Invariant densities for piecewise linear maps

Paweł Góra

Concordia University

June 2008
Contents

Piecewise linear map

τ-expansion of numbers

Matrix $S$

τ-invariant density

Conjecture

Ergodic properties

Application

References
Rediscovery, by a different method, of the results of Christoph Kopf (Innsbruck)

Piecewise linear map

N branches, three vectors:

1. slopes $\beta = [3, 3, -4, -5, -2]$
2. lengths $\alpha = [1, 0.35, 0.8, 1, 0.3]$
3. heights of lower end $\gamma = [0, 0.2, 0.1, 0, 0.7]$
Map \( \tau \) can be conveniently represented using "digits"

if \( \beta_j > 0 \), then \( a_j = \beta_j b_j - \gamma_j \),

if \( \beta_j < 0 \), then \( a_j = \beta_j b_j - (\gamma_j + \alpha_j) \), \( j = 1, \ldots, N \).

Then, map \( \tau \) is

\[
\tau(x) = \beta_j \cdot x - a_j, \quad \text{for} \quad x \in l_j, \ j = 1, 2, \ldots, N.
\]

In the example the digits are:

\[
a = \{0, 0.8, -2.7, -4.25, -2.7\}.
\]
\(\tau\)-expansion of numbers

For any \(x \in [0, 1]\) we define its "index" \(j(x)\) and its "digit" \(a(x)\):

\[
j(x) = j \quad \text{for} \quad x \in I_j, \ j = 1, 2, \ldots, N,
\]

and

\[
a(x) = a_{j(x)}.
\]

We define the cumulative slopes for iterates of points as follows:

\[
\beta(x, 1) = \beta_{j(x)};
\]

\[
\beta(x, n) = \beta(x, n - 1) \cdot \beta_{j(\tau^{n-1}(x))}, \quad n \geq 2.
\]

Then, the following expansion holds:

\[
x = \sum_{n=1}^{\infty} \frac{a(\tau^{n-1}(x))}{\beta(x, n)}.
\]
Example: Binary expansion

\[ \tau(x) = \begin{cases} 
2x & \text{if } 0 \leq x < 1/2; \\
2x - 1 & \text{if } 1/2 \leq x \leq 1. 
\end{cases} \]
Example: Binary expansion 2

\[ x = 0.23, \quad \tau(x) = 0.46, \quad \tau^2(x) = 0.92, \]
\[ \tau^3(x) = 0.84, \quad \tau^4(x) = 0.68, \quad \tau^5(x) = 0.36, \]
\[ \tau^6(x) = 0.72, \quad \tau^7(x) = 0.44, \quad \tau^8(x) = 0.88, \]
\[ \tau^9(x) = 0.76, \quad \tau^{10}(x) = 0.52, \quad \ldots \]

\[ x = \frac{0}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} \]
\[ + \frac{1}{2^7} + \frac{0}{2^8} + \frac{1}{2^9} + \frac{1}{2^{10}} + \frac{1}{2^{11}} + \ldots \]
Example: Classical $\beta$-map, $\beta = 3.3$

\[ \tau(x) = \beta \cdot x \mod 1 \]

\begin{align*}
    x &= 0.23, \tau(x) = 0.759, \tau^2(x) = 0.505, \\
    \tau^3(x) &= 0.666, \tau^4(x) = 0.196, \tau^5(x) = 0.674, \\
    \tau^6(x) &= 0.136, \tau^7(x) = 0.450, \tau^8(x) = 0.486, \\
    \tau^9(x) &= 0.603, \tau^{10}(x) = 0.989, \ldots
\end{align*}
Classical $\beta$-map, $\beta = 3.3$

\[ x = \frac{0}{3.3} + \frac{2}{3.3^2} + \frac{1}{3.3^3} + \frac{2}{3.3^4} + \frac{0}{3.3^5} + \frac{2}{3.3^6} + \frac{0}{3.3^7} + \frac{1}{3.3^8} + \frac{1}{3.3^9} + \frac{1}{3.3^{10}} + \frac{3}{3.3^{11}} + \ldots \]

Parry’s invariant density:

\[ h(x) = 1 + \sum_{n=1}^{\infty} \chi[0,\tau^n(1)] \frac{1}{\beta^n} \]
Classical $\beta$-map, $\beta = 3.3$

\[
x = \frac{0}{3.3} + \frac{2}{3.3^2} + \frac{1}{3.3^3} + \frac{2}{3.3^4} + \frac{0}{3.3^5} + \frac{2}{3.3^6} \\
+ \frac{0}{3.3^7} + \frac{1}{3.3^8} + \frac{1}{3.3^9} + \frac{1}{3.3^{10}} + \frac{3}{3.3^{11}} + \cdots
\]

Parry’s invariant density:

\[
h(x) = 1 + \sum_{n=1}^{\infty} \chi[0,\tau^n(1)] \frac{1}{\beta^n}.
\]
Back to the first example:

\[ x = 0.23, \quad \tau(x) = 0.69, \quad \tau^2(x) = 0.80, \]
\[ \tau^3(x) = 0.25, \quad \tau^4(x) = 0.75, \quad \tau^5(x) = 0.50, \]
\[ \tau^6(x) = 0.70, \quad \tau^7(x) = 0.75, \quad \tau^8(x) = 0.50, \]
\[ \tau^9(x) = 0.70, \quad \tau^{10}(x) = 0.75, \quad \ldots \]

indices:

\[ 1, 4, 4, 1, 4, 3, 4, 4, 3, 4, 4, \ldots \]

\[ x = \frac{0}{3} + \frac{-4.25}{-15} + \frac{-4.25}{75} + \frac{0}{225} + \frac{-4.25}{-1125} + \frac{-2.7}{4500} \]
\[ + \frac{-4.25}{-22500} + \frac{-4.25}{112500} + \frac{-2.7}{-450000} + \frac{-4.25}{2250000} + \frac{-4.25}{-11250000} + \ldots \]
Special points: $c_i$, $i = 1, 2, \ldots, K + L$

"Greedy", "lazy" and "hanging" branches. $K$ - number of shorter branches, $L$ - number of hanging branches.

c_i’s are endpoints of partition intervals whose image is not 0 or 1. Some of them are duplicated.

c_i’s are grouped into "left" $U_l$ and "right" $U_r$ and also into "upper" $W_u$ and "lower" $W_l$ points.
Special points: $c_i, i = 1, 2, \ldots, K + L$

"Greedy", "lazy" and "hanging" branches. $K$ - number of shorter branches, $L$ - number of hanging branches.

$c_i$'s are endpoints of partition intervals whose image is not 0 or 1. Some of them are duplicated.

$c_i$'s are grouped into "left" $U_l$ and "right" $U_r$ and also into "upper" $W_u$ and "lower" $W_l$ points.
Special points: $c_i, i = 1, 2, \ldots, K + L$

"Greedy", "lazy" and "hanging" branches. $K$ - number of shorter branches, $L$ - number of hanging branches.

$c_i$’s are endpoints of partition intervals whose image is not 0 or 1. Some of them are duplicated.

$c_i$’s are grouped into "left" $U_l$ and "right" $U_r$ and also into "upper" $W_u$ and "lower" $W_l$ points.
Numbers $S_{i,j}$, $1 \leq i, j \leq K + L$, $\tau$ increasing

Matrix $S$ is constructed in a way somewhat similar to the construction of kneading matrix.

If all branches are increasing: $U_r = \mathcal{W}_u$ and $U_l = \mathcal{W}_l$.

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_i, n)} \delta(\tau_u^n(c_i) > c_j), \text{ for } c_i \in \mathcal{W}_u \text{ and all } c_j,$$

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_i, n)} \delta(\tau_l^n(c_i) < c_j), \text{ for } c_i \in \mathcal{W}_l \text{ and all } c_j.$$
Numbers $S_{i,j}$, $1 \leq i, j \leq K + L$, $\tau$ general

In general:

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i, n)|} \left[ \delta(\beta(c_i, n) > 0)\delta(\tau^n(c_i) > c_j) + \delta(\beta(c_i, n) < 0)\delta(\tau^n(c_i) < c_j) \right],$$

for $c_i \in U_r$ and all $c_j$,

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i, n)|} \left[ \delta(\beta(c_i, n) < 0)\delta(\tau^n(c_i) > c_j) + \delta(\beta(c_i, n) > 0)\delta(\tau^n(c_i) < c_j) \right],$$

for $c_i \in U_l$ and all $c_j$. 
Equation for coefficients $D = [D_1, \ldots, D_{K+L}]$

\[
(-S^T + \text{Id})D = D_0 v ,
\]

where $v = [1, 1, \ldots, 1]$.

