A Taylor series for the function arctan

The integral If we invert $y = \arctan(x)$ to obtain $x = \tan y$, then, by differentiating with respect to y, we find $dx/dy = \sec^2 y = 1 + \tan^2 y = 1 + x^2$. Thus we have (ignoring the constant of integration)

$$y = \arctan(x) = \int \frac{dx}{1+x^2}. \tag{1}$$

If we now differentiate $y = \arctan(x/a)$ with respect to x, where a is a constant, we have, by the chain rule,

$$y' = \left(\frac{1}{a}\right) \frac{1}{1+(x/a)^2} = \frac{a}{x^2+a^2}. \tag{2}$$

Thus we obtain the indefinite integral

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan(x/a). \tag{3}$$

The Taylor Series By expanding the integrand in (3) as a geometric series $1/(1-r) = 1 + r + r^2 + \ldots$, $|r| < 1$, and then integrating, we can obtain a series to represent the function $\arctan(x/a)$. We use the dummy variable t for the integration on $[0, x]$ and we first write

$$\arctan(x/a) = a \int_0^x \frac{dt}{t^2+a^2} = \frac{1}{a} \int_0^x \frac{dt}{1+(t/a)^2} \tag{4}$$

Substituting the geometric series with $r = -(t/a)^2$, we find

$$\arctan(x/a) = \frac{1}{a} \int_0^x \sum_{n=0}^{\infty} (-t^2/a^2)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{x}{a}\right)^{2n+1}. \tag{5}$$

The radius of convergence of this series is the same as that of the original geometric series, namely $R = 1$, or, in terms of x, $|x/a| < 1$. The series is a convergent alternating series at the right-hand end point $x = a$; and it can be shown that sum equals the value of $\arctan(1) = \pi/4$ (as we might hope). Thus we have the nice (but slowly converging) series for π given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots. \tag{6}$$

By choosing partial sums of (5) we obtain a sequence of Taylor polynomial approximations to arctan. For example, the sum of three terms yields the Taylor polynomial of degree five given by:

$$\arctan(x/a) \approx T(x) = \frac{x}{a} - \frac{1}{3}\left(\frac{x}{a}\right)^3 + \frac{1}{5}\left(\frac{x}{a}\right)^5. \tag{7}$$