The present paper investigates the structure of stable sets of maximum size in certain vertex-transitive subgraphs of powers of complete graphs. Motivated by the seminal result by Erdős, Ko and Rado [6] and by Greenwell and Lovász [7] characterising stable sets of maximum size respectively in Kneser graphs and in powers of complete graphs, J. Körner (see [10]) suggested the following framework that underscores the similarity in flavour of these two results. A family of vertex-transitive graphs is defined whose elements are tuples, and where non-adjacency of two tuples is governed by the appearance of certain symbols in a common coordinate. More precisely, let $b \geq 1$ (the number of symbols that can appear in the tuples) and let $d_1 \geq d_2 \geq \cdots \geq d_b \geq 0$ be integers (the distribution of the symbols). Let $n = \sum d_i$ (the number of coordinates in the tuples) and let $m$ denote the the largest index $i$ such that $d_i > 0$. Let $P$ be any subgroup of the symmetric group $S_b$, and let $C$ be a nonempty subset of $[b] = \{1, \ldots, b\}$. The graph $G(P; C; d_1, \ldots, d_b)$ is defined as follows: its vertices are the $n$-tuples $(a_1, \ldots, a_n) \in [b]^n$ such that there exists a permutation $\sigma \in P$ and a permutation $\tau \in S_n$ for which $(a_1, \ldots, a_n) = (x_{\tau(1)}, \ldots, x_{\tau(n)})$ where $(x_1, \ldots, x_n) = (\sigma(1), \ldots, \sigma(1), \sigma(2), \ldots, \sigma(2), \ldots, \sigma(m), \ldots, \sigma(m))$ and $\sigma(i)$ appears exactly $d_i$ times. Two such tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are not adjacent if and only if there exists a coordinate $i$ such that $x_i = y_i \in C$ (the two tuples coincide for a symbol belonging to $C$). It is easy to see that, provided the permutations in $P$ preserve the set $C$, this graph is vertex-transitive under the combined actions of $S_n$ (permutation of the coordinates) and $P$ (substitution of the symbols). In what follows we always assume $C$ is invariant under $P$.

For $G$ a graph let $\alpha(G)$ denote its stability number, i.e. the size of a stable (independent) set of maximum size in $G$. We’ll refer to the number $\frac{\alpha(G)}{|V(G)|}$ as the independence ratio of $G$. It is proved in [10] that for $G = G(P; C; d_1, \ldots, d_b)$ the...
independence ratio satisfies
\[
\frac{\alpha(G)}{|V(G)|} \geq \frac{1}{|O|} \sum_{i \in O} \frac{d_i}{n}
\]
where \(d = \max\{d_i : i \in C\}\) and \(O\) is the orbit under \(P\) of any \(q \in C\) such that \(d_q = d\). It is proved in the same paper that for many special cases this bound is tight. In particular, if \(P\) is trivial, then the independence ratio is \(d/n\), and if \(P = S_b\) then the ratio is \(1/b\), a result first obtained by Deza and Frankl (see \([5]\) and also \([4]\)). Assuming the bound is tight, then the sets \(I_q^p\) consisting of all tuples \((x_1, \ldots, x_n)\) such that \(x_p = q\) are stable sets of maximal size in \(G\), provided \(q \in C\).

This suggests the following problem:

For which parameters \(P, C, d_1, \ldots, d_b\) are all the stable sets of maximum size in the graph \(G(P; C; d_1, \ldots, d_b)\) of the form \(I_q^p\)?

This problem has been investigated by several authors, for various subfamilies (see also, for instance \([1]\), \([8]\)):

(1) Let \(1 \leq r \leq n\) be integers such that \(2r \leq n\). Then \(K(r, n)\), the Kneser graph of \(r\)-subsets of \([n]\), is isomorphic to \(G(P; C; d_1, d_2)\) where \(b = 2\), \(P\) is the trivial group, \(C = \{2\}\), \(d_1 = n-r\) and \(d_2 = r\). The isomorphism is given by the correspondence which sends every \(r\)-subset \(X\) of \([n]\) to the \(n\)-tuple \((x_1, \ldots, x_n)\) where \(x_i = 2\) if \(i \in X\) and \(x_i = 1\) otherwise. The celebrated Erdős-Ko-Rado Theorem \([6]\) states that the maximum stable sets in this case are precisely the \(I_q^p\).

(2) Let \(d_1 = \cdots = d_b = 1\) (so that \(b = n\)), let \(P = S_b\) (hence \(C = [b]\)). The vertices of this graph are the permutations in \(S_n\), and was first investigated by Cameron and Ku \([3]\) (see also \([10]\)). More recently, Ku and Wong \([9]\) have investigated the case where \(P = A_b\), the alternating group on \(b\) symbols. In both cases, it is shown that the maximum stable sets are of the expected form.

(3) Although powers of complete graphs are not captured directly by this family of graphs, all subgraphs invariant under permutation of coordinates and substitution of symbols are. In fact, these are precisely the graphs \(G(P; [C], d_1, \ldots, d_b)\) with \(P = S_b\) (and hence \(C = [b]\)). Greenwell and Lovász \([7]\) show that the maximum stable sets in powers of complete graphs are precisely the \(I_q^p\). Partial results in this direction on the above subgraphs were obtained in \([10]\).

The present paper investigates this last family of graphs, and shows that, except for obvious boundary cases, the maximum independent sets of \(G\) are of the expected form. Furthermore, this result may be reformulated, in the spirit of Greenwell and Lovász, as a unique colourability result for these vertex-transitive graphs.

2. Preliminaries and statement of the main result

All graphs in this paper are finite, undirected, without loops nor multiple edges. Let \(G = (V(G), E(G))\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\), and let \(I \subseteq V(G)\). We say that \(I\) is stable or independent if no edge of \(G\) has both endpoints
in $I$. We shall be interested in stable sets of maximal size, and we’ll often refer to those as *maximum stable sets*.\(^1\)

For convenience, we shall denote from now on the graph $G(S_b, [b], d_1, \ldots, d_b)$ simply as $G(d_1, \ldots, d_b)$. Recall that this graph is defined as follows: its vertices are the $n$-tuples $(\alpha_1, \ldots, \alpha_n) \in [b]^n$ such that there exists a permutation $\tau \in S_n$ and a permutation $\sigma \in S_b$ for which $(\alpha_1, \ldots, \alpha_n) = (x_{\tau(1)}, \ldots, x_{\tau(n)})$ where

$$(x_1, \ldots, x_n) = (\sigma(1), \ldots, \sigma(1), \sigma(2), \ldots, \sigma(2), \ldots, \sigma(m), \ldots, \sigma(m))$$

and $\sigma(i)$ appears exactly $d_i$ times. Two such tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are not adjacent if and only if there exists a coordinate $i$ such that $x_i = y_i$ (the two tuples coincide in some coordinate). In other words, the graph is the induced subgraph of $K^n_b$ whose vertices consist of all tuples whose entries follow the prescribed distribution $d_1, \ldots, d_b$: some symbol appears $d_i$ times, another $d_2$ times, and so on.

Before we state our main result, we need to introduce some terminology and require a few preliminary results. If $G$ and $H$ are graphs, a map $f : V(G) \to V(H)$ is a *graph homomorphism* if it is edge-preserving, i.e. if $f(x)f(y) \in V(H)$ whenever $xy \in V(G)$. Notice that a homomorphism from a graph $G$ to the complete graph $K_b$ is the same thing as a $b$-colouring. In particular, Corollary 3 of [7] may be rephrased as follows:

**Lemma 2.1 ([7]).** The only homomorphisms from $K^n_b$ to $K_b$ are those of the form $(x_1, \ldots, x_n) \mapsto \sigma(x_i)$ where $\sigma$ is some permutation in $S_b$.