Parameter $D_0$ is taken to be 1 if the system is solvable with $D_0 = 1$ and we take $D_0 = 0$ otherwise. The system always has non-vanishing solution with one of the values of the parameter.
Invariant density, $\tau$ piecewise increasing

For $\tau$ piecewise increasing:

$$h(x) = D_0 + \sum_{i \in W_u} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta(c_i, n)}$$

$$+ \sum_{i \in W_l} D_i \sum_{n=1}^{\infty} \chi_{[\tau^n(c_i), 1]} \frac{1}{\beta(c_i, n)} ,$$
Invariant density, $\tau$ general

Let us define:

$$
\chi(\beta, x) = \begin{cases} 
\chi[0,x], & \text{for } \beta > 0, \\
\chi[x,1], & \text{for } \beta < 0.
\end{cases}
$$

For general $\tau$:

$$
h(x) = D_0 + \sum_{c_i \in U_r} D_i \sum_{n=1}^{\infty} \frac{\chi(\beta(c_i, n), \tau^n(c_i))}{|\beta(c_i, n)|}
+ \sum_{c_i \in U_l} D_i \sum_{n=1}^{\infty} \frac{\chi(-\beta(c_i, n), \tau^n(c_i))}{|\beta(c_i, n)|},
$$
Invariant density, $\tau$ general

Let us define:

$$\chi(\beta, x) = \begin{cases} 
\chi[0, x] , & \text{for } \beta > 0 , \\
\chi[x, 1] , & \text{for } \beta < 0 .
\end{cases}$$

For general $\tau$:

$$h(x) = D_0 + \sum_{c_i \in U_r} D_i \sum_{n=1}^{\infty} \frac{\chi(\beta(c_i, n), \tau^n(c_i))}{|\beta(c_i, n)|}$$

$$+ \sum_{c_i \in U_l} D_i \sum_{n=1}^{\infty} \frac{\chi(-\beta(c_i, n), \tau^n(c_i))}{|\beta(c_i, n)|} ,$$
The invariant density of $\tau$ of our main example.
Conjecture: Let $\tau$ be piecewise linear, piecewise increasing and eventually piecewise expanding map. Then, 1 is not an eigenvalue of matrix $S \implies$ dynamical system $(\tau, h \cdot m)$ is ergodic on $[0, 1]$.

The conjecture is proved for greedy maps (all shorter branches touch 0). (Thus, it also holds for lazy maps, i.e., maps with all shorter branches touching 1.)
Conjecture fails for maps with decreasing branches

$N = 2,$

$\alpha = [1, 0.8]$, $\beta = [1.8, -1.8]$, $\gamma = [0, 0.2]$.

$\tau$ is ergodic on a smaller interval $[0.2, 1]$. 
Conjecture fails for maps with decreasing branches

Matrix $S = [S_{1,1}] = [1.125]$ has an eigenvalue 1.125 and system (15) is solvable for $D_0 = 1$. We have $D_1 = -0.8$.

For the corresponding piecewise increasing map, i.e., if we keep the same $\alpha$'s and $\gamma$'s and change $\beta$ to $\beta = [1.8, 1.8]$, matrix $S = [S_{1,1}] = [1]$ has an eigenvalue 1.
The slope $\beta$ is constant. Then, $\tau$ is ergodic on $[0,1]$ and 1 is an eigenvalue of $S$. 
Theorem

Let $\tau$ be a piecewise linear and eventually piecewise expanding map which admits an invariant density supported on $[0, 1]$. Then, if at least one branch of $\tau$ is onto then $\tau$ has at most two ergodic components. If at least two branches are onto, then $\tau$ is exact.

A map without hanging branches has at most two ergodic components.

