

Free Energy of the Two–Matrix Model/dToda Tau–Function

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Abstract

We provide an integral formula for the free energy of the two-matrix model with polynomial potentials of arbitrary degree (or formal power series). This is known to coincide with the τ -function of the dispersionless two-dimensional Toda hierarchy. The formula generalizes the case of conformal maps of Jordan curves studied by Kostov, Krichever, Mineev-Weinstein, Wiegmann, Zabrodin and separately Takhtajan. Finally we generalize the formula found in genus zero to the case of spectral curves of arbitrary genus with certain fixed data.

1 Introduction

Many instances of integrable systems are obtained by means of a suitable limit (dispersionless or semi-classical) of a statistical theory. The departing point of our analysis in this paper is the random 2-matrix model [21, 9], which is attracting growing attention due to its applications to solid state physics [15] (e.g., conduction in mesoscopic devices, quantum chaos and, lately, crystal growth[22]), particle physics [29], 2d-quantum gravity and string theory [11, 13, 3]. The model under inspection consists of two Hermitian matrices M_1, M_2 of size $N \times N$ with a probability distribution given by the formula

$$\begin{aligned} d\mu(M_1, M_2) &= \frac{1}{\mathcal{Z}_N} dM_1 dM_2 \exp \left[-\frac{1}{\hbar} \text{Tr} (V_1(M_1) + V_2(M_2) - M_1 M_2) \right], \\ V_1(x) &= \sum_{K=1}^{\infty} \frac{u_K}{K} x^K; \quad V_2(y) = \sum_{J=1}^{\infty} \frac{v_J}{J} y^J, \end{aligned} \quad (1-1)$$

where V_i are formal power series but soon will be restricted to polynomials for simplicity. The partition function \mathcal{Z}_N is known to be a τ -function for the KP hierarchy in each set of deformation parameters (coefficients of V_1 or V_2) and to provide solutions of the two-Toda hierarchy [28, 1, 2]. This model has been previously investigated in the series of paper [5, 6, 7] where a duality of spectral curves and differential systems for the relevant biorthogonal polynomials has been unveiled and analyzed in the case of polynomial potentials. In [8] the mixed correlation functions of the model (traces of powers of the two non-commuting matrices) have been reduced to a formal Fredholm-like determinant without any assumption on the nature of the potentials and using the recursion coefficients for the biorthogonal polynomials. We briefly recall that the biorthogonal polynomials are two sequences of monic polynomials ([5] and references therein)

$$\pi_n(x) = x^n + \dots, \quad \sigma_n(y) = y^n + \dots, \quad n = 0, 1, \dots \quad (1-2)$$

that are “orthogonal” (better say “dual”) w.r.t. to the coupled measure on the product space

$$\int_{\mathbb{R}} \int_{\mathbb{R}} dx dy \pi_n(x) \sigma_m(y) e^{-\frac{1}{\hbar}(V_1(x) + V_2(y) - xy)} = h_n \delta_{mn}, \quad h_n \neq 0 \quad \forall n \in \mathbb{N} \quad (1-3)$$

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where $V_1(x)$ and $V_2(y)$ are the functions (called *potentials*) appearing in the two-matrix model measure (1-1). It is convenient to introduce the associated quasipolynomials defined by the formulas

$$\psi_n(x) := \frac{1}{\sqrt{h_{n-1}}} \pi_{n-1}(x) e^{-\frac{1}{\hbar} V_1(x)} \quad (1-4)$$

$$\phi_n(y) := \frac{1}{\sqrt{h_{n-1}}} \sigma_{n-1}(y) e^{-\frac{1}{\hbar} V_2(y)} . \quad (1-5)$$

In terms of these two sequences of quasipolynomials the multiplications by x and y respectively are represented by semiinfinite square matrices $Q = [Q_{ij}]_{i,j \in \mathbb{N}^*}$ and $P = [P_{ij}]_{i,j \in \mathbb{N}^*}$ according to the formulae

$$\begin{aligned} x\psi_n(x) &= \sum_m Q_{n,m} \psi_m(x) ; & y\phi_n(y) &= \sum_m P_{m,n} \phi_m(y) \\ Q_{n,m} &= 0 = P_{m,n}, & \text{if } n > m + 1. \end{aligned} \quad (1-6)$$

The matrices P and Q have a rich structure and satisfy the “string equation”

$$[P, Q] = \hbar \mathbf{1} . \quad (1-7)$$

We refer for further details to [4, 5, 6, 7] where these models are studied especially in the case of polynomial potentials. We also point out that the model can easily be generalized to accommodate contours of integration other than the real axes [4, 5].

The partition function is believed to have a large N expansion according to the formula

$$-\frac{1}{N^2} \ln \mathcal{Z}_N = \mathcal{F} = \mathcal{F}^{(0)} + \frac{1}{N^2} \mathcal{F}^{(1)} + \dots . \quad (1-8)$$

This expansion in powers of N^{-2} has been repeatedly advocated for the 2-matrix model on the basis of physical arguments [13, 14] and has been rigorously proven in the one-matrix model [16]. In the two-matrix model this expansion is believed to generate 2-dimensional statistical models of surfaces triangulated with ribbon-graphs [13, 11, 17], where the powers of N^{-1} are the Euler characteristics of the surfaces being tessellated. From this point of view the term $\mathcal{F}^{(0)}$ corresponds to a genus 0 tessellation and the next to a genus one tessellation.

The object of this paper is the leading term of the free energy, $\mathcal{F}^{(0)}$. It is the generating function of the expectations of the powers of the two matrices in the model

$$\langle M_1^K \rangle = K \partial_{u_K} \mathcal{F}^{(0)} + \mathcal{O}(N^{-2}) , \quad \langle M_2^K \rangle = J \partial_{v_J} \mathcal{F}^{(0)} + \mathcal{O}(N^{-2}) . \quad (1-9)$$

Integral formulas for these partial derivatives at the leading order are known and involve integrals over a certain spectral curve (see below [18]), however a closed formula for the function $\mathcal{F}^{(0)}$ itself was so far missing; this paper fills the gap (Thm. 2.1). Remarkably, an algorithm for the computation of the subleading terms is known and also a closed expression of the genus 1 correction $\mathcal{F}^{(1)}$ [14], and therefore this paper precedes logically and complements [14]. The paper is organized as follows: in section 2 we recall the main formulas known in the literature and set up the notation, linking our result with the relevant other approaches [23, 24, 25, 27, 30, 20, 19, 31, 26]. The core of the paper is section 2.1 where the formula for the leading term of the free energy (dToda tau function) is presented and proved (Thm. 2.1). In section 2.2 we apply our result to the generating function of the canonical change of coordinates represented by the Lax operators of the dToda hierarchy.

Finally in section 3 we extend the result obtained in section 2.1 and find an integral expression for the free energy of the two matrix model in the case the spectral curve is of arbitrary genus (Thm. 3.1).

2 Planar limit (dToda hierarchy)

In this section we investigate the planar free energy ($\mathcal{F}^{(0)}$ in the notation of the introduction) and we will make soon the common “one-cut” assumption (to be lifted in section 3) which amounts to saying that the multiplication

operators tend to meromorphic functions over a spectral curve of genus zero. