

Harish-Chandra integrals as nilpotent integrals

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Abstract

Recently the correlation functions of the so-called Itzykson-Zuber/Harish-Chandra integrals were computed (by one of the authors and collaborators) for all classical groups using an integration formula that relates integrals over compact groups with respect to the Haar measure and Gaussian integrals over a maximal nilpotent Lie subalgebra of their complexification. Since the integration formula *a posteriori* had the same form for the classical series, a conjecture was formulated that such a formula should hold for arbitrary semisimple Lie groups. We prove this conjecture using an abstract Lie-theoretic approach.

1 Introduction and setting

In random matrix theory [1, 2] a particularly important role is played by the so-called Itzykson-Zuber/Harish-Chandra measure. Its integral is [3, 4]

$$\int_{U(N)} dU e^{\text{tr}(XUYU^\dagger)} = C_N \frac{\det(e^{x_i y_j})_{i,j}}{\Delta(X)\Delta(Y)} \quad (1.1)$$

where dU is the Haar measure on $U(N)$, X, Y are diagonal matrices and $\Delta(X) = \prod_{i < j} (x_i - x_j)$, with $X = \text{diag}(x_1, \dots, x_N)$ and C_N is some proportionality constant.

In a recent paper [5] integrals of invariant functions on such type measures were computed for all the classical series (A_n, B_n, C_n, D_n) corresponding to unitary, orthogonal and symplectic ensembles, generalizing an earlier result [6]. An outstanding conjecture was formulated in [5] relating certain spherical integrals

Conjecture 1 (Conjecture 1.1 in [5]) *Let \mathfrak{g} a semisimple Lie algebra over \mathbb{C} , \mathfrak{h} a Cartan subalgebra, K the maximal compact group in $\exp(\mathfrak{g})$, \langle, \rangle the Killing form and \mathfrak{W} the Weyl group. Let $F(X, Y)$ be a polynomial on $\mathfrak{g} \times \mathfrak{g}$ invariant under diagonal adjoint action $F(\text{Ad}_g(X), \text{Ad}_g(Y)) = F(X, Y)$. Then the following identity holds $\forall H, J \in \mathfrak{h}$:*

$$\int_K dk F(H, \text{Ad}_k J) e^{-\langle H, \text{Ad}_k J \rangle} = \frac{C}{|\mathfrak{W}|} \sum_{w \in \mathfrak{W}} \epsilon_w \frac{e^{-\langle H, J_w \rangle}}{\prod_{\alpha > 0} \alpha(H)\alpha(J)} \int_{\mathfrak{n}_+} dN F(H + N, J_w + N^\dagger) e^{-\langle N, N^\dagger \rangle}, \quad (1.2)$$

where J_w stands for the action of the Weyl group on \mathfrak{h} , ϵ_w is the usual sign homomorphism and C is a suitable constant depending only on the Lie algebra under consideration. (All the symbols will be defined more in detail later)

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Such conjecture was verified a posteriori for all the classical series but the authors of [5] failed to provide a general proof that would apply also to exceptional Lie algebras. This identity was the main initial step towards an effective computation of all correlation functions for spherical integrals over the compact forms of the classical groups, $SU(N)$, $SO(2n, \mathbb{R})$, $SO(2n+1, \mathbb{R})$, $Sp(n, \mathbb{R})$. In this short note we provide a Lie-algebra-theoretical proof of Conjecture 1 (Thm. 5.1) that does not rely on any specificity of the Lie algebra as long as it is semisimple and provides a precise value for the proportionality constant C .

We will need to prove a slight generalization of Weyl-integration formula (which may well be known in the literature but we could not find in any of the standard references). The proof of Conjecture 1 is contained in Thm. 5.1.

We will liberally use known facts about (semi)-simple Lie algebras, all of which can be found in standard reference books like [7]. Let \mathfrak{g} be a complex semisimple Lie algebra over \mathbb{C} , G the corresponding simply connected group, $G = \exp(\mathfrak{g})$. Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}$ be a Cartan subalgebra (over \mathbb{C} unless otherwise specified) and $\mathfrak{R} \subset \mathfrak{h}_{\mathbb{C}}^{\vee}$ be the set of roots. Let an ordering of the roots be chosen: it fixes the set of positive roots \mathfrak{R}_+ . The set of simple positive roots with respect to this ordering will be denoted by Φ . We will use the Chevalley basis $\{E_{\alpha}, H_{\alpha}\}_{\alpha \in \mathfrak{R}}$ (called *root vectors* and *coroots* respectively) of \mathfrak{g} where E_{α} spans \mathfrak{g}_{α} and the set H_{α} for $\alpha \in \Phi$ spans \mathfrak{h} . Such a basis has the properties $[E_{\alpha}, E_{-\alpha}] = H_{\alpha}$, $[H_{\alpha}, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$, $\alpha \in \mathfrak{R}$. Here and in the following we will use the notations

$$\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Phi} \mathbb{R}\{H_{\alpha}\}, \quad \mathfrak{h} := \mathfrak{h}_{\mathbb{C}} := \sum_{\alpha \in \Phi} \mathbb{C}\{H_{\alpha}\}, \quad \mathfrak{n}_+ = \sum_{\alpha > 0} \mathbb{C}\{E_{\alpha}\}, \quad \mathfrak{b}_+ = \mathfrak{h}_{\mathbb{C}} + \mathfrak{n}_+. \quad (1.3)$$

The compact form $\mathfrak{k} \subset \mathfrak{g}$ will be chosen as the span of

$$\mathfrak{k} = i\mathfrak{h}_{\mathbb{R}} + \sum_{\alpha > 0} \mathbb{R}\{X_{\alpha}, Y_{\alpha}\}, \quad X_{\alpha} := (E_{\alpha} - E_{-\alpha}), \quad Y_{\alpha} := i(E_{\alpha} + E_{-\alpha}) \quad (1.4)$$

On each of the above Lie algebras we will use the Lebesgue measure such that the unit cube in the coordinates given by the specified basis has unit volume.

