$$\|f\|_\infty \leq \gamma_1 \frac{d}{\varepsilon(1-r)} \ . \tag{4}$$

**Theorem 2.** *If $\tau \in \mathcal{T}_\Sigma$ and satisfies the harmonic average of slopes condition $s_H < 1$, then it satisfies the assumptions of Rychlik's Theorem.*

*Proof.* Condition (1): Note that $\sup g \leq \frac{1}{s}$. Let $J \in \mathcal{P}^{(n)}$, $x, y \in J$. We have

$$\frac{g_n(x)}{g_n(y)} = \frac{g(\tau^{n-1}(x))g(\tau^{n-2}(x))\dots g(\tau(x))g(x)}{g(\tau^{n-1}(y))g(\tau^{n-2}(y))\dots g(\tau(y))g(y)}.$$

For any $k = 0, \dots, n-1, \tau^k(x)$ and $\tau^k(y)$ belong to the same element $J_k$ of $\mathcal{P}^{(n-k)}$. Using the inequality

$$\frac{a}{b} = 1 + \frac{a-b}{b} \leq \ \exp\ \left(|\frac{a-b}{b}|\right),$$

we get

$$\begin{aligned}\frac{g(\tau^k(x))}{g(\tau^k(y))} &= \frac{\frac{1}{g(\tau^k(y))}}{\frac{1}{g(\tau^k(x))}}\\ &\le \exp\left(\frac{1}{g\left(\tau^k(x)\right)}\left|\frac{1}{g(\tau^k(x))}-\frac{1}{g(\tau^k(y))}\right|\right)\\ &\le \exp\left(M\cdot d_{n-k}\right),\end{aligned}$$

and thus,

$$\frac{g_n(x)}{g_n(y)} \le \exp\left(M\cdot\sum_{k=0}^{n-1} d_{n-k}\right) \le \exp\left(M\cdot D\right).$$

We have established condition (1) with

$$d = \exp\left(M\cdot D\right).$$

We now invoke the harmonic average of slopes condition to prove condition (2): let $\varepsilon = \frac{1}{2}\delta$ and $r = s_H < 1$. (It is important to note that we did not use the lengths of the first and the last interval of the partition to define $\delta$.) It is enough to notice that, for any $J' \in \mathcal{P}^{(n)}$, $\tau^n(J')$ is an interval and if $m(\tau^n J') < \varepsilon$, then $\tau^n J'$ can intersect at most two intervals of $\mathcal{P}$. Thus, $\sum_{J\in\mathcal{P}(\tau^n J')} \sup g \le s_H = r < 1$.

Condition (3) is satisfied by definition. This completes the proof. □

## 4. Stability of ACIM for Families of Maps

The main motivation for this investigation is to prove stability of the acim for maps with turning fixed or periodic points. The general setting is as follows. Let $\tau_0$ be a map with an invariant density $f_0$ and and $\{\tau_\gamma\}_{\gamma>0}$ a family of maps with invariant densities $f_\gamma$ such that $\tau_\gamma$ converge to $\tau_0$ in some sense as $\gamma$ converges to 0. Question: under what conditions does $f_\gamma \to f_0$ in some sense? Such problems were investigated in many articles but usually using bounded variation technique [7, 8].

**Theorem 3.** *Let the family $\{\tau_\gamma\}_{\gamma>0} \subset \mathcal{T}_\Sigma$ satisfies the assumptions of Rychlik's Theorem in a uniform way, i.e, with the same constants and $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$. If $\tau_0$ has exactly one acim, then $f_\gamma \to f_0$ in $L^1$ as $\gamma \to 0$. In the general case every limit point of the family $\{f_\gamma\}$, as $\gamma \to 0$, is an invariant density of $\tau_0$.*

*Proof.* The proof follows from Theorem 11.2.3 of [1] which we recall below with appropriate changes. □

**Theorem 4.** *Let $\tau_\gamma \in \mathcal{T}$, $\gamma \geq 0$. Let the invariant densities of $\{f_\gamma\}_{\gamma\geq 0}$ be uniformly bounded in $L^\infty$. If $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$, then any limit point of $\{f_\gamma\}_{\gamma>0}$, as $\gamma \to 0$, is a $\tau_0$-invariant density. If $\{\tau_0, f\cdot m\}$ is ergodic, then $f_\gamma \to f_0$ in $L^1$.*

We now describe two families of maps for which Theorem 3 applies.

