Maps with Memory

Iterated Function Systems (IFS), Fractals, Invariant Measures and Applications

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Map with memory

Let $f : \mathbb{R} \to \mathbb{R}$ be a map. We want to consider a process, which is not a map, and represents situation when f on each step uses not only current information but also some information from the past. For current state x_n and $0 < \alpha < 1$ we define:

$$x_{n+1} = T(x_n) = f(\alpha x_n + (1-\alpha)x_{n-1}) .$$

"Invariant measure" and map G

We are interested in something we could call an "invariant measure" of the process. We consider ergodic averages

$$A_g(x_0, x_{-1}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(x_i).$$

They are related to ergodic averages of the map $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x,y) = (y, f(\alpha y + (1-\alpha)x).$$

"Invariant measure"

Let us assume that G has an ergodic invariant measure ν on $\mathfrak{B}(\mathbb{R}^2)$. Measure ν defines a marginal measure μ on the first coordinate: $\mu(A) = \nu(A \times \mathbb{R})$. In particular, if $\nu = h(x,y)dxdy$ is an absolutely continuous measure with density $h(x,y)$, then

$$\mu = \left(\int_{\mathbb{R}} h(x,y)dy \right) dx$$

is also absolutely continuous with density $h_1(x) = \int_{\mathbb{R}} h(x,y)dy$.

"Invariant measure"

Since we assume that G is ν-ergodic, the Birkhoff's Ergodic Theorem holds. Thus, for any integrable function g and almost every pair (x,y) we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(G^i(x,y)) = \int g(x,y)\,d\nu(x,y)\,.$$

If the function g depends only on the first coordinate, we can rewrite this as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\Pi_1(G^i(x,y))) = \int g(x)\,d\mu(x)\,.$$

"Invariant measure"

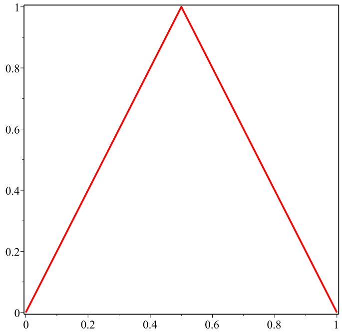
If we accept inconsistency in the first few terms of the sum (which does not change the limit), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i(x)) = \int g(x)d\mu(x) .$$

Since the limit is independent of initial condition the history used by T is unimportant.
This shows that the marginal of the G-invariant measure determines the behaviour of ergodic averages of trajectories of the process T. Thus, μ is a good candidate for an "invariant" measure of T.

Example: f is the tent map

We considered the example where $f : [0,1] \to [0,1]$ is the tent map.



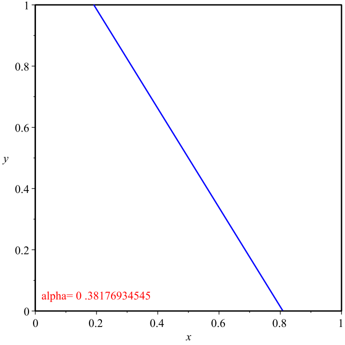
Tent map

$$\begin{cases} f(x) = 2x, & \text{if } ,x < 1/2 \\ 2 - 2x, & \text{if } ,x \geq 1/2. \end{cases}$$

Example: f is the tent map

Then, G is a piecewise linear map:

$$G(x,y) = \begin{cases} 2\alpha y + 2(1-\alpha)x, & \text{if } \alpha y + (1-\alpha)x < 1/2; \\ 2 - 2\alpha y - 2(1-\alpha)x, & \text{if } \alpha y + (1-\alpha)x \geq 1/2. \end{cases}$$



Partition for the map G.

Derivative matrices D_1, D_2

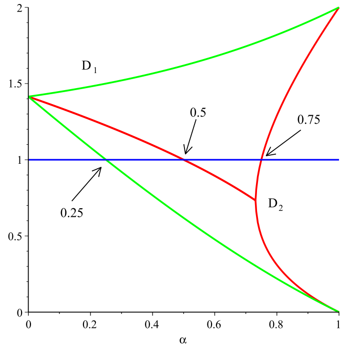
Below (to the left of) the line $\alpha y + (1-\alpha)x = 1/2$ the derivative matrix of G is

$$D_1 = \begin{bmatrix} 0 & 1 \\ 2(1-\alpha) & 2\alpha \end{bmatrix},$$

and above (to the right of) this line it is

$$D_2 = \begin{bmatrix} 0 & 1 \\ -2(1-\alpha) & -2\alpha \end{bmatrix},$$

Eigenvalues of matrices D_1, D_2



Green: absolute value of eigenvalues of D_1, red: of D_2.

Conjectures and Theorems

Computer experiments show that G behaves in very different manners depending on α. We conjecture:

$0 < \alpha < 1/2$: map G preserves absolutely continuous invariant measure. Proven for $0 < \alpha \leq 0.46$.

$\alpha = 1/2$: every point is eventually periodic with period 3. Proven.