We finally recall that any semisimple Lie algebra admits a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}_+$ where \mathfrak{k} is a compact Lie algebra, $\mathfrak{a} \subset \mathfrak{h}_{\mathbb{R}}$ is Abelian. This decomposition subtends the *Iwasawa decomposition* of the Lie group $G = KAN$.

2 Schur decompositions

The first fact we need is a generalization of the Schur decomposition to arbitrary Lie algebras: Schur decomposition is a widely known decomposition of matrices and states that any complex square matrix M can be written in the form UTU^{\dagger} with $U \in U(N)$ and T an upper semi-triangular matrix.

Theorem 2.1 *Any ad-regular element $M \in \mathfrak{g}$ is K -conjugate to an element in $\mathfrak{b}_+ = \mathfrak{h}_{\mathbb{C}} + \mathfrak{n}_+$.*

Proof. It is known [7] that any ad-regular element in \mathfrak{g} is conjugate to an element in \mathfrak{h} , namely $M = Ad_g H \in \mathfrak{h}_{\mathbb{C}}$. There are (generically) $|\mathfrak{W}|$ such ways of representing M . Using the Iwasawa decomposition $g = kan$ we have immediately

$$M = Ad_k(H + (Ad_{an}H - H)) = Ad_k B \quad (2.1)$$

with $B = Ad_{an}H \in \mathfrak{b}_+$ and $Ad_{an}H - H \in \mathfrak{n}_+$. **Q.E.D.**

Since we will be concerned with integration formulæ, the above theorem suffices since the set of ad-regular elements is a Zariski open set, dense in \mathfrak{g} : in particular nonregular elements are a set of Lebesgue measure zero. However, the following more general theorem can also be proved (but we will not prove it here in the interest of conciseness and also because it is completely irrelevant to our main purpose).

Theorem 2.2 *Any element $M \in \mathfrak{g}$ is conjugated by an element of the maximal compact subgroup K to an element $H + N$ with $H \in \mathfrak{h}$, $N \in \mathfrak{n}_+$*

3 Complex Weyl integration formula

The goal of this section is to write an integral formula for functions on \mathfrak{g} in terms of integrals on \mathfrak{b} and K .

Define

$$\mathcal{M} := (K \times \mathfrak{b})/T \tag{3.1}$$

where the action of the Cartan torus is $t \cdot (k, V) := (kt^{-1}, tVt^{-1})$, $V \in \mathfrak{b}$ $k \in K$. We will write $V = H + N$ with $iH \in \mathfrak{h}_{\mathbb{C}}$ and $N \in \mathfrak{n}_+$. The action of T is then $t \cdot (k, H, N) = (kt^{-1}, H, tNt^{-1})$ so that we can also think of \mathcal{M} as $\mathcal{M} = \mathfrak{h} \times (K \times \mathfrak{n}_+)/T$. The tangent space to \mathcal{M} at $[(k, V)]$ is identified with $\mathfrak{k}/i\mathfrak{h}_{\mathbb{R}} + \mathfrak{b}$ by $[(k_s, V_s)] := [(ke^{sX}, V + sW)]$.

Consider the map

$$\begin{aligned} \pi : \mathcal{M} &\longrightarrow \mathfrak{g} \\ [(k, V)] &\mapsto kV k^{-1} \end{aligned} \tag{3.2}$$

The topological degree of π is the cardinality of the Weyl group \mathfrak{W} : to see this it is sufficient to note that $Ad_K \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}$ has the advocated degree and then use a continuity argument.

The differential of the map $\pi : \mathcal{M} \rightarrow \mathfrak{g}$ at a point $[(k, V)]$ is then computed as

$$\frac{d}{ds} Ad_{ke^{sX}}(V + sW) \Big|_{s=0} = k([X, V] + W)k^{-1}, \quad X \in \mathfrak{k}/i\mathfrak{h}_{\mathbb{R}}, \quad W \in \mathfrak{b}. \tag{3.3}$$

In order to write a matrix representation of the above map we write it in the natural basis of $T_M \mathfrak{g} = \mathfrak{n}_- + \mathfrak{b}_+$

$$\begin{aligned} d\pi : T_{[(k, V)]} \mathcal{M} \sim \mathfrak{k}/i\mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{b}_+ &\longrightarrow \mathfrak{n}_- + \mathfrak{b}_+ \\ (X, W) &\mapsto d\pi(X, W) = Ad_k([X, V] + W) \end{aligned} \tag{3.4}$$

We compute the determinant of the above map without the Ad_k term (which does not change its value) and we think of \mathfrak{n}_- as a vector space over \mathbb{R} with a real basis provided by $\mathfrak{n}_- := \sum_{\alpha < 0} \mathbb{R}\{E_{\alpha}\} + i \sum_{\alpha < 0} \mathbb{R}\{E_{\alpha}\}$

$$ad_{H+N}(X_{\alpha}) = \Re(\alpha(H))E_{-\alpha} + i\Im(\alpha(H))E_{-\alpha} + \sum_{-\beta > -\alpha} \mathbb{C}\{E_{-\beta}\} \bmod \mathfrak{b}_+ \tag{3.5}$$

$$ad_{H+N}(Y_{\alpha}) = -\Im(\alpha(H))E_{-\alpha} + i\Re(\alpha(H))E_{-\alpha} + \sum_{-\beta > -\alpha} \mathbb{C}\{E_{-\beta}\} \bmod \mathfrak{b}_+ \tag{3.6}$$

$$V := H + N$$

It appears that the matrix has a block-uppertriangular shape and these ‘‘upper triangular’’ parts do not contribute to the determinant. The latter becomes then the product of the determinants of the above 2×2 blocks which are simply $|\alpha(H)|^2$.