Let $1 \leq p \leq n$ and $1 \leq q \leq b$ be integers. Let $I^q_p$ denote the induced subgraph of $K^n_b$ that consists of all tuples $(x_1, \ldots, x_n)$ such that $x_p = q$.

**Lemma 2.2 ([7]).** The sets $I^q_p$ are the only stable sets of maximal size in $K^n_b$.

The group $S_n$ acts naturally on the graph $K^n_b$ by permuting the entries of the tuples. Furthermore it is easy to see that the correspondence

$$(x_1, \ldots, x_n) \mapsto (\sigma(x_1), \ldots, \sigma(x_n))$$

for each $\sigma \in S_b$ defines an action of $S_b$ on the graph as a group of automorphisms.

Let $G$ denote the group of automorphisms of $K^n_b$ generated by these actions. Clearly the graph $G(d_1, \ldots, d_b)$ is the subgraph of $K^n_b$ induced by the orbit under $G$ of the vertex $(1, \ldots, 1, \ldots, m, \ldots, m)$ where each $i$ appears $d_i$ times, $1 \leq i \leq b$. In particular, $G(d_1, \ldots, d_b)$ is vertex-transitive.

It is clear that the set $I^q_p \cap V(G(d_1, \ldots, d_b))$ is stable in the graph $G(d_1, \ldots, d_b)$; for convenience we shall denote it simply by $I^q_p$. We shall use the same convention in all graphs considered, when there is no possibility of confusion. We shall show that, assuming certain minimal restrictions on the parameters, the sets $I^q_p$ are the only stable sets of maximal size in $G(d_1, \ldots, d_b)$. We first need to prove that they are indeed maximum, and for this we require a result of Albertson and Collins [2], often referred to as the ‘no-homomorphism lemma’.

**Lemma 2.3 ([2]).** Let $G$ and $H$ be graphs such that $H$ is vertex-transitive and there exists a homomorphism $\phi : G \to H$. Then

$$\frac{\alpha(G)}{|V(G)|} \geq \frac{\alpha(H)}{|V(H)|}. \quad (1)$$

---

\(^1\)Stable sets that are maximal for inclusion play no role here.
Furthermore, if equality holds in (1), then for any stable set $I$ of cardinality $\alpha(H)$ in $H$, $\phi^{-1}(I)$ is a stable set of cardinality $\alpha(G)$ in $G$.

We now compute the independence ratio of the graph $G = G(d_1, \ldots, d_b)$. First we find an embedding of the complete graph $K_b$ in $G$: choose any vertex $v = (x_1, \ldots, x_n)$ of $G$ and let $\sigma \in S_b$ be defined by $\sigma(i) = i + 1$ (modulo $b$). The orbit of $v$ under the action of the subgroup generated by $\sigma$ is clearly isomorphic to $K_b$. Secondly, it is clear that the projection onto a coordinate is a $b$-colouring of the graph $G$, and thus we have a homomorphism from $G$ to $K_b$. Since all the graphs involved are vertex-transitive, it follows from Lemma 2.3 that the independence ratio of $G(d_1, \ldots, d_b)$ is equal to $1/b$, and that the sets $I^p_b$ are of maximal size.

There are two main families of maximum stable sets different from the $I^p_b$, both obtained by a simple pigeonhole argument. Recall that $d_1$ is the maximal number of occurrences of a symbol of a tuple in $G(d_1, \ldots, d_b)$ and $m$ is the number of different symbols that appear in a tuple.

- Suppose that $d_1 > n/2$. Let $I$ consist of all tuples that contain the entry 1 exactly $d_1$ times. Obviously this is a stable set, and it has maximal size.
- Suppose that $m = 2$. Then either $d_1 > n/2$ and we have the previous case, otherwise $d_1 = d_2 = n/2$ and $d_i = 0$ for all $i \geq 3$. In this case let $I$ consist of all tuples that have $n/2$ 1’s appearing in the first $n - 1$ coordinates. Obviously this is a stable set, and it is easy to see that it has maximal size.

In all other cases we have that $d_1 \leq n/2$ and $m \geq 3$. Our main result states that in this case the $I^p_b$ are the only maximum stable sets. To rephrase this result in terms of extension of colourings, we require the following lemma:

**Lemma 2.4.** Let $m \geq 2$. If $d_1 \geq 2$ or if $m < n$, then the following conditions are equivalent:

1. every $b$-colouring of $G(d_1, \ldots, d_b)$ extends to a unique $b$-colouring of $K^n_b$;
2. the sets $I^p_b$ are the only stable sets of $G(d_1, \ldots, d_b)$ of maximal size.

**Proof.** Let $G = G(d_1, \ldots, d_b)$. Suppose that (1) holds. Let $I$ be an independent set of $G$ of maximal size. For each $1 \leq i \leq b$ consider the set

$$I + i = \{(x_1 + i, \ldots, x_n + i) : (x_1, \ldots, x_n) \in I\}$$

where the sum is taken modulo $b$. Obviously all these sets are independent sets of maximal size, and any two are disjoint; furthermore they partition the graph. Thus we have a $b$-colouring $\phi$ of $G$ defined by $\phi(v) = i$ if $v \in I + i$. Then $\phi$ extends to a unique $b$-colouring $\psi$ of $K^n_b$, which by Lemma 2.1 must be of the form $\psi((x_1, \ldots, x_n)) = \sigma(x_i)$ for some $i$ and some $\sigma \in S_b$. Then $I$ is the intersection of $G$ with $\psi^{-1}(b)$, which is precisely $I^\sigma_{p^{-1}(b)}$.

Now suppose that (2) holds, and let $\phi$ be a homomorphism from $G$ to $K_b$. By Lemma 2.3 $\phi^{-1}(i)$ is of the form $I^p_{b'}$ for every $i$. It is easy to see that the sets $I^p_b$ and $I^q_{b'}$ are disjoint if and only if either (i) $p = p'$ and $q \neq q'$ or (ii) $p \neq p'$ and $q = q'$, this second case occurring only when $d_1 = 1$. It follows that the set $\{\phi^{-1}(1), \phi^{-1}(2), \ldots, \phi^{-1}(b)\}$ is of the form $\{I^p_{b'}, \ldots, I^p_{b'}\}$ or $\{I^q_{b'}, \ldots, I^q_{b'}\}$. However, the second case can occur only if $m = n = b$ contrary to our hypothesis. Thus there exists an index $p$ and a permutation $\sigma \in S_b$ such that

$$\phi((x_1, \ldots, x_n)) = \sigma(x_p)$$
which clearly extends to $K^n_b$. This extension, call it $\psi$, is unique by Lemma 2.1.

Notice that the lone exception to the last result is the graph $G(1, 1, \ldots, 1)$: even though its maximum stable sets are all of the form $I^n_1$ (see Lemma 3.1 below), the graph $G(1, 1, \ldots, 1)$ admits $b$-colourings that do not extend to any colourings of $K^n_b$; indeed, let $f(x_1, \ldots, x_b) = i$ when $x_i = 1$.

We summarise our results:

**Theorem 2.5.** Let $b \geq 3$, let $d_1 \leq n/2$ and let $m \geq 3$. Then the stable sets of maximal size in $G(d_1, \ldots, d_b)$ are obtained by fixing a coordinate. Furthermore, unless $d_i = 1$ for all $1 \leq i \leq b$, every $b$-colouring of $G(d_1, \ldots, d_b)$ extends to a unique $b$-colouring of $K^n_b$.

### 3. Proof of the main result

From now on, $G$ shall denote a fixed graph $G = G(d_1, \ldots, d_b)$ with the following conditions on the parameters: $n = \sum_{i=1}^b d_i$, $2 \leq d_1 \leq n/2$, $3 \leq m \leq b$; recall that $m$ denotes the largest index $i$ such that $d_i > 0$. Set $G_b = G(1, \ldots, 1)$, where the number of $1$’s is $b$. Furthermore, we fix a stable set $I$ of maximal size in $G$. The restriction of the $k^{th}$ projection $K^n_b \to K_b$ to $G(d_1, \ldots, d_b)$ will be denoted by $\pi_k$.