**Example 1.** *Let $\tau_0 \in \mathcal{T}_\Sigma$ satisfy the harmonic average condition $s_H < 1$. Let $\tau_\gamma$ be defined on the same partition $\mathcal{P} = \{I_1, I_2, \ldots, I_q\}$ and $\tau_\gamma \to \tau_0$, as $\gamma \to 0$, in $C^1(\mathrm{int}(I_i))$ for all $i = 1, 2, \ldots, q$. We also assume that the summable oscillation condition is satisfied uniformly for $\{\tau_\gamma\}_{\gamma\geq 0}$. Then, the family $\{\tau_\gamma\}_{\gamma\geq 0}$ satisfies the assumptions of Theorem 3.*

**Example 2.** *Let $\tau_0 \in \mathcal{T}_\Sigma$ satisfy the harmonic average condition $s_H < 1$. Let each $\tau_\gamma$ be piecewise expanding on the partition $\mathcal{P}_\gamma = \{I_0^{(\gamma)}, I_2^{(\gamma)}, \ldots, I_{q+1}^{(\gamma)}\}$, $I_i^{(\gamma)} = [a_{i-1}^{(\gamma)}, a_i^{(\gamma)}]$, $i = 0, 1, 2, \ldots, q+1$. We allow the possibility that $I_0^{(\gamma)}$ or $I_{q+1}^{(\gamma)}$ or both of them are empty. We assume $a_i^{(\gamma)} \to a_i^{(0)}$ as $\gamma \to 0$, $i = 0, 1, 2, \ldots, q$. Then, automatically $a_{-1}^{(\gamma)} \to a_0^{(0)}$ and $a_{q+1}^{(\gamma)} \to a_q^{(0)}$ as $\gamma \to 0$. We also assume that the summable oscillation condition and harmonic average condition are satisfied uniformly for $\{\tau_\gamma\}_{\gamma\geq 0}$. If $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$, then the family $\{\tau_\gamma\}_{a\geq 0}$ satisfies the assumptions of Theorem 3.*

Stability of acim was obtained in [6] under additional much stronger conditions on the family of transformations. The two main stronger conditions assumed in [6] are

(i) There exists a constant $\delta > 0$ such that for any $\tau_\gamma$ in the family of maps there exists a finite partition $\mathcal{K}_\gamma$ such that for any $J \in \mathcal{K}_\gamma$, $\tau_{\gamma|_J}$ is one-to-one, $\tau_\gamma(J)$ is an interval, and

$$\min_{J\in K_\gamma} \mathrm{diam}(J) \quad > \quad \delta$$

(ii) For any $m \geq 1$, there exists $\delta_m > 0$ such that if

$$\mathcal{K}_\gamma^{(m)} \quad = \quad \bigvee_{j=0}^{m-1} \tau_\gamma^{-j}(\mathcal{K}_\gamma)$$

then

$$\min_{J\in K_\gamma^{(m)}} \mathrm{diam}(J_m) \quad \geq \quad \delta_m > 0.$$

From these conditions it follows that the family of densities is weakly compact in $L^1$.

## 5. Examples

The results of this paper allow us to answer a question posed in [4].

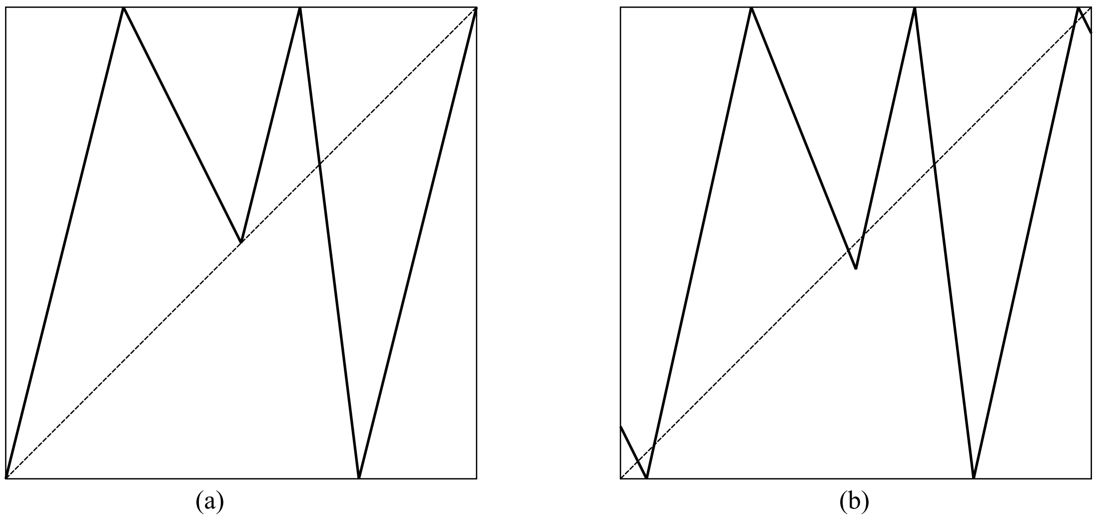


Figure 1. The 3rd iterates of maps of Example 3

**Example 3.**

Let $\tau_\gamma$, $0 \le \gamma < \varepsilon_0 < 1/2$, be a map defined by

$$\tau_\gamma(t) = \begin{cases} \frac{1}{2} - \gamma + (1+2\gamma)t & , \quad 0 \le t < \frac{1}{2} ; \\ 2 - 2t & , \quad \frac{1}{2} \le t \le 1 \end{cases} . \tag{5}$$