The Jacobian of $d\tau$ at the point $[k, V] \in \mathcal{M}$ ($V = H + N$) is thus

$$J(k, V) = \prod_{\alpha \in \mathfrak{R}_+} |\alpha(H)|^2 =: |\Delta(H)|^2. \quad (3.7)$$

The notation $\Delta(H) := \prod_{\alpha \in \mathfrak{R}_+} \alpha(H)$ is used in analogy with the case of $\mathfrak{g} = sl(n, \mathbb{C})$ where it reduces to the Vandermonde determinant. A well known property is that

$$\Delta(H_w) = (-)^w \Delta(H) \quad (3.8)$$

where $w \in \mathfrak{W}$ and H_w stands for the action of the Weyl group on $\mathfrak{h}_{\mathbb{C}}$ and the notation $(-)^w$ means the parity of the Weyl-transformation (i.e. the parity of the number of elementary reflections along walls of Weyl chambers in which w can be decomposed). Collecting these pieces of information we have proved the following

Theorem 3.1 (Complex-Weyl integration formula) *Let $F : \mathfrak{g} \rightarrow \mathbb{C}$ be a smooth integrable function invariant under the adjoint action of K . Then*

$$\int_{\mathfrak{g}} dM F(M) = c_{\mathfrak{k}} \int_{\mathfrak{h}_{\mathbb{C}} \times \mathfrak{n}_+} dH dN F(H + N) |\Delta(H)|^2, \quad c_{\mathfrak{k}} := \frac{\mu(K)/\mu(T)}{|\mathfrak{W}|}$$

where dM , dH , dN are the Lebesgue measures on \mathfrak{g} , $\mathfrak{h}_{\mathbb{C}}$, \mathfrak{n}_+ respectively defined above, $\mu(K)$ and $\mu(T)$ are the induced measures on the compact group K and the maximal torus T , \mathfrak{W} is the Weyl group and $|\mathfrak{W}|$ is its cardinality.

There is one more piece of information that we can extract from the above and is contained in the following

Corollary 3.1 *For any Ad_K -invariant smooth integrable function $F : \mathfrak{g} \rightarrow \mathbb{C}$ the function*

$$\widehat{F}(H) := \int_{\mathfrak{n}_+} dN F(H + N) : \mathfrak{h}_{\mathbb{C}} \rightarrow \mathbb{C} \quad (3.9)$$

is Weyl-invariant.

Proof. By the generalized Schur decomposition (Thm. 2.1) a regular element M can be represented modulo the Ad_K action as $H + N \in \mathfrak{h} + \mathfrak{n}_+$ or $H_w + \tilde{N}$ where H_w is in the same \mathfrak{W} -orbit through H and $\tilde{N} \in \mathfrak{n}_+$ is some other element in the *same* nilpotent subalgebra \mathfrak{n}_+ . In general the dependence of \tilde{N} on N, H is a complicated expression. Consider a small ball $M \in \mathcal{U} \subset \mathfrak{g}$ consisting of regular elements. This ball can be mapped *diffeomorphically* to some neighborhood $\mathcal{H} \times \mathcal{L}$ in $\mathfrak{h}_{\mathbb{C}} \times (K \times \mathfrak{n}_+)/T$ with \mathcal{H} lying in a suitable Weyl chamber and containing H . Choosing another Weyl chamber \mathcal{H}_w we have a distinct diffeomorphism between \mathcal{U} and $\mathcal{H}_w \times \tilde{\mathcal{L}}$. Since the Jacobian computed above is always $|\Delta(H)|^2 = |\Delta(H_w)|^2$ and *independent* of N , we conclude that the transformation $N \mapsto \tilde{N}$ preserves the Lebesgue measure of \mathfrak{n}_+ , namely $d\tilde{N}/dN = 1$. Since regular elements are open and dense (and the complement has zero measure) we can then write

$$\widehat{F}(H_w) = \int_{\mathfrak{n}_+} d\tilde{N} F(H_w + \tilde{N}) = \int_{\mathfrak{n}_+} d\tilde{N} F(Ad_k(H_w + \tilde{N})) = \int_{\mathfrak{n}_+} F(H + N) dN = \widehat{F}(H). \quad (3.10)$$

Q.E.D.

The integration formula of Thm. 3.1 should be considered a mild generalization of the standard Weyl integration formula which we state here for functions on the Lie algebra of a compact Lie group K .

Theorem 3.2 (Weyl integration formula) *Let $F : \mathfrak{k} \rightarrow \mathbb{C}$ be a smooth function integrable with respect to the Lebesgue measure. Then*

$$\int_{\mathfrak{k}} dX F(X) = c_{\mathfrak{k}} \int_{i\mathfrak{h}_{\mathbb{R}}} dH |\Delta(H)|^2 \int_K dk F(Ad_k(H)) \quad (3.11)$$

In case of an Ad_K -invariant function the above reduces to

$$\int_{\mathfrak{k}} dX F(X) = c_{\mathfrak{k}} \int_{i\mathfrak{h}_{\mathbb{R}}} dH |\Delta(H)|^2 F(H) \quad (3.12)$$

where we need to put the absolute-value sign because $\Delta(H)^2$ may be negative on $i\mathfrak{h}_{\mathbb{R}}$ if $\dim n_+$ is odd ($c_{\mathfrak{k}}$ has the same meaning and value as in Thm. 3.1).