A greedy map with an invariant density supported on $[0, 1]$ is exact.
Examples of maps without hanging branches

Let $N = 4$ and $\tau$ be defined by vectors

$$\alpha = \left[ \frac{4}{6}, \frac{1}{6}, \frac{2}{6}, \frac{1}{6} \right] \quad , \quad \beta = [1, 2, 2, 2] \quad , \quad \gamma = \left[ \frac{2}{6}, 0, 0, \frac{5}{6} \right].$$

$\tau$ is eventually expanding and

$$C_0 = [0, \frac{1}{6}] \cup \left[ \frac{2}{6}, \frac{3}{6} \right] \cup \left[ \frac{4}{6}, \frac{5}{6} \right], \quad C_1 = \left[ \frac{1}{6}, \frac{2}{6} \right] \cup \left[ \frac{3}{6}, \frac{4}{6} \right] \cup \left[ \frac{5}{6}, 1 \right]$$

are its ergodic components.
Supports of ergodic components

\[ C_0 = \left[ 0, \frac{1}{6} \right] \cup \left[ \frac{2}{6}, \frac{3}{6} \right] \cup \left[ \frac{4}{6}, \frac{5}{6} \right] \]

\[ C_1 = \left[ \frac{1}{6}, \frac{2}{6} \right] \cup \left[ \frac{3}{6}, \frac{4}{6} \right] \cup \left[ \frac{5}{6}, 1 \right] \]
Application: approximation of acim for arbitrary map

Map modeling the movement of rotary drill (A. Lasota and P. Rusek [15], also [3]): $\tau_\Lambda$ depends on Froude number

$$\Lambda = \frac{v^2 M}{FR}.$$  

The more uniform is the invariant density of $\tau_\Lambda$ the more efficient is the use of the drill.
Map modeling the movement of rotary drill

\[ \tau_{3.5} \text{ rescaled from } [0, 0.9]. \]
Piecewise linear approximation on the partition \{0, 0.111, 0.138, 0.2, 0.333, 0.383, 0.5, 0.6, 0.75, 0.9, 1\}.
Approximations of the $\tau_{3.5}$-invariant density obtained: as invariant density $h$ of the piecewise linear approximation (green); $h_U$ by Ulam’s method on the same partition (red).
Errors: $|P_{\tau_{3.5}} h - h|$ - green
$|P_{\tau_{3.5}} h_U - h_U|$ - red.
Integrals of the errors functions are 0.19 and 0.12, correspondingly.
Ulam’s method in a nutshell

Let $\mathcal{P} = \{I_1, I_2, \ldots, I_N\}$ be a partition of $[0, 1]$. Map $\tau$ is modeled by a Markov chain with transition matrix

$$\mathbb{P} = \left[ \frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_i)} \right].$$

If $v = [v_1, v_2, \ldots, v_N]$ is the stationary (left) vector of $\mathbb{P}$, then

$$h_U = \sum_{i=1}^{N} \frac{v_i}{m(I_i)} \chi_{I_i},$$

is an approximation of $\tau$-invariant density.
Ulam’s method uses Markov linear approximation on finer partition, which can be seen from the transition matrix:

\[
\begin{bmatrix}
0.397368 & 0.089818 & 0.196142 & 0.316671 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.589628 & 0.347872 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.184216 & 0.467794 & 0.299714 & 0.048277 & 0.000000 \\
0.275252 & 0.102510 & 0.217319 & 0.376598 & 0.028320 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.390801 & 0.382753 & 0.226446 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.633478 & 0.203218 & 0.163305 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.058187 & 0.718809 & 0.223005 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.771484 & 0.228516 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
1.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
1.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000
\end{bmatrix}
\]
Piecewise linear approximation on actual Ulam’s partition

Using this finer partition for our piecewise linear approximation we obtain density $h$ (green)
Errors of the approximations

Errors: $|P_{\tau^{3.5}} h - h|$ - green

$|P_{\tau^{3.5}} h_U - h_U|$ - red.

Integrals of the errors functions are 0.105 and 0.117, correspondingly.


Dajani, Karma; Hartono, Yusuf; Kraaikamp, Cor, *Mixing properties of $(\alpha, \beta)$-expansions*, preprint.


Dajani, Karma; Kalle, Charlene *A note on the greedy $\beta$-transformations with deleted digits*, to appear in SMF Séminaires et Congres, Number 19, 2008.