$1/2 < \alpha < 3/4$: the fixed point $(2/3, 2/3)$ is the global attractor. Proven.

$\alpha = 3/4$: every point on the line $x + y = 4/3$ is periodic with period two. Any other point is attracted to a one of these periodic trajectories. Proven.

Conjectures and Theorems

$3/4 < \alpha < 1$: map G preserves a singular continuous measure, so called SRB-measure μ. (Sinaj-Ruelle-Bowen).

$$\frac{1}{n}\sum_{k=0}^{n-1} g(G^k(x,y)) \to \int g d\mu,$$

for any continuous g and Lebesgue almost every (x,y).

ACIM - Absolutely Continuous Invariant Measure

Tsujii Theorem: If $T : B \to B$ is piecewise real analytic and for any vector v

$$\|DT(v)\| \geq \lambda \|v\| \ , \ \lambda > 1,$$

then T preserves an ACIM.
Singular values of the matrix M are square roots of eigenvalues of M^*M. In dimension 2, the smaller of them

$$\sigma_2 = \inf_{v \neq 0} \frac{\|M(v)\|}{\|v\|}.$$

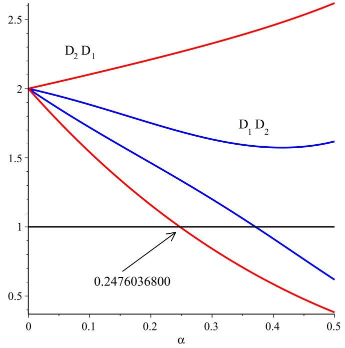
Useful inequalities:

$$\sigma_2(M_1 \cdot M_2) \geq \sigma_2(M_1)\sigma_2(M_2),$$

$$\sigma_2(\prod_{i=1}^{n} M_i) \geq \frac{\det(\prod_{i=1}^{n} M_i)}{\| \prod_{i=1}^{n} M_i \|}.$$

ACIM - Absolutely Continuous Invariant Measure

For $0 < \alpha \leq 0.2476036800$ it is easy to see that $\sigma_2(D_i D_j) > 1$ for all combinations of $1 \leq i,j \leq 2$.



Singular values $\sigma_2(D_1 D_2) = \sigma_2(D_1 D_1)$ (blue) and $\sigma_2(D_2 D_1) = \sigma_2(D_2 D_2)$ (red).

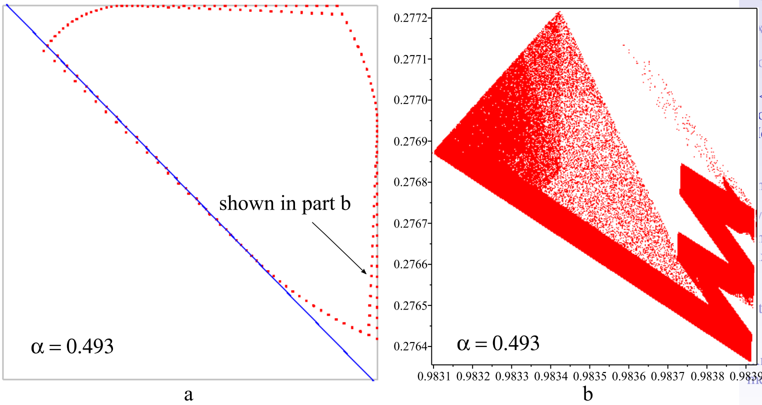
ACIM - Absolutely Continuous Invariant Measure

Computer experiments using the inequality

$$\sigma_2\left(\prod_{i=1}^{n} M_i\right) \geq \frac{\det\left(\prod_{i=1}^{n} M_i\right)}{\left\|\prod_{i=1}^{n} M_i\right\|}.$$

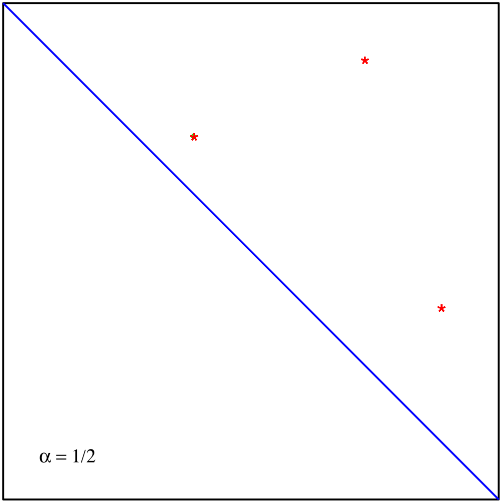
show that for any $0 < \alpha < 1/2$ some iterate of G satisfies Tsujii's condition.
Proven for $0 < \alpha \leq 0.46$.
For α's in a very narrow window around $\alpha = 0.493$, the support of conjectured acim consists of 175 clusters which under action of G move by 58 positions in the clockwise direction. Since $3 \cdot 58 = 174$, G^{175} preserves every cluster.
We observed similar behaviour for
$\alpha = 0.4883$ (106 clusters moving by 35 positions),
$\alpha = 0.4943$ (214 clusters moving by 71 positions) and
$\alpha = 0.4973$ (448 clusters moving by 149 positions).