Remark 3.1 *The explicit value of $\mu(K)/\mu(T)$ was computed by Macdonald [8] for a slightly different choice of normalization for the Lebesgue measure. The actual value of this constant is irrelevant for our purpose and does not directly enter the computation of the proportionality constant in the conjecture.*

4 Gaussian integrals

Let $V_{\mathbb{R}}$ be a real vector space and $\langle \cdot, \cdot \rangle : V_{\mathbb{R}} \times V_{\mathbb{R}} \rightarrow \mathbb{R}$ a positive definite bilinear pairing (an inner product). Let $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes \mathbb{C}$ be its complexification. Denote by dx the Lebesgue measure on $V_{\mathbb{R}}$ and dz the Lebesgue measure on $V_{\mathbb{C}}$ (the normalizations of which are irrelevant at this point). The inner product $\langle \cdot, \cdot \rangle$ extends to an inner-product (linearly over \mathbb{C}) on $V_{\mathbb{C}}$. Moreover the real form $V_{\mathbb{R}} \subset V_{\mathbb{C}}$ defines also a natural conjugation $z \rightarrow \bar{z}$ which fixes $V_{\mathbb{R}}$.

Lemma 4.1 *With the notations and definitions above, for any polynomial function F on $V_{\mathbb{C}} \times V_{\mathbb{C}}$ define*

$$\begin{aligned} \langle F \rangle_{\mathbb{R}} &:= \frac{1}{Z_{\mathbb{R}}} \int_{V_{\mathbb{R}}} \int_{V_{\mathbb{R}}} dx dy e^{-a\langle x,x \rangle - c\langle y,y \rangle - 2b\langle x,y \rangle} F(x, y) \\ \langle F \rangle_{\mathbb{C}} &:= \frac{1}{Z_{\mathbb{C}}} \int_{V_{\mathbb{C}}} dz e^{-a\langle z,z \rangle - c\langle \bar{z},\bar{z} \rangle - 2b\langle z,\bar{z} \rangle} F(z, \bar{z}) \end{aligned} \quad (4.1)$$

where $Z_{\mathbb{R}}$ and $Z_{\mathbb{C}}$ are determined⁴ by the requirement that $\langle 1 \rangle = 1$. While the convergence of the integrals in the two cases imposes different conditions on the numbers a, b, c , nevertheless (i) both $\langle F \rangle_{\mathbb{R}, \mathbb{C}}$ are polynomials in $a/\delta, b/\delta, c/\delta$, $\delta := ac - b^2$ and (ii) as polynomials they coincide.

Proof. The key is in showing that the generating functions for the moments of the two integrals in either cases are identical, namely that for $A, B \in V_{\mathbb{C}}$

$$G_{\mathbb{R}}(A, B) := \left\langle e^{\langle x,A \rangle + \langle y,B \rangle} \right\rangle_{\mathbb{R}} = \exp \left[\frac{1}{4} \left(\frac{c}{\delta} \langle A, A \rangle + \frac{a}{\delta} \langle B, B \rangle - 2 \frac{b}{\delta} \langle A, B \rangle \right) \right] = G_{\mathbb{C}}(A, B) := \left\langle e^{\langle z,A \rangle + \langle \bar{z},B \rangle} \right\rangle_{\mathbb{C}} \quad (4.2)$$

for then $\langle F \rangle_{\mathbb{R}} = F(\partial_A, \partial_B) G_{\mathbb{R}}(A, B)|_{A=0=B} = \langle F \rangle_{\mathbb{C}}$, which proves both points of the lemma at the same time. In order to show (4.2) we use an orthonormal coordinate basis for $\langle \cdot, \cdot \rangle$ so that –writing $A = (\alpha_1, \dots, \alpha_n)$ and $B = (\beta_1, \dots, \beta_n)$ in this basis– the integral $G_{\mathbb{R}}$ factorizes as $G_{\mathbb{R}} = \prod G_1(\alpha_j, \beta_j)$ with

$$G_1(\alpha, \beta) := \frac{1}{Z_{1,\mathbb{R}}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \exp \left[-(x, y) M \begin{pmatrix} x \\ y \end{pmatrix} + x\alpha + y\beta \right], \quad M := \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (4.3)$$

⁴It is an easy exercise that we leave to the interested reader to verify that $Z_{\mathbb{R}} = (2\pi)^n \delta^{\frac{n}{2}}$, $Z_{\mathbb{C}} = (2\pi)^n (-\delta)^{\frac{n}{2}}$, $n = \dim_{\mathbb{R}} V_{\mathbb{R}} = \dim_{\mathbb{C}} V_{\mathbb{C}}$, $\delta = ac - b^2$. These precise expressions are nevertheless irrelevant for our purposes.

Define $(x', y') = (x, y) - \frac{1}{2}(\alpha, \beta)M^{-1}$, where now the contours of integration may be some lines parallel to the real axis in the complex x' and y' planes. However the ensuing integrals can be deformed (by Cauchy theorem) back to the real axis and the integral yields $G_1(\alpha, \beta) = g(\alpha, \beta) := \exp \frac{1}{4} \left[\frac{c}{\delta} \alpha^2 + \frac{a}{\delta} \beta^2 - 2\frac{b}{\delta} \alpha \beta \right]$.