The general strategy will be as follows: we pull back the stable set $I$ of $G$ under a homomorphism $h : G' \to G$. By choice of $G'$ and an application of Lemma 2.3, the stable set thus produced will be of the form $I^n_\beta$; this will impose conditions on the stable set $I$. By repeating this procedure for various conveniently chosen homomorphisms, we’ll manage to prove that $I$ is also of the correct form. Unfortunately, we seem unable to avoid distinguishing three distinct cases: the graphs $G(2, 1, 1)$, $G(2, 2, 1)$ and $G(2, 2, 2)$ will be handled in section 3 by a separate approach. For the remaining graphs, we’ll consider two cases throughout:

- **If** $b \geq 4$ we will work with homomorphisms $h : G_b \to G$ of the form
  
  $x = (x_1, \ldots, x_b) \mapsto h(x) = (x_{i_1}, \ldots, x_{i_m})$

  where for each $1 \leq i \leq b$, $x_i$ appears exactly $d_i$ times as a coordinate function.

- **If** $b = 3$ we will use homomorphisms $K^3_d \times G(1, 1, 0) \to G$ of the form
  
  $x = (x_1, \ldots, x_{d_3}, x_{d_3+1}, x_{d_3+2}) \mapsto h(x) = (y_1, \ldots, y_n)$

  where for each pair $(i, j)$ with $1 \leq i \leq d_3, 0 \leq j \leq 2$, exactly one of the coordinate functions $y_k$ is equal to $x_{i+j}$ (sum is modulo 3), and $(d_1 - d_3)$ of the coordinate functions will be $x_{d_3+1}$ and $(d_2 - d_3)$ of the coordinate functions will be $x_{d_3+2}$.

In either case, we denote the set of such homomorphisms by $\mathcal{H}(G)$.

**Example:** Let $b = 3, d_1 = d_2 = 3, d_3 = 2$. Now $G' = K^2_d \times G(1, 1, 0)$ and $G = G(3, 3, 2)$. Set $h(x_1, x_2, x_3, x_4) = (x_1 + 1, x_3, x_1, x_2 + 2, x_2, x_1 + 2, x_4, x_2 + 1)$. This tuple is really in $G(3, 3, 2)$ for every $x_1, x_2, x_3, x_4 \in K_b$ if $x_3 \neq x_4$, since $x_1, x_1 + 1$ and $x_1 + 2$ are three different vertices in $K_3$, similarly the three vertices $x_2, x_2 + 1, x_2 + 2$ and also the vertices $x_3$ and $x_4$ are different.
Our first step is to show that the stable sets of maximal size in the graphs $G_b$ and $K_3^{d_3} \times G(1,1,0)$ are of the right form. For $G_b$ the result can be found in [3] (see also [10]).

**Lemma 3.1** ([3]). Let $b \geq 3$. The stable sets of maximal size in $G_b$ are of the form $I_p^q$ for some $1 \leq p, q \leq b$.

**Lemma 3.2.** Let $k \geq 0$ and consider the subgraph $L = K_3^k \times G(1,1,0)$ of $K_3^{k+2}$. Then every maximum stable set in $L$ is of the form $I_p^q$ for some $1 \leq p \leq k+2$, $1 \leq q \leq 3$.

**Proof.** Since $L$ is vertex-transitive, it is easy to see by Lemma 2.3 that $\frac{\alpha(L)}{|V(L)|} = \frac{1}{3}$. Consider the following subgraphs of $L$: $L_1 = K_3^2 \times \{(1,2), (2,3), (3,1)\}$ and $L_2 = K_3^2 \times \{(2,1), (3,2), (1,3)\}$. Both are isomorphic to $K_3^{3+1}$, and $V(L)$ is the disjoint union of $V(L_1)$ and $V(L_2)$. A simple count shows that every maximum stable set $I$ of $L$ is the union of two maximum stable sets $I_1 \subseteq L_1$ and $I_2 \subseteq L_2$. In $L_1$ and $L_2$ every maximum stable set is of the proper form: $I_1 = I^1_{i_1}$, $I_2 = I^2_{i_2}$. Notice further that if $i_s > k$ then we may choose it to be $k+1$ or $k+2$, whichever is more convenient. Clearly $i_1 = i_2$, otherwise there is a vertex in $I_1$ adjacent to a vertex in $I_2$. If this is not one of the last two coordinates then $j_1 = j_2$ by the same reason. If $i_1 = i_2$ is $k+1$ or $k+2$ then we have reduced the problem to the case $k = 0$, and this is an easy exercise for the reader. \hfill \Box

Fix a homomorphism $h \in \mathcal{H}(G)$. Since the independence ratio of its domain and of $G$ are equal, the pre-image of $I$ under $h$ is a maximum stable set by Lemma 2.3, and by Lemmas 3.1 and 3.2 it is of the form $I_p^q$. Consider the subset of the coordinates that depend on $x_p$, more precisely the coordinates with coordinate function of the form $\nu(x_p)$, where $\nu$ is a permutation of the complete graph $K_b$. We call this set the support of $h$ and we denote it by $\text{supp}(h)$. Set $\text{col}(h) = q$ and $\text{cor}(h) = p$. We will call these the main colour and main coordinate of the homomorphism $h$, respectively.

**Example:** Let $b = 3, d_1 = d_2 = 3, d_3 = 2$. Now $G' = K_3^2 \times G(1,1,0)$ and $G = G(3,3,2)$. Set $h((x_1, x_2, x_3, x_4)) = (x_1 + 1, x_3, x_1, x_2 + 2, x_2, x_1 + 2, x_4, x_2 + 1)$.

Set $I = I_6^2$. Now $h^{-1}(I) = I_7^1$, since $h_6 = x_1 + 2$ so $I$ consists of the elements of $G(3,3,2)$ with sixth coordinate equal to 1 (and $2 + 2 = 1 \mod 3$). The pre-image of the set $I$ contains exactly those elements in $G'$ with first coordinate equal to 2. Hence $\text{cor}(h) = 1$ and $\text{col}(h) = 2$. And $\text{supp}(h) = \{1, 3, 6\}$, since the first, third and sixth coordinate functions are $x_1 + 1, x_1$ and $x_1 + 2$. (We say that these coordinate functions depend on $x_1$, while the others on either $x_2$ or $x_3$ or $x_4$, respectively.)

We will investigate the set system that consists of all the supports of the homomorphisms in $\mathcal{H}(G)$. (These homomorphisms only differ by a permutation of the coordinate functions.) The following lemma, which can be found in a slightly different form as Lemma 5.6 of [10], will be used to show that the supports have a common element.

**Proposition 3.3** ([10]). Let $e_1 \geq e_2 \geq \cdots \geq e_r$ be positive integers, where $r \geq 3$ and $2e_1 \leq \sum_{i=1}^r e_i = n$. Consider a set system $\mathcal{S}$ on $[n] = \{1, \ldots, n\}$ such that the following conditions hold:

1. Every set in $\mathcal{S}$ has cardinality $e_i$ for some $1 \leq i \leq r$;

(2) No two sets in \( S \) are disjoint;
(3) For any partition \([n] = \bigcup_{i=1}^{t} A_i\), where \(|A_i| = e_i\) one of the sets \( A_i \) is in \( S \). Then there is an \( x \in [n] \) such that every set in \( S \) contains \( x \).