ACIM - Absolutely Continuous Invariant Measure



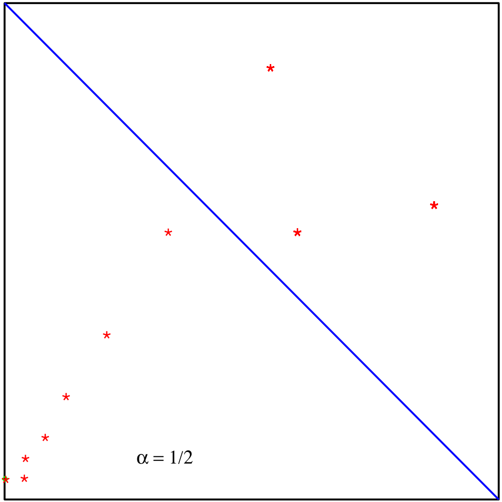
Support of conjectured ACIM for $\alpha = 0.4930$ and one of the clusters.

$\alpha = 1/2$ Period Three



Initial point in the upper half.

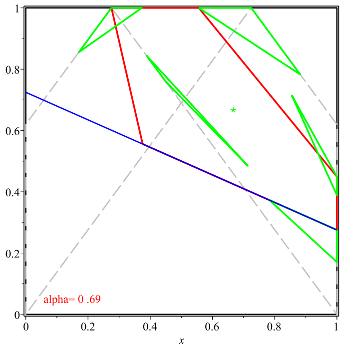
$\alpha = 1/2$ Period Three



Initial point in the lower half.

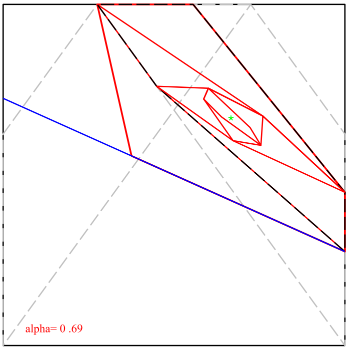
$1/2 < \alpha < 3/4$ Globally Attracting Fixed Point

For $1/2 < \alpha < 3/4$ the fixed point $(2/3, 2/3)$ attracts all other points (except $(0, 0)$).



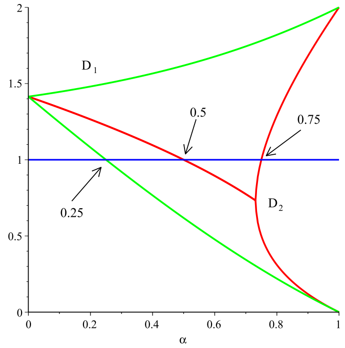
Trapping region.

$1/2 < \alpha < 3/4$ Globally Attracting Fixed Point



The evolution of the trapping region.

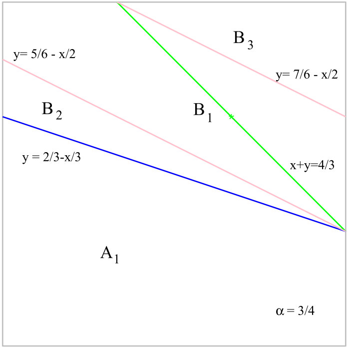
Eigenvalues of matrices D_1, D_2



Green: absolute value of eigenvalues of D_1, red: of D_2.

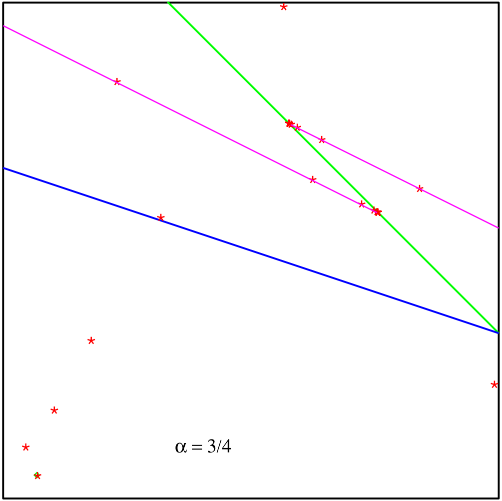
$\alpha = 3/4$: Globally Attracting Segment

For $\alpha = 3/4$, every point on the line $x + y = 4/3$ is periodic with period two. Any other point is attracted to a one of these periodic trajectories.



Attracting segment (green) for $\alpha = 3/4$.

$\alpha = 3/4$: Globally Attracting Segment



Typical trajectory.

$\alpha > 3/4$: Sinaj-Ruelle-Bowen measure

For $\alpha > 3/4$, one eigenvalue of D_2 is larger than 1, another smaller than 1, with $\det(D_2) < 1$.
We conjecture that G preserves singular continuous SRB measure, with absolutely continuous conditional measures on unstable segments.

References I

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