$\tau_0$ is exact with invariant density $f_0 = \frac{2}{3}\chi_{[0,1/2]} + \frac{4}{3}\chi_{[1/2,1]}$. Is this acim stable under perturbation given by the family $\{\tau_\gamma\}_{\gamma>0}$? $\tau_0$ has a turning point 1/2 which is periodic with period 3. Previously known methods did not give an answer to this question.

We will consider the family of third iterates $\{\tau_\gamma^3\}_{\gamma>0}$. $\tau_0^3$ is shown in Fig. 1 (a) and a typical $\tau_\gamma^3$ is shown in Fig. 1 (b). The slopes of $\tau_\gamma^3$ are $s_1 = s_3 = s_7 = 2 + 8\gamma + 8\gamma^2$, $s_2 = s_4 = s_6 = 4 + 8\gamma$, and $s_5 = 8$. Since $\tau_0$ is exact, $\tau_0^3$ is also exact with the same acim and stability of acim for $\tau_0^3$ implies the same for $\tau_0$. We can see that the family $\{\tau_\gamma^3\}_{\gamma>0}$ satisfies the conditions of Example 2. Thus, $\tau_0$ has a stable acim.

**Example 4.**

Let us consider the following W-shaped map:

$$\tau_0(t) = \begin{cases} 1-6t & , \quad 0 \le t < \frac{1}{6} \ ; \\ -\frac{2}{9} + \frac{5}{3}t - \frac{\sqrt{6}}{81}(3-9t)^{3/2} & , \quad \frac{1}{6} \le t \le \frac{1}{3} \ ; \\ -3t + \frac{4}{3} & , \quad \frac{1}{3} \le t < \frac{4}{9} \ ; \\ \frac{9}{5}t - \frac{4}{5} & , \quad \frac{4}{9} \le t \le 1 \ . \end{cases} \tag{6}$$

The derivative of the second branch is

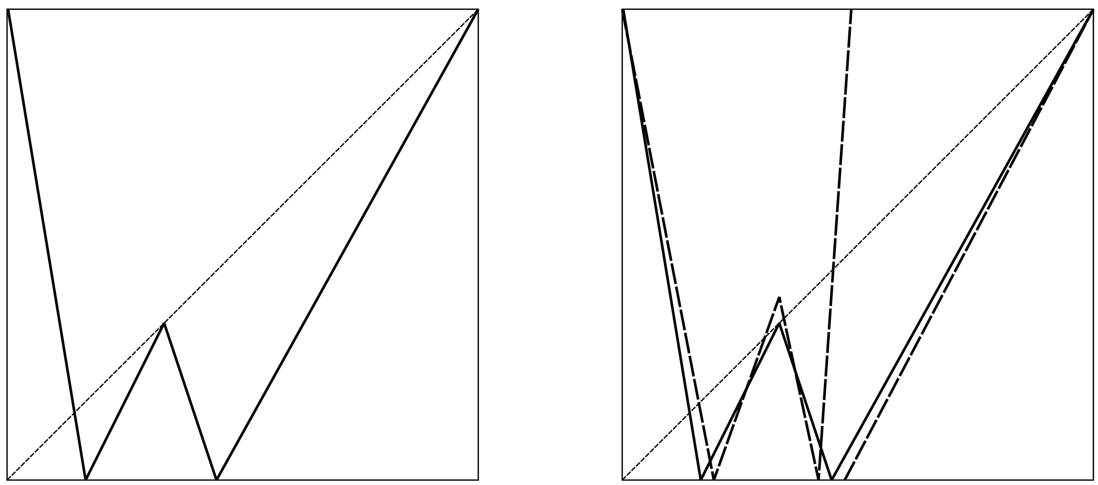

FIGURE 2. Example of W-map not in class $C^{1+1}$ and its untypical perturbation.