We shall prove that the set of all supports of homomorphisms in \( \mathcal{H}(G) \) contains no pairwise disjoint sets. To do this we need the following two lemmas that will guarantee that some special permutations of the coordinate functions do not change the support. If \( f \in \mathcal{H}(G) \) and \( u, v \) are distinct elements of \([1, 2, \ldots, n]\), let \( f^{uv} \in \mathcal{H}(G) \) be obtained from \( f \) by exchanging its \( u \)-th and \( v \)-th coordinate functions, i.e. if \( f(x) = (y_1, \ldots, y_u, \ldots, y_v, \ldots, y_n) \) then \( f^{uv}(x) = (y_1, \ldots, y_v, \ldots, y_u, \ldots, y_n) \).

**Lemma 3.4.** Suppose that \( b \geq 4 \). Let \( f \in \mathcal{H}(G) \) and let \( u, v \) be distinct elements of \([1, 2, \ldots, n]\). Then \( \text{col}(f^{uv}) = \text{col}(f) \). Furthermore if \( u, v \notin \text{supp}(f) \) then \( \text{supp}(f^{uv}) = \text{supp}(f) \) and \( \text{cor}(f^{uv}) = \text{cor}(f) \).

**Proof.** Let \( g = f^{uv}, \pi_u \circ f = x_s \) and \( \pi_v \circ f = x_t \). Consider the graph \( H \) with \( V(H) = \{a_1, \ldots, a_b, c_1, \ldots, c_e\} \): the vertices \( a_i \) and \( a_j \) \((c_i \) and \( c_j)\) are adjacent if \( i \neq j \), and the vertices \( a_i \) and \( c_j \) are adjacent if there is a coordinate \( k \) such that \( \pi_k \circ f = x_i \) and \( \pi_k \circ g = x_j \). Call a homomorphism \( h : H \to K_b \) valid if \( a_{\text{cor}(f)} \) is mapped to \( \text{col}(f) \) and \( c_{\text{cor}(g)} \) is mapped to \( \text{col}(g) \). If \( h \) is a valid homomorphism then both \( f(h(a_1), \ldots, h(a_b)) \) and \( g(h(c_1), \ldots, h(c_e)) \) are in \( I \), since the vertices \( h(a_1), \ldots, h(a_b) \) and also \( h(c_1), \ldots, h(c_e) \) form a complete graph. On the other hand these elements are adjacent, since \( f \) is a homomorphism and the entries are coordinatewise adjacent. We can conclude that there is no valid homomorphism \( h : H \to K_b \).

The edges of the form \((a_i, c_j)\) can be at most \((a_i, c_j), (a_i, c_k)\) and \((a_i, c_l) \) \((1 \leq i \leq b)\). Denote the graph of vertex set \( V(H) \) having these edges plus the edges \((a_i, a_j), (c_i, c_j) \) \((i \neq j)\) by \( H' \). By the above, there is an fortiori no valid homomorphism \( h : H' \to K_b \). This implies that \( a_{\text{cor}(f)} \) and \( c_{\text{cor}(g)} \) are adjacent and \( \text{col}(f) = \text{col}(g) \): in this case we clearly do not have any valid homomorphism, since a valid homomorphism is a homomorphism \( h \) with the property \( h(a_{\text{cor}(f)}) = \text{col}(f) \) and \( h(c_{\text{cor}(g)}) = \text{col}(g) \), but this property pretends \( h \) to be a homomorphism. If these did not hold then it would be easy to find such an \( h \) since \( b \) is large enough: this needs some very easy case analysis with more complicated notations.
(See Figure 1: the \( u \)th and \( v \)th coordinate functions depend on \( x_1 \) and \( x_2 \), respectively.)

It is clear that, since \( a_{\text{cor}(f)} \) and \( c_{\text{cor}(g)} \) are adjacent, then either (i) \( \text{cor}(g) = \text{cor}(f) \) or (ii) \( \{\text{cor}(f), \text{cor}(g)\} = \{s, t\} \). If neither \( u \) nor \( v \) is in the support of \( f \) then neither \( s \) nor \( t \) is equal to \( \text{cor}(f) \), and hence (i) must hold, and by definition of \( g \) its support is equal to that of \( f \).

\[ \square \]

The technical condition on the parameters \( d_1, d_2, d_3 \) in the statement of the next lemma is necessary to ensure that homomorphisms \( f \in \mathcal{H}(G) \) depend on at least three variables, and hence ensure that Proposition 3.3 applies to the system of supports of these homomorphisms. The only graphs that manage to escape are \( G(2, 1, 1) \), \( G(2, 2, 1) \) and \( G(2, 2, 2) \): these will be taken care of in section 4 by different methods.\(^2\)

**Lemma 3.5.** Suppose that \( b = 3 \). Suppose moreover that either (i) \( d_3 \geq 3 \) or (ii) \( d_3 \geq 2 \) and \( d_1 > d_3 \) or (iii) \( d_3 \geq 1 \) and \( d_2 > d_3 \). Let \( f \in \mathcal{H}(G) \) and let \( u, v \) be distinct elements of \( \{1, 2, \ldots, n\} \) such that the coordinate functions \( \pi_u \circ f \) and \( \pi_v \circ f \) do not depend on the same variable. Let \( g = f^{uv} \).

1. Suppose that one of \( \pi_u \circ f \) or \( \pi_v \circ f \) depends on \( x_i \) for some \( i \leq d_3 \).
   (a) If \( u, v \notin \text{supp}(f) \) then \( \text{supp}(f) = \text{supp}(g) \), \( \text{cor}(f) = \text{cor}(g) \) and \( \text{col}(f) = \text{col}(g) \).
   (b) If \( \{u, v\} \cap \text{supp}(f) \neq \emptyset \) then \( \{u, v\} \cap \text{supp}(g) \neq \emptyset \), say \( w_f \in \text{supp}(f) \cap \{u, v\}, w_g \in \text{supp}(g) \cap \{u, v\} \). Now \( \pi_{w_f} \circ f(x) = \pi_{w_g} \circ g(y) \) for any \( x \in f^{-1}(I) \) and \( y \in g^{-1}(I) \).

2. Assume \( \pi_u \circ f = x_{d_3+1} \) and \( \pi_v \circ f = x_{d_3+2} \).
   (a) If \( u, v \notin \text{supp}(f) \) then \( \text{supp}(f) = \text{supp}(g) \), \( \text{cor}(f) = \text{cor}(g) \) and \( \text{col}(f) = \text{col}(g) \).
   (b) If \( \text{supp}(f) \cap \{u, v\} \neq \emptyset \) then \( \text{supp}(g) \cap \{u, v\} \neq \emptyset \) and \( \text{col}(f) = \text{col}(g) \).

**Proof.** We define the graph \( H \) in the following way: \( V(H) = \{a_i + j, c_i + j : 1 \leq i \leq d_3, 0 \leq j \leq 2\} \cup H_1 \cup H_2 \), where \( a_i + 0 = a_i \), if \( d_1 > d_3 \) then \( H_1 = \{a_{d_3+1}, c_{d_3+1}\} \), and \( H_1 = \emptyset \) otherwise, and if \( d_2 > d_3 \) then \( H_2 = \{a_{d_3+2}, c_{d_3+2}\} \), \( H_2 = \emptyset \) else. The vertices \( a_i + j \) and \( c_i + j' \) are adjacent if there is a coordinate \( k \) such that \( \pi_k \circ f = x_i + j \) and \( \pi_k \circ g = x_i + j' \). If present, the vertices \( a_{d_3+1} \) and \( a_{d_3+2} \) are adjacent, as are \( c_{d_3+1} \) and \( c_{d_3+2} \).

Call a mapping \( h : H \to K_3 \) valid if \( h(a_i + j) = h(a_i) + j \) and \( h(c_i + j) = h(c_i) + j \) for all \( i, j \), and \( h(a_{\text{cor}(f)}) = \text{col}(f) \) and \( h(c_{\text{cor}(g)}) = \text{col}(g) \). There is no valid homomorphism \( h \) of \( H \) to \( K_3 \), for otherwise the vertices \( f(h(a_1), h(a_2), \ldots) \) and \( g(h(c_1), h(c_2), \ldots) = f^{uv}(h(c_1), h(c_2), \ldots) \) would be adjacent vertices of \( I \), since \( f \) is a homomorphism and the above entries are pairwise adjacent (after flipping coordinate \( i \) and \( j \) of \( (h(c_1), h(c_2), \ldots) \)).