For the second case we have $G_{\mathbb{C}} = \prod \tilde{G}_1(\alpha_j, \beta_j)$ with $\tilde{G}_1(\alpha, \beta)$ given below: we need to express the integration in the real/imaginary part of $z = x + iy$

$$\tilde{G}_1(\alpha, \beta) := \frac{1}{Z_{1,\mathbb{C}}} \int_{\mathbb{C}} d^2z \exp \left[-(z, \bar{z}) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + z\alpha + \bar{z}\beta \right] = \quad (4.4)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} dx dy \exp \left[-(x, y) \begin{pmatrix} a+c+2b & i(a-c) \\ i(a-c) & 2b-a-c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + x(\alpha + \beta) + iy(\alpha - \beta) \right] \quad (4.5)$$

In this case we perform the shift $(x', y') = (x, y) - \frac{1}{2}(\alpha + \beta, i(\alpha - \beta)) \begin{pmatrix} a+c+2b & i(a-c) \\ i(a-c) & 2b-a-c \end{pmatrix}^{-1}$, followed by deforming back the integration contours on the real x' and y' axes. Straightforward linear algebra gives the same result $g(\alpha, \beta)$ as above. **Q.E.D.**

5 Proof of Conjecture 1

Let $\vartheta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the **Cartan involution** (antilinear)

$$\vartheta(cE_{\alpha}) = -\bar{c}E_{-\alpha}, \quad \vartheta(cH_{\alpha}) = -\bar{c}H_{\alpha}. \quad (5.1)$$

Definition 5.1 For a (semi)simple Lie algebra \mathfrak{g} over \mathbb{C} , given the Cartan involution ϑ defined above, we will denote by $M^{\dagger} = -\vartheta(M)$, and by $M^{\vartheta} = \vartheta(M)$.

Remark 5.1 The notation M^{\dagger} has been defined to coincide with the usual hermitian conjugate in the standard fundamental representation of $sl(n, \mathbb{C})$.

The Cartan involution ϑ (or \dagger) defines a real form of \mathfrak{g} which is precisely \mathfrak{k} , the *compact real form* as the 1-eigenspace of ϑ . Two properties are immediate ($\langle \cdot, \cdot \rangle$ is the Killing form):

- $\langle M, M^{\vartheta} \rangle \leq 0$ is a **negative** definite **sesquilinear** quadratic form for $M \in \mathfrak{g}$;
- $\langle X, X^{\dagger} \rangle = -\langle X, X^{\vartheta} \rangle = -\langle X, X \rangle \geq 0$ is a *positive* definite quadratic form for $X \in \mathfrak{k}$ (as a **real** vector space).

Consider the following quadratic form on $\mathfrak{g} \times \mathfrak{g}$

$$Q_A(X, Y) := a \langle X, X \rangle + 2b \langle X, Y \rangle + c \langle Y, Y \rangle, \quad A := \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.2)$$

We leave to the reader to verify the following easy

Lemma 5.1 There exist two open domains $\mathbb{D}_{\mathfrak{k}}$ and $\mathbb{D}_{\mathfrak{g}}$ for the parameters a, b, c such that

- if $(a, b, c) \in \mathbb{D}_{\mathfrak{k}}$ then $\Re Q_A(X, Y)$ is positive definite on $\mathfrak{k} \times \mathfrak{k}$;
- if $(a, b, c) \in \mathbb{D}_{\mathfrak{g}}$ then $\Re Q_A(M, M^{\vartheta})$ is positive definite on \mathfrak{g} .

The specific form of these domains is largely irrelevant for our considerations and in the interest of conciseness we will not specify them further.

Definition 5.2 *Define*

$$Z_{\mathfrak{k}} := \int_{\mathfrak{k}} \int_{\mathfrak{k}} dXdY e^{-Q_A(X,Y)} , \quad (a, b, c) \in \mathbb{D}_{\mathfrak{k}} , \quad Z_{\mathfrak{g}} := \int_{\mathfrak{g}} dM e^{-Q_A(M, M^{\vartheta})} , \quad (a, b, c) \in \mathbb{D}_{\mathfrak{g}} . \quad (5.3)$$

Then, for any polynomial function F on $\mathfrak{g} \times \mathfrak{g}$ we define

$$\langle F \rangle_{\mathfrak{k}} := \frac{1}{Z_{\mathfrak{k}}} \int_{\mathfrak{k}} \int_{\mathfrak{k}} dXdY F(X, Y) e^{-Q_A(X, Y)} , \quad \langle F \rangle_{\mathfrak{g}} := \frac{1}{Z_{\mathfrak{g}}} \int_{\mathfrak{g}} dM F(M, M^{\vartheta}) e^{-Q_A(M, M^{\vartheta})} \quad (5.4)$$

As an application of Lemma 4.1 with $V_{\mathbb{R}} = \mathfrak{k}$, $V_{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}$ and ϑ as the involution leaving \mathfrak{k} invariant, we have

Proposition 5.1 *For any polynomial function F on $\mathfrak{g} \times \mathfrak{g}$ both $\langle F \rangle_{\mathfrak{k}}$ and $\langle F \rangle_{\mathfrak{g}}$ are **polynomials** in $a/\delta, b/\delta, c/\delta$ with $\delta = ac - b^2$. As polynomials they **coincide***

The next theorem contains the proof of Conjecture 1 with precise values of the proportionality constants.

Theorem 5.1 *Let F be an Ad_K invariant (polynomial) function on $\mathfrak{g} \times \mathfrak{g}$, where the action of Ad_K is the diagonal action*

$$F(X, Y) = F(Ad_k(X), Ad_k(Y)) , \quad \forall X, Y \in \mathfrak{g} \quad (5.5)$$

Then, for any $H, J \in \mathfrak{h}_{\mathbb{C}}$

$$\int_K dk F(H, Ad_k(J)) e^{\gamma \langle H, Ad_k(J) \rangle} = \frac{C_{\mathfrak{g}}}{|\mathfrak{W}|} \sum_{w \in \mathfrak{W}} \frac{e^{\gamma \langle H, J_w \rangle}}{\Delta(H) \Delta(J_w)} \frac{\int_{\mathfrak{n}_+} dN F(H + N, J_w + N^{\vartheta}) e^{\gamma \langle N, N^{\vartheta} \rangle}}{\int_{\mathfrak{n}_+} dN e^{\gamma \langle N, N^{\vartheta} \rangle}} \quad (5.6)$$