$$\tau'_{0,2}(t) = \frac{5}{3} + \frac{\sqrt{6}}{2}(\frac{1}{3} - t)^{1/2} \ .$$

The minimal moduluses of the slopes are $s_1 = 6$, $s_2 = \frac{5}{3}$, $s_3 = 3$ and $s_2 = \frac{9}{5}$ and we have

$$s_H = \max\left\{\frac{1}{6} + \frac{3}{5}, \frac{3}{5} + \frac{1}{3}, \frac{1}{3} + \frac{5}{9}\right\} = \max\left\{\frac{23}{30}, \frac{14}{15}, \frac{24}{27}\right\} = \frac{24}{27} < 1 \ .$$

The map $\tau$ is not $C^{1+1}$ because the derivative of $\tau_2$ is not Lipschitz. Thus, no previous result can prove the stability of acim for map $\tau$. On the right hand side of Fig. 2 we present an untypical perturbation of

$\tau$. We define it as follows

$$\tau_\gamma(t) = \begin{cases} 1 - \frac{6}{1+6\gamma}t & , \quad 0 \le t < \frac{1}{6} + \gamma \ ; \\ -\frac{2}{9} + \frac{5}{3}t - \frac{\sqrt{6}}{81}(3-9t)^{3/2} + A(\gamma)t + B(\gamma) & , \quad \frac{1}{6} + \gamma \le t \le \frac{1}{3} \ ; \\ -\frac{3+18\gamma}{1-9\gamma}(t - 4/9 + \gamma) & , \quad \frac{1}{3} \le t < \frac{4}{9} - \gamma \ ; \\ \frac{1}{2\gamma}(t - 4/9 + \gamma) & , \quad \frac{4}{9} - \gamma \le t < \frac{4}{9} + \gamma \ , \\ \frac{9}{5-9\gamma}(t - 4/9 - \gamma) & , \quad \frac{4}{9} + \gamma \le t \le 1 \ , \end{cases} \tag{7}$$

where $A(\gamma), B(\gamma)$ are constants such that $\tau_{\gamma,2}(1/3) = 1/3 + 2\gamma$ and $\lim_{\gamma\to 0} A(\gamma) = \lim_{\gamma\to 0} B(\gamma) = 0$. It is easy to check that for $\gamma$ small enough the family $\{\tau_\gamma\}$ satisfies uniformly the assumptions of Rychlik's Theorem and that $\tau_\gamma$ converge almost uniformly to $\tau_0$. Thus, the acims of the $\tau_\gamma$'s converge to the acim of $\tau_0$.

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## References

[1] A. Boyarsky and P. Góra, *Laws of Chaos. Invariant Measures and Dynamical Systems in One Dimension*, Probability and its Applications, Birkhaüser, Boston, MA, 1997.

[2] P. Collet and J.P. Eckmann, *Measures invariant under mappings of the unit interval*, in book Regular and Chaotic Motions in Dynamic Systems, edited by G. Velo, A.S. Wightman, NATO ASI Series, Series B: Physics **118**, Plenum Press, N.Y., 1985, 233–265.

[3] P. Eslami and P. Góra, *Stronger Lasota-Yorke inequality for piecewise monotonic transformations*, preprint.

[4] P. Eslami and M. Misiurewicz, *Singular limits of absolutely continuous invariant measures for families of transitive map*, Journal of Difference Equations and Applications 18,no. 4 (2012), 739–750, DOI:10.1080/10236198.2011.590480.

[5] P. Góra, *Properties of invariant measures for piecewise expanding one-dimensional transformations with summable oscillations of derivative*, Ergodic Theory Dynam. Systems **14** (1994), no. 3, 475–492.

[6] P. Góra and A. Boyarsky, *Compactness of invariant densities for families of expanding, piecewise monotonic transformations*, Can. J. Math, **XLI** (1989), 855–869.

[7] G. Keller, *Stochastic stability in some chaotic dynamical systems*, *Monatshefte für Mathematik* **94** (4) (1982) 313–333.

[8] G. Keller and C. Liverani, *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **28** (1)(1999), 141–152.

[9] A. Lasota; J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations* , Trans. Amer. Math. Soc. **186** (1973), 481–488 (1974); MR0335758 (49 #538).

[10] Zhenyang Li, *W-like maps with various instabilities of acim's*, available at http://arxiv.org/abs/1109.5199
[11] Zhenyang Li, P. Góra, A. Boyarsky, H. Proppe and P. Eslami, *A Family of Piecewise Expanding Maps having Singular Measure as a limit of ACIM's*, accepted to Ergodic Th. and Dyn. Syst, DOI:10.1017/S0143385711000836.
[12] M.R. Rychlik, *Invariant measures and the variational principle for Lozi mappings*, Ph.D. Thesis, University of California, Berkeley, 1983.
[13] B. Schmitt, *Contributions a l'étude de systemes dynamiques unidimensionnels en théorie ergodique*, Ph.D. Thesis, University of Bourgogne, 1986.

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