1. This is similar to the proof of the last lemma. The graph \( H' \) is defined in the following way: \( V(H') = V(H) \), and \( E(H') = V(H')^2 \cap \{(a_i + j, c_i + j) : 1 \leq i \leq d_3+2, 0 \leq j \leq 2\} \cup \{(a_{d_3+1}, a_{d_3+2}), (c_{d_3+1}, c_{d_3+2}), (a_s + k_s, c_t + k_t), (a_t + k_t, c_s + k_s)\} \), where \( \pi_u \circ f = x_s + k_s \), \( \pi_v \circ f = x_t + k_t \) (see Figures 2 and 3: in Figure 2 one of the

\(^2\)Note that the parameters of the graph \( G(2, 2, 1) \) actually satisfy the conditions of Lemma 3.5, but we can’t apply Proposition 3.3 because \( n < 6 \).
$u^\text{th}$ or the $v^\text{th}$ coordinate function depends on $x_{d_3+1}$ or $x_{d_3+2}$, in Figure 3 neither does.)

$$\begin{align*}
\pi_u \circ f &= x_{d_3+1}, \quad \pi_v \circ f = x_{d_3+2} \\
\text{and that } \cor(f) &= d_3 + 1.
\end{align*}$$

Let $w$ be a coordinate such that $\pi_w \circ f = x_1 + 2$, and let $g' = f^{uv}$. We know from the previous case that $\cor(f) = \cor(g')$ and $\col(f) = \col(g')$. Now $g = (g'^{uw})^{uv}$.
Consider the graph $H'$ with vertex set $V(H') = V(H)$ and edge set $E(H') = \{(i + j, c_i + j) : 1 \leq i \leq d_3, 0 \leq j \leq 2\} \cup \{(a_{d_3 + 1}, a_{d_3 + 2}), (c_{d_3 + 1}, c_{d_3 + 2}), (a_{d_3 + 1}, c_1 + 2), (a_{d_3 + 2}, c_{d_3 + 1}), (a_1 + 2, c_{d_3 + 2})\}$.

![Diagram of graph $H'$ with vertex set $V(H') = V(H)$ and edge set $E(H')$.]

**Figure 4.** The graph $H', d_3 = 1$.

We call a colouring $h : H' \to K_3$ valid if $c_{d_3 + 1}$ gets colour $\text{col}(g)$, $a_{\text{cor}(g')}(+0)$ gets colour $\text{col}(g')$ and for every $1 \leq i \leq d_3$ and $0 \leq j \leq 2$ the colour of $a_i + j$ ($c_i + j$) is the colour of $a_i$ ($c_i$) plus $j$. If $H$ had a valid colouring $h$ then $g'(h(a_1), \ldots), g(h(c_1), \ldots)$ would be adjacent elements in the independent set $I$. It is easy to check that the only colouring of $c_{d_3 + 1}$ and one of the vertices $a_i + j$ that can not be extended is when we colour $a_{d_3 + 1}$ or $a_{d_3 + 2}$ (i.e. an adjacent vertex) with the colour of $c_{d_3 + 1}$.

Now we prove (a). Assume $\pi_u \circ f = x_{d_3 + 1}, \pi_v \circ f = x_{d_3 + 2}$ and that $\text{cor}(f) = 1$. Assume for a contradiction that $\text{cor}(f) \neq \text{cor}(g)$. We have two subcases, first assume $\text{cor}(g) > d_3$, say $\text{cor}(g) = d_3 + 1$. Let $w$ be a coordinate such that $\pi_w \circ f = x_1 + 2$, and $g' = g^{uw}$. We know by the previously proved case (a) that $\text{cor}(g') = \text{cor}(g) = d_3 + 1$ and $\text{col}(g') = \text{col}(g)$. Observe that $f = (g^{uw})^{uw}$. We have seen in the proof of the previous case that $\text{cor}(f) = \text{cor}(g')$, contradicting $\text{cor}(f) = 1$ and $\text{cor}(g') = d_3 + 1$.

The other subcase is when $\text{cor}(f) \neq \text{cor}(g) \leq d_3$, say $\text{cor}(g) = 2$. There is a sequence of functions $h_0 = f, \ldots, h_k = g$ such that $h_i$ and $h_{i+1}$ differ only by a transposition of two coordinates, one depending on $x_2$ and the other on $x_{d_3 + 1}$ or $x_{d_3 + 2}$, respectively. We have proved in case (1) that if $\text{cor}(f) = 1$ then for every $i$ the main coordinate $\text{cor}(h_i)$ can only be $1, d_3 + 1$ or $d_3 + 2$ in contradiction with $\text{cor}(g) = 2 \leq d_3$. So we have proved $\text{cor}(f) = \text{cor}(g)$.

Finally we have to show $\text{col}(f) = \text{col}(g)$. We use again a sequence of functions $h_0 = f, \ldots, h_k = g$ such that $h_i$ and $h_{i+1}$ differ only by a transposition of two coordinates, one depending now on $x_1$ and the other on $x_{d_3 + 1}$ or $x_{d_3 + 2}$, respectively. We have proved in case (1) that the colour is preserved when doing such transpositions. This completes the proof of the lemma.

Given $x \in G$ and a permutation $\sigma \in S_n$ let $\sigma^x$ denote the tuple obtained from $x$ by permuting its coordinates according to $\sigma$, i.e. $x^\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

**Lemma 3.6.**

1. Let $x$ and $y$ be two vertices of $G$ and $a \in [b]$ such that the sets of coordinates $U = \{u : x_u = a\}, V = \{v : y_v = a\}$ are disjoint. Then there are permutations $\nu, \mu \in S_n$ such that $\nu$ fixes $U$ and $\mu$ fixes $V$ pointwise and $x^\nu$ is adjacent to $y^\mu$ in $G$. 

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(2) Consider the graph $G = G(d_1, d_2, d_3)$, where $d_1 \leq n/2, d_3 > 0$ and let $i_1, i_2, i_3, j_1, j_2, j_3 \in [n]$ be distinct elements. Then there are adjacent vertices $x, y \in G$ such that $x_{i_k} = k$ and $y_{j_k} = k$, $(k = 1, 2, 3)$.

Proof. (1) Choose permutations $\mu$ and $\nu$ that fix $U$ and $V$ pointwise, respectively, such that the number of identical coordinates in $x' = x^\mu$ and $y' = y^\nu$ is minimal. Suppose for a contradiction that $x'$ and $y'$ have a coordinate with a common entry, call it $i$. Since we obviously cannot exchange this entry for another to reduce the number of identical coordinates, it means that $x'_v = i = y'_u$ for all $u \in U$ and all $v \in V$ and either $x'_k$ or $y'_k$ is equal to $i$ for $k \not\in U \cup V$. Thus the total number of occurrences of $i$ in both tuples is at least $n+1$, contradicting the fact that $d_1 \leq n/2$.

(2) Consider a vertex $x$ such that $x_{i_k} = k$ and a vertex $y$ such that $y_{j_k} = k$. Following the idea of the previous case choose permutations $\mu$ and $\nu$ with the following properties: $\nu$ fixes $i_1, i_2, i_3$, $\mu$ fixes $j_1, j_2, j_3$ and the number of identical coordinates in $x' = x^\nu$ and $y' = y^\nu$ is minimal. Suppose indirectly that $x'_i = y'_i$ for some $i$, and denote $x'_i$ by $w$. If $i \not\in \{i_1, \ldots, j_3\}$ then $x'_j = w$ or $y'_j = w$ holds for every $j \neq i$. Hence the total number of occurrences of $w$ is $> n$ in $x'$ and $y'$, a contradiction.