The normalization constant $C_{\mathfrak{g}}$ is given by

$$C_{\mathfrak{g}} = |\mathfrak{W}| \prod_{j=1}^{\dim \mathfrak{h}} m_j! \prod_{\alpha > 0} \frac{\langle \alpha, \alpha \rangle}{2\gamma} . \quad (5.7)$$

where m_j are the exponents of the Weyl group. Moreover we have

$$\int_{\mathfrak{n}_+} dN e^{\gamma \langle N, N^{\vartheta} \rangle} = \prod_{\alpha > 0} \frac{\pi \langle \alpha, \alpha \rangle}{2\gamma} \quad (5.8)$$

Remark 5.2 *By simplifying the values of the constants (note that $\dim \mathfrak{n}_+$ is the number of positive roots) we obtain*

$$\int_K dk F(H, Ad_k(J)) e^{\gamma \langle H, Ad_k(J) \rangle} = \frac{\prod_{j=1}^{\dim \mathfrak{h}} m_j!}{\pi^{\dim \mathfrak{n}_+}} \sum_{w \in \mathfrak{W}} \frac{e^{\gamma \langle H, J_w \rangle}}{\Delta(H) \Delta(J_w)} \int_{\mathfrak{n}_+} dN F(H + N, J_w + N^{\vartheta}) e^{\gamma \langle N, N^{\vartheta} \rangle} \quad (5.9)$$

Remark 5.3 *Note that the convergence of the Gaussian integral in the formula demands $\Re(\gamma) > 0$, but the identity is one between analytic functions of γ .*

Proof. We start from the proof of (5.8): since $\langle E_\alpha, E_\beta \rangle = \frac{2}{\langle \alpha, \alpha \rangle} \delta_{\alpha, -\beta}$ ⁵, writing $N = \sum_{\alpha > 0} n_\alpha E_\alpha$ the integral is recast into the form

$$\int_{\mathfrak{n}_+} dN e^{\gamma \langle N, N^\vartheta \rangle} = \prod_{\alpha > 0} \int_{\mathbb{C}} d^2 n_\alpha e^{\frac{-2\gamma}{\langle \alpha, \alpha \rangle} |n_\alpha|^2} = \prod_{\alpha > 0} \frac{\pi \langle \alpha, \alpha \rangle}{2\gamma} \quad (\Re(\gamma) > 0). \quad (5.10)$$

The value of $C_{\mathfrak{g}}$ is computed by evaluating explicitly the integrals on both sides for $F \equiv 1$, which reduces the formula to the famous Harish–Chandra expression

$$\int_K dk e^{\gamma \langle H, Ad_k(J) \rangle} = \frac{C_{\mathfrak{g}}}{|\mathfrak{W}|} \sum_{w \in \mathfrak{W}} \frac{e^{\gamma \langle H, J_w \rangle}}{\Delta(H) \Delta(J_w)} \quad (5.11)$$

In this case the equality was established in (Thm. 2, pag 104 [3])⁶ where the value of the constant $C_{\mathfrak{g}}$ was given by $C_{\mathfrak{g}} = (\gamma)^{-\dim \mathfrak{n}_+} (\Delta, \Delta)$. The bracket $(p(H), q(H))$ was defined ibidem for any polynomials p, q over \mathfrak{h} by writing them in a *orthonormal* basis $\langle \omega_j, \omega_k \rangle = \delta_{jk}$ ⁷

$$p(H) := \sum_{\vec{n}} a_{\vec{n}} \prod_{\ell=1}^{\dim(\mathfrak{h})} \omega_\ell^{n_\ell}(H), \quad q(H) := \sum_{\vec{n}} b_{\vec{n}} \prod_{\ell=1}^{\dim(\mathfrak{h})} \omega_\ell^{n_\ell}(H) \quad (5.12)$$

$$(p, q) := \sum_{\vec{n}} a_{\vec{n}} b_{\vec{n}} \prod_{\ell=1}^{\dim(\mathfrak{h})} m_\ell! \quad (5.13)$$

The number (Δ, Δ) ($\Delta = \prod_{\alpha > 0} \alpha$) has been computed in [8]⁸ and is given by

$$(\Delta, \Delta) = 2^{\frac{\dim \mathfrak{h} - \dim \mathfrak{g}}{2}} |\mathfrak{W}| \prod_{j=1}^n m_j! \prod_{\alpha > 0} \langle \alpha, \alpha \rangle \quad (5.14)$$

which proves the expression for the constants noticing that $\frac{\dim \mathfrak{g} - \dim \mathfrak{h}}{2} = \dim \mathfrak{n}_+$.

We now turn to the proof of the equality: consider first the integral over $\mathfrak{k} \times \mathfrak{k}$

$$\begin{aligned} \frac{1}{\mathcal{Z}_{\mathfrak{k}}} \int_{\mathfrak{k}} \int_{\mathfrak{k}} dX dY F(X, Y) e^{-a \langle X, X \rangle - c \langle Y, Y \rangle - 2b \langle X, Y \rangle} = \\ = \frac{c_{\mathfrak{k}}^2}{\mathcal{Z}_{\mathfrak{k}}} \int_{i\mathfrak{h}_{\mathbb{R}}} \int_{i\mathfrak{h}_{\mathbb{R}}} dH dJ \Delta(H)^2 \Delta(J)^2 e^{-a \langle H, H \rangle - c \langle J, J \rangle} \underbrace{\int_K dk F(H, kJk^{-1}) e^{-2b \langle H, kJk^{-1} \rangle}}_{=: I(H, J)} \end{aligned} \quad (5.15)$$

where we have used Weyl integration formula (Thm. 3.2) twice. On the other hand for the integral over \mathfrak{g} , using the complex Weyl integration formula (Thm. 3.1) we have

$$\begin{aligned} T(a, b, c) &:= \frac{1}{\mathcal{Z}_{\mathfrak{g}}} \int_{\mathfrak{g}} dM F(M, M^\vartheta) e^{-a \langle M, M \rangle - c \langle M^\vartheta, M^\vartheta \rangle - 2b \langle M, M^\vartheta \rangle} = \\ &= \frac{c_{\mathfrak{k}}}{\mathcal{Z}_{\mathfrak{g}}} \int_{\mathfrak{h}_{\mathbb{C}}} dZ |\Delta(Z)|^2 e^{-a \langle Z, Z \rangle - c \langle Z^\vartheta, Z^\vartheta \rangle} \int_{\mathfrak{n}_+} dN F(Z + N, Z^\vartheta + N^\vartheta) e^{-2b \langle Z + N, Z^\vartheta + N^\vartheta \rangle} = \end{aligned}$$