So we may suppose $i \in \{i_1, \ldots, j_3\}$, say $i = i_w$. Now $x'_{j_k} = w$ ($k = 1, 2, 3$). And for any coordinate $u \not\in \{i_1, \ldots, j_3\}$ we know that $y'_u = i$ is equal to $w$: else we could exchange $y'$ by $y'^u$ not increasing the number of identical coordinates but having an identical pair not in $\{i_1, \ldots, j_3\}$ like in the previous case. Altogether we have at least $(n-2)$ occurrences of $w$ in $y'$, contradicting $n \geq 6$.

\[\square\]

Proposition 3.7. Let $d_1 \geq \cdots \geq d_b \geq 0$ with $b \geq 3$ and $d_1 \leq n/2$. Then the sets $I_p^b$ are the only stable sets of maximal size in $G = G(d_1, \ldots, d_b)$.

Proof. The cases $G(2, 1, 1)$, $G(2, 2, 1)$ and $G(2, 2, 2)$ will be dealt with in section 4 below. Hence from now on we may assume that $b \geq 4$ or that $b = 3$ and the conditions on $d_1, d_2, d_3$ from Lemma 3.5 hold; in this last case, since we know the result holds for $G(1, 1, 1)$ and we deal with $G(2, 2, 1)$ separately, we may also assume that $n \geq 6$.

(1) Suppose that $b \geq 4$. Let $I \subset G$ be a stable set of maximal size. We first prove that the supports of members of $\mathcal{H}(G)$ pairwise intersect. Suppose for a contradiction that there are homomorphisms $f, g \in \mathcal{H}(G)$ such that $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. Notice first that $\text{cor}(f) = \text{cor}(g)$. Indeed, if this is not the case then we may interchange, using only transpositions, the coordinates of $f$ outside its support to obtain $h \in \mathcal{H}(G)$ such that $h$ and $g$ coincide on $\text{supp}(g)$; and by the iterated use of Lemma 3.4 for this sequence we have $\text{cor}(f) = \text{cor}(h)$. It is clear that we may proceed similarly from $g$, this time transposing coordinates outside the support of $g$, to obtain $h$; thus $\text{cor}(f) = \text{cor}(h) = \text{cor}(g)$, a contradiction.

Let $p = \text{cor}(f) = \text{cor}(g)$. We know from Lemma 3.4 that $\text{col}(f) = \text{col}(g)$, denote it by $q$. Let $z \in G_h$ be any tuple with $z_p = q$. Apply Lemma 3.6 (1) to $x = f(z)$ and $y = g(z)$, with $U = \text{supp}(f)$ and $V = \text{supp}(g)$, to obtain adjacent elements $x^\mu$ and $y^\nu$ in $G$. Since $\mu$ fixes $U$ pointwise, it follows that $x^\mu \in f'(I_p^b)$ where $f'$ is obtained from $f$ by transpositions of its coordinates outside the support; in particular $\text{col}(f') = \text{col}(f) = q$ so that $x^\mu \in I$. The same argument shows that $y^\nu \in I$, a contradiction.
So the set system of all supports satisfies the conditions of Proposition 3.3 with \( r = m \) and \( e_i = d_i \) for all \( 1 \leq i \leq m \), hence there is an element \( p \in [n] \) contained in all supports. By Lemma 3.4 the main colour is the same for all homomorphisms, denote it by \( q \). We will prove that \( I = I_p^q \); indeed, consider an arbitrary element \( x \in I_p^q \). There is a homomorphism \( h \in \mathcal{H}(G) \) such that \( x \) is in the image of \( h \) and such that the set of coordinates of \( x \) to \( \text{col}(h) \) is exactly \( \text{supp}(h) \). Hence \( x \in I \), \( I \supseteq I_p^q \). On the other hand \( I_p^q \) is a stable set of maximal size, so \( I = I_p^q \). This completes the proof of the case \( b = 4 \).

(2) Now suppose that \( b = 3 \) and that the conditions of Lemma 3.5 hold. Let \( I \subseteq G \) be a stable set of maximal size. We prove that the supports of members of \( \mathcal{H}(G) \) pairwise intersect. Suppose for a contradiction that there are homomorphisms \( f, g \in \mathcal{H}(G) \) such that \( \text{supp}(f) \cap \text{supp}(g) = \emptyset \). The proof that \( \text{cor}(f) = \text{cor}(g) \) will be identical to the one above, this time invoking Lemma 3.5.

Let \( \nu \) denote a permutation that fixes \( \text{supp}(f) \). We will prove \( \text{cor}(f^\nu) = \text{cor}(f) \). There is a sequence of homomorphisms \( f = f_0, \ldots, f_k \) such that \( f_k = f_k^\nu \), \( f_k = f^\nu \) and \( u, v \notin \text{supp}(f) \). We prove \( \text{cor}(f_i) = \text{cor}(f) \) by induction on \( i \). \( f_{i+1} = f_i^\nu \), and \( u, v \notin \text{supp}(f) = \text{supp}(f_i) \). In this case Lemma 2.5 implies \( \text{cor}(f_{i+1}) = \text{cor}(f_i) \). We can prove the same statement for \( g \) and a permutation \( \mu \) fixing \( \text{supp}(g) \). If \( \text{supp}(f) \) and \( \text{supp}(g) \) are disjoint and \( \text{cor}(f) \neq \text{cor}(g) \) then by Lemma 3.6 there are permutations \( \nu \) and \( \mu \) fixing the appropriate support such that \( f^\nu = g^\mu \). This contradicts \( \text{cor}(f) \neq \text{cor}(g) \).

Consider the case \( \text{cor}(f) > d_3 \); we will prove that \( \text{col}(f) = \text{col}(g) \). We know that \( f \) and \( g \) only differ by a permutation of the coordinate functions. Hence there exists a sequence \( f = h_1, \ldots, h_r = g \) such that for each \( i \) there is a transposition \( (uv) \) such that \( h_i = h_i^{uv} \), where the coordinate functions \( \pi_u \circ h_i \) and \( \pi_v \circ h_i \) depend on different variables, and either \( \pi_u \circ h_i = x_{d+1} \) or \( \pi_v \circ h_i = x_{d+2} \) or \( u, v \notin \text{supp}(h_i) \). Hence \( \text{col}(h_i) = \text{col}(h_{i+1}) \) for every \( 1 \leq i \leq r \), so \( \text{col}(f) = \text{col}(g) \).

Consider the vertices \( x \in \text{im}(f) \cap I \) and \( y \in \text{im}(g) \cap I \). By Lemma 3.6 (1) there are permutations \( \nu \) and \( \mu \) that \( \mu \) fixes \( \text{supp}(f) \), \( \nu \) fixes \( \text{supp}(g) \) and \( x^\mu \) is adjacent to \( y^\nu \). Lemma 3.5 implies \( \text{cor}(\mu \circ f) = \text{cor}(f) \) and \( \text{col}(f^\nu) = \text{col}(f) \), hence \( x^\mu \in I \). Similarly \( y^\nu \in I \), but these are adjacent vertices, a contradiction.

Now assume \( \text{cor}(f) = \text{cor}(g) \leq d_3 \). By Lemma 3.6 (2) there are adjacent elements \( u, v \) such that \( u_i = \pi_i \circ f(f^{-1}(I)) \) if \( i \in \text{supp}(f) \) and \( v_i = \pi_i \circ g(g^{-1}(I)) \) if \( i \in \text{supp}(g) \). We get \( u, v \in I \) by Lemma 3.5, contradicting that \( I \) is stable.

So the set system of all supports satisfies condition (2) of Proposition 3.3. By definition, the support of a map \( f \in \mathcal{H}(G) \) is a block of a partition of \( [n] \) with blocks of size 3, or \( d_1 - d_3 \) or \( d_2 - d_3 \); let \( e_1, \ldots, e_r \) denote the sizes, in decreasing order, of the blocks of this partition. Since \( n \geq 6 \) and by the technical condition on the parameters, we have that \( e_1 \leq n/2 \) and \( r \geq 3 \). Thus we may apply Proposition 3.3: there is an element \( p \) contained in all supports. And Lemma 3.5 guarantees that \( \pi_i \circ f(f^{-1}(I)) \) is the same for all homomorphism \( f \), say \( q \). The proof of \( I = I_p^q \) is the same as in the previous section.

\[ \square \]

4. The graphs \( G(2,1,1) \), \( G(2,2,1) \) and \( G(2,2,2) \).

Lemma 4.1. The sets \( I_p^q \) are the only stable sets of maximal size in \( G(2,1,1) \).
Proof. Let $I$ be a maximum stable set in $G = G(2, 1, 1)$; it is of size 12. Let $I(q)$ denote the subset of $I$ whose tuples contain the symbol $q$ twice. Using homomorphisms $f : G_3 \to G$ of the form $f(x_1, x_2, x_3) = (x_i, x_j, x_k, x_l)$ where \{i, j, k, l\} = \{1, 2, 3\} it follows from no-homomorphism Lemma 2.3 that $I$ contains precisely two tuples of any fixed pattern $(x_i, x_j, x_k, x_l)$ such as $(a, b, c, a)$ for instance.

Claim 1. For any $q = 1, 2, 3$, if $I(q)$ contains 5 or more tuples with $q$ in some coordinate $p$ then $I = I^q_p$.

Proof of Claim 1. We may assume without loss of generality that $I$ contains the tuples $(1, 1, 2, 3), (1, 1, 3, 2), (1, 2, 1, 3), (1, 3, 1, 2), (1, 2, 3, 1)$. It is easy to verify that the only tuple of the form $(a, b, c, a)$ not adjacent to any of these tuples is $(1, 3, 2, 1)$, and hence $I$ must also contain this tuple. It is then a routine matter to verify that $I = I^1_1$.

Claim 2. For any $q = 1, 2, 3$, if $I(q)$ contains 5 or more tuples then $I = I^q_p$ for some $p \in \{1, \ldots, n\}$.

Proof of Claim 2. Suppose without loss of generality that $q = 1$. There are exactly 6 choices for the position of the two 1’s in a tuple, and two tuples for each choice; notice also that $I(1)$ cannot contain tuples whose positions for 1’s are disjoint sets. Hence, if $I(1)$ contains at least 6 tuples and it is not of the form $I^q_p$ then by Claim 1 and the preceding discussion, $I$ must, up to permutation of the coordinates, consist of the tuples $(1, 1, 2, 3), (1, 1, 3, 2), (1, 2, 1, 3), (1, 3, 1, 2), (2, 1, 1, 3)$ and $(3, 1, 1, 2)$. Any other tuple of $I$ must contain exactly one 1, and it is a simple matter to verify that any such tuple will be adjacent to one in the above list, a contradiction.

Now suppose that $I(1)$ contains exactly 5 tuples. As noted above, if $I$ is not of the desired form, then we have without loss of generality that $I(1)$ consists of the tuples $(1, 1, 2, 3), (1, 1, 3, 2), (1, 2, 1, 3), (1, 3, 1, 2)$ and $(2, 1, 1, 3)$; all other tuples of $I$ contain a single 1, and it is easy to see that any such tuple can only have its single 1 in position 1. This in turn forces the 4th coordinate to be 3 since $(2, 1, 1, 3)$ is in $I$. Hence there are 3 possibilities, whence $|I| \leq 8$, a contradiction.

We may now assume by Claim 2 that $|I(q)| = 4$ for all $q$. Suppose for a contradiction that $I$ is not of the desired form. Without loss of generality, there are only 3 cases to consider:

Case 1: $I(1)$ consists of the tuples $(1, 1, 2, 3), (1, 1, 3, 2), (1, 2, 1, 3), (1, 3, 1, 2)$. It is easy to see that any tuple of $I$ with a single 1 has it in the first coordinate; since there are only 6 of these, we get $|I| \leq 10$, a contradiction.

Case 2: $I(1)$ consists of the tuples $(1, 1, 2, 3), (1, 1, 3, 2), (1, 2, 1, 3), (1, 2, 3, 1)$. We know there is another tuple of the form $(a, b, c, a)$ in $I$ and by inspection we see it must be $(1, 3, 2, 1)$; thus we have $|I(q)| \geq 5$.

Case 3: $I(1)$ consists of the tuples $(1, 1, 2, 3), (1, 1, 3, 2), (1, 2, 1, 3), (u, 1, 1, v)$. Again one verifies easily that any tuple of $I$ not in this list will have a single 1 in position 1 or 2. Since there are exactly two vertices in $I$ of the form $(a, b, c, a)$ they must be $(3, 1, 2, 3)$ and $(2, 1, 3, 2)$; however this last tuple is adjacent to $(1, 2, 1, 3)$, a contradiction.

\[ \Box \]

Lemma 4.2. The sets $I^q_p$ are the only stable sets of maximal size in $G(2, 2, 2)$.

Proof. Let $I$ be a maximum stable set in $G = G(2, 2, 2)$; clearly $|I| = 30$. We’ll say that $I$ has the right form if there exist $p$ and $q$ such that $I = I^q_p$. For any
3-element subset $T \subseteq \{1, \ldots, 6\}$ define $G(T)$ to be the set of tuples in $G$ whose coordinates in $T$ are all distinct; for instance $G(\{1, 2, 3\})$ consists of all tuples of the form $(a, b, c, d, e, f)$ with $\{a, b, c\} = \{d, e, f\} = \{1, 2, 3\}$. Let $I(T) = I \cap G(T)$. For any $T$ there is an obvious isomorphism of $G_3 \times G_3$ with $G(T)$. Since $G_3 \times G_3$ is vertex-transitive, is 3-colourable and contains a triangle, it has stability ratio equal to $1/3$ by Lemma 2.3. Applying Lemma 2.3 to the embedding of $G(T)$ into $G$ we conclude that the cardinality of $I(T)$ is 12.

Claim 1. If there exists a $T$ such that $I(T)$ has the right form then $I$ has the right form.

Proof of Claim 1. Without loss of generality it suffices to show that if $I(\{1, 2, 3\})$ has the right form then so does $I(B)$ for any other 3-element set $B$ of indices. Suppose without loss of generality that all the tuples in $I(\{1, 2, 3\})$ have their first coordinate equal to 1. A routine computation shows that every tuple in $G(\{2, 3, 4\})$ that has its first coordinate equal to 2 or 3 has a neighbour in $I(\{1, 2, 3\})$. Since $I(\{2, 3, 4\})$ contains 12 elements we conclude it must be of the right form, i.e. it consists of all tuples in $G(\{2, 3, 4\})$ with 1 in the first coordinate. The very same argument applies to $I(\{2, 3, x\})$ for $x = 4, 5, 6$. Since $I(\{2, 3, 4\}) = I(\{1, 5, 6\})$, the same line of thought shows that $I(\{5, 6, y\})$ is also of the desired form, for $y = 2, 3, 4$. Repeating for $I(\{2, 3, 5\}) = I(\{1, 4, 6\})$ and $I(\{2, 3, 6\}) = I(\{1, 4, 5\})$ we obtain the desired result.

Claim 2. If $I$ contains a pair of tuples that share only one coordinate then $I$ has the right form.

Proof of Claim 2. Without loss of generality, by permuting entries and coordinates if necessary, we may assume that $I$ contains the tuples $(1, 1, 2, 3, 3, 3)$ and $(1, 2, 3, 3, 2, 1)$. We shall prove that $I(\{1, 2, 3\})$ has the right form and the claim will follow from Claim 1.