⁵ Indeed, $\langle [E_\alpha, E_{-\alpha}], H \rangle = \langle E_\alpha, [E_{-\alpha}, H] \rangle = \alpha(H) \langle E_\alpha, E_{-\alpha} \rangle$. On the other hand $[E_\alpha, E_{-\alpha}] = H_\alpha$ and thus $\alpha(H) \langle E_\alpha, E_{-\alpha} \rangle = \langle H_\alpha, H \rangle$. Evaluating on $H = H_\alpha$ ($\alpha(H_\alpha) = 2$, $\langle H_\alpha, H_\alpha \rangle = \frac{4}{\langle \alpha, \alpha \rangle}$) we get the assertion.

⁶In loc. cit. the exponent has a plus sign and no constant γ , which means that we have to map $H \mapsto \gamma H$ in Harish–Chandra’s formula, thus yielding the factor $(\gamma)^{-\dim \mathfrak{n}_+}$ due to the homogeneity of Δ .

⁷We denote by the same symbol \langle, \rangle the induced inner product on \mathfrak{h}^\vee .

⁸ The formula is quoted as reported in an appendix of a paper of Harder cited ibidem, due to a private communication of Steinberg.

$$= \frac{c_{\mathfrak{k}}}{Z_{\mathfrak{g}}} \int_{\mathfrak{h}_{\mathbb{C}}} dZ |\Delta(Z)|^2 e^{-a\langle Z, Z \rangle - c\langle Z^{\vartheta}, Z^{\vartheta} \rangle - 2b\langle Z, Z^{\vartheta} \rangle} \int_{\mathfrak{n}_{+}} dNF(Z + N, Z^{\vartheta} + N^{\vartheta}) e^{-2b\langle N, N^{\vartheta} \rangle} \quad (5.16)$$

Here we have used that $f(M) := F(M, M^{\vartheta})e^{-QA(M, M^{\vartheta})}$ is Ad_K -invariant (but not Ad_G -invariant!) and then the simple fact that $\langle Z + N, Z^{\vartheta} + N^{\vartheta} \rangle = \langle Z, Z^{\vartheta} \rangle + \langle N, N^{\vartheta} \rangle$.

We now point out that $T(a, b, c)$ is a polynomial in $a/\delta, b/\delta, c/\delta$ by Lemma 4.1. Since $|\Delta(Z)|^2 = (-)^{n_{+}} \Delta(Z) \Delta(Z^{\vartheta})$ (where $Z^{\vartheta} = -Z^{\dagger}$ is again the natural conjugation w.r.t. $i\mathfrak{h}_{\mathbb{R}}$), applying once more Lemma 4.1 with $V_{\mathbb{R}} = i\mathfrak{h}_{\mathbb{R}}$ and $V_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$ we obtain

$$\begin{aligned} T(a, b, c) &= \frac{(-)^{n_{+}} c_{\mathfrak{k}}}{Z_{\mathfrak{g}}} \int_{i\mathfrak{h}_{\mathbb{R}}} dH \int_{i\mathfrak{h}_{\mathbb{R}}} dJ \Delta(H) \Delta(J) e^{-a\langle H, H \rangle - c\langle J, J \rangle - 2b\langle H, J \rangle} \int_{\mathfrak{n}_{+}} dNF(H + N, J + N^{\vartheta}) e^{-2b\langle N, N^{\vartheta} \rangle} = \\ &= \frac{(-)^{n_{+}} c_{\mathfrak{k}} Z_{\mathfrak{n}_{+}}}{Z_{\mathfrak{g}}} \int_{i\mathfrak{h}_{\mathbb{R}}} dH \int_{i\mathfrak{h}_{\mathbb{R}}} dJ \Delta(H)^2 \Delta(J)^2 e^{-a\langle H, H \rangle - c\langle J, J \rangle} \underbrace{\frac{e^{-2b\langle H, J \rangle}}{\Delta(H) \Delta(J)} \frac{1}{Z_{\mathfrak{n}_{+}}} \int_{\mathfrak{n}_{+}} dNF(H + N, J + N^{\vartheta}) e^{-2b\langle N, N^{\vartheta} \rangle}}_{=: G(H, J)} = \end{aligned} \quad (5.17)$$

where the last line is just a different way of rewriting the previous line with $Z_{\mathfrak{n}_{+}} = \int_{\mathfrak{n}_{+}} \exp -2b\langle N, N^{\vartheta} \rangle dN$.

Comparison of formulæ (5.15) and (5.17) suggests the naïve observation that $I(H, J) = \frac{(-)^{n_{+}} c_{\mathfrak{k}} Z_{\mathfrak{k}} Z_{\mathfrak{n}_{+}}}{c_{\mathfrak{k}}^2 Z_{\mathfrak{g}}} G(H, J)$ but this cannot possibly be the case since $I(H, J)$ is Weyl-invariant in **both** variables while in general $G(H, J)$ is not.