Let $A$ be the graph with $V(A) = \{0, 1, 2, 3, 4\}$ such that $\{0, 1, 2, 3, 4\}$ is a path and 1 and 3 are adjacent (see Figure 5.) Let $f$ be a homomorphism from $A$ to

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3Unfortunately $G_3 \times G_3$ has quite a few maximum stable sets not of the right form, a fact that complicates matters a bit.
if we apply Lemma 2.3 to its restriction to the triangle \{1,2,3\} we conclude that one of these vertices is mapped to \(I\). In particular, if \(f(0)\) and \(f(4)\) belong to \(I\) then \(f(2) \in I\). Set \(f(0) = (1,1,2,3,3)\) and \(f(4) = (1,2,3,3,2,1)\). A routine computation shows that this partial map may be extended to a homomorphism when \(f(2)\) is one of the following tuples (and hence all these tuples must belong to \(I\)):

\[
\begin{align*}
(1,3,2,1,2,3) \\
(1,3,1,2,2,3) \\
(1,2,3,2,1,3) \\
(1,2,3,1,3,2) \\
(1,2,2,3,1,3) \\
(1,2,1,3,3,2);
\end{align*}
\]

we may repeat the argument with the first and last of these tuples since they share only one coordinate: setting \(f(0) = (1,2,1,3,3,2)\) and \(f(4) = (1,3,2,1,2,3)\) we obtain (just permute the columns appropriately) that the following tuples are also in \(I\):

\[
\begin{align*}
(1,1,3,3,2,2) \\
(1,3,2,3,1,2) \\
(1,3,2,2,3,1).
\end{align*}
\]

Finally we may repeat with the second and fourth tuples of the first list to produce the following element of \(I\):

\[
(1,3,3,2,1,2).
\]

Of the 12 tuples we found to be in \(I\), 6 are in \(I(\{1,2,3\})\):

\[
\begin{align*}
(1,2,3,3,2,1) \\
(1,2,3,2,1,3) \\
(1,2,3,1,3,2) \\
(1,3,2,1,2,3) \\
(1,3,2,3,1,2) \\
(1,3,2,2,3,1).
\end{align*}
\]

Let \(X\) denote the set of 6 tuples in \(G(\{1,2,3\})\) with a 1 in the first coordinate that are not in the above list. We proceed to show that \(X \subseteq I\).

A routine calculation shows that the only tuples in \(G(\{1,2,3\})\) that are not adjacent to any of the 12 tuples found above are \(u = (2,3,1,1,2,3)\), \(v = (3,1,2,2,3,1)\), \(w = (3,2,1,1,3,2)\) and the tuples in \(X\); notice that \(u\) and \(v\) are adjacent. Suppose that \(u \in I\). Then \((1,2,3,3,1,2)\) and \((1,2,3,2,3,1)\) cannot be in \(I\), and since \(|I(\{1,2,3\})| = 12\), the remaining 4 tuples of \(X\) are in \(I\ and \ w \in I\). But \((1,3,2,2,1,3) \in \ I\) is adjacent to \(w\), a contradiction. A similar argument with \(v\) and \(w\) shows that none of them can be in \(I\) and the proof of Claim 2 is complete.

By Claims 1 and 2 it now remains to show that \(I\) must contain a pair of tuples that share only one coordinate. We’ll say that the tuples \((x_1, \ldots, x_6)\) and \((y_1, \ldots, y_6)\) are friendly if there exist distinct indices \(i\) and \(j\) such that \(x_i = x_j = y_i = y_j\) and \(x_k \neq y_k\) for all \(k \neq i, j\).

**Claim 3.** The set \(I\) contains either a pair of tuples that share only one coordinate or a friendly pair.
Proof of Claim 3. Define a graph $G'$ with vertices $V(G') = V(G)$ where $x$ and $y$ are adjacent if either (i) $x$ and $y$ are adjacent in $G$ or (ii) $x$ and $y$ share a single coordinate or (iii) $x$ and $y$ are friendly. It is clear that $G'$ is vertex-transitive under the actions of $S_3$ and $S_6$ described in section 1. Furthermore, one verifies readily that the tuples of the form $(a,b,c,a,b,c)$ with $\{a,b,c\} = \{1,2,3\}$ form a clique of size 6 in $G'$; by Lemma 2.3 this implies that $\alpha(G')/|V(G')| \leq 1/6$ and thus a maximum independent set in $G'$ has size at most 15; since $|I| = 30$ we conclude it contains an edge of $G'$ and this proves the claim.

We can now complete the proof of the lemma. If $I$ contains a pair of tuples with a single coincidence then we’re done by Claim 2, so by Claim 3 we may assume without loss of generality that $I$ contains the friendly pair $u = (1,1,2,2,3,3)$ and $v = (1,1,3,3,2,2)$. Consider the set $J$ obtained from $I$ by removing $u$ and adding $w = (2,1,1,2,3,3)$, obtained from $u$ by exchanging coordinates 1 and 3. $J$ contains $v$ and $w$ which have a single coincidence; if $w$ was already in $I$ then we’re done; otherwise, if $J$ is stable it is a maximum stable set and it must be of the right form, in this case $J = I_2^1$, and it follows that $I = I_2^1$. The last possibility is that there exists a tuple in $I$ which is adjacent to $w$, which we may assume coincides with $u$ in two places. Hence one of the 2 tuples of the form $(1,3,2,3,x,y)$ with $\{x,y\} = \{1,2\}$ is in $I$. Repeating the same argument this time replacing $v$ by $w' = (3,1,1,3,2,2)$ we obtain that $I$ contains one of the 2 tuples of the form $(1,2,3,2,x,y)$ with $\{x,y\} = \{1,3\}$. Of the 4 possibilities, 2 will yield tuples with a single coincidence, and the other 2 are symmetric by a permutation of the last 2 columns; hence we may assume without loss of generality that $I$ contains $a = (1,3,2,3,1,2)$ and $b = (1,2,3,2,1,3)$.

Now we repeat the above process by exchanging coordinates 2 and 4: construct $J$ from $I$ by removing $u$ and adding $w = (1,2,2,1,3,3)$. Since $v$ and $w$ have a single coincidence we obtain that $I$ contains one of the 2 tuples $(3,1,3,2,x,y)$ with $\{x,y\} = \{1,2\}$. However the presence of $a$ in $I$ means $I$ contains $c = (3,1,3,2,1,2)$. Similarly, by using $v$ instead of $u$, we obtain that $I$ contains $d = (2,1,3,2,1,3)$.

Finally we repeat the same procedure, this time obtaining $J$ from $I$ by replacing $d$ by $(2,2,1,3,1,3)$ (exchanging coordinates 2 and 3). Since this tuple and $c$ share a single coordinate we conclude as before that $I$ contains one of the 2 tuples of the form $(3,1,2,x,3,y)$ with $\{x,y\} = \{1,2\}$. The presence of $b$ in $I$ forces the tuple to be $(3,1,2,2,3,1)$; and this tuple coincides with $b$ only in the fourth coordinate, which (finally) ends our proof. □

Lemma 4.3. The sets $I_p^3$ are the only stable sets of maximal size in $G(2,2,1)$.

Proof. Consider the homomorphism $f : G(2,2,2) \rightarrow G(2,2,1)$ defined by

$$f(x_1,\ldots,x_6) = (x_1,\ldots,x_5).$$

Let $I$ be a maximum stable set of $G(2,2,1)$. By Lemma 2.3 and the last result $f^{-1}(I) = I_p^3$. We cannot have $p = 6$ for otherwise $I$ would contain the adjacent tuples $(a,a,b,b,q)$ and $(b,q,a,a,b)$ (where $\{a,b,q\} = \{1,2,3\}$). Thus $I = I_p^3$ and we’re done. □

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