What will be shown instead is that the *symmetrization* of $G(H, J)$ is proportional to $I(H, J)$, namely

$$I(H, J) = \frac{(-)^{n_{+}} c_{\mathfrak{k}} Z_{\mathfrak{k}} Z_{\mathfrak{n}_{+}}}{c_{\mathfrak{k}}^2 Z_{\mathfrak{g}}} \frac{1}{|\mathfrak{W}|} \sum_{w \in \mathfrak{W}} G(H, J_w) \quad (5.18)$$

which is precisely the assertion of our theorem. Note that $G(H_w, J_w)$: indeed from Corollary 3.1 the integral⁹

$$f(Z, Z^{\vartheta}) := \int_{\mathfrak{n}_{+}} dNF(Z + N, Z^{\vartheta} + N^{\vartheta}) e^{-2b\langle N, N^{\vartheta} \rangle} \quad (5.19)$$

is a polynomial (since F is a polynomial in both variables) with invariance $f(Z_w, Z_w^{\vartheta}) = f(Z, Z^{\vartheta})$. The polynomial $f(H, J)$ can be written as

$$f(H, J) = e^{H\partial_Z} e^{J\partial_{Z^{\vartheta}}} f(Z, Z^{\vartheta}) \Big|_{Z=0=Z^{\vartheta}} \quad (5.20)$$

where $H\partial_Z, J\partial_{Z^{\vartheta}}$ stand for the vector-fields $H\partial_Z Z = H, H\partial_Z Z^{\vartheta} = 0$ and similarly $J\partial_{Z^{\vartheta}} Z = 0, J\partial_{Z^{\vartheta}} Z^{\vartheta} = J$. Therefore it is sufficient to symmetrize $G(H, J)$ with respect to –say– J in order to obtain a completely Weyl-invariant function. The symmetrization can be carried under the integral sign without changing its value since the measures dH, dJ and the exponential factors that precede G in (5.17) are all \mathfrak{W} -invariant. We have thus obtained

$$\frac{c_{\mathfrak{k}} Z_{\mathfrak{n}_{+}}}{(-)^{n_{+}} Z_{\mathfrak{g}}} \int_{i\mathfrak{h}_{\mathbb{R}} \times i\mathfrak{h}_{\mathbb{R}}} dH dJ \Delta(H)^2 \Delta(J)^2 e^{-a\langle H, H \rangle - c\langle J, J \rangle} \sum_{w \in \mathfrak{W}} \frac{G(H, J_w)}{|\mathfrak{W}|} = \frac{c_{\mathfrak{k}}^2}{Z_{\mathfrak{k}}} \int_{i\mathfrak{h}_{\mathbb{R}} \times i\mathfrak{h}_{\mathbb{R}}} dH dJ \Delta(H)^2 \Delta(J)^2 e^{-a\langle H, H \rangle - c\langle J, J \rangle} I(H, J)$$

Of course this equality *per se* does not imply eq. 5.18. However we can use the following argument. We replace the invariant polynomial $F(X, Y)$ by $h(X)g(Y)F(X, Y)$ with h, g arbitrary Ad_K invariant polynomials over \mathfrak{k}

⁹The prefactor of which already has the advocated invariance.

Note that $g(H + N) = g(H)$ (and so for h): indeed any Ad_K -invariant polynomial on \mathfrak{k} is automatically Ad_G -invariant (on $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}$) and for a generic H , $H + N$ is Ad_G -conjugate to H itself since ad_{H+N} is semisimple (in the adjoint representation). Thus, in eq. 5.17, the two extra factors h, g will be independent of N and thus factorizable outside of the integral over the nilpotent algebra \mathfrak{n}_+ .

On the other hand, in eq. 5.15 they clearly and immediately factor out of the K -integral thus yielding the identity

$$\int_{i\mathfrak{h}_{\mathbb{R}}} dH \int_{i\mathfrak{h}_{\mathbb{R}}} dJ \Delta(H)^2 \Delta(J)^2 e^{-a\langle H, H \rangle - c\langle J, J \rangle} h(H) g(J) \underbrace{\left(\frac{c_{\mathfrak{k}}(-)^{n_+} \mathcal{Z}_{\mathfrak{n}_+}}{\mathcal{Z}_{\mathfrak{g}}} \frac{1}{|\mathfrak{W}|} \sum_{w \in \mathfrak{W}} G(H, J_w) - \frac{c_{\mathfrak{k}}^2}{\mathcal{Z}_{\mathfrak{k}}} I(H, J) \right)}_{=: R(J, H)} = 0 \quad (5.21)$$

valid for *arbitrary* Weyl-invariants polynomials h, g on $\mathfrak{h}_{\mathbb{R}}$. Note that in $\mathcal{H} := \mathcal{L}^2(i\mathfrak{h}_{\mathbb{R}} \times i\mathfrak{h}_{\mathbb{R}}, dH dJ e^{-a\langle H, H \rangle - c\langle J, J \rangle})$ the set of all polynomials is dense and that the bracket expression above belongs to this space (we can take $a, c \in \mathbb{R}_-$ for this computation). The projector onto the subspace of \mathfrak{W} -invariant functions

$$\mathcal{H}_{\mathfrak{W}} := \{f(H, J) \in \mathcal{H} : f(H_w, J_{w'}) = f(H, J), \forall w, w' \in \mathfrak{W}\} \quad (5.22)$$

is self-adjoint and hence the range is a closed subspace, to which $R(J, H)$ belongs. The space of Weyl invariant polynomials form a basis in this space and in particular are dense. Thus the vanishing of eq. 5.21 says that $R(J, H)$ is orthogonal to such a dense set, thus is identically vanishing. The last detail is that the identity so far has been proved only for $H, J \in i\mathfrak{h}_{\mathbb{R}}$; however, being an identity between polynomials, it must hold for its complexification as well, namely on the whole $\mathfrak{h}_{\mathbb{C}}$. The theorem is proved and so is Conjecture 1, with $\gamma = -2b$. **Q.E.D.**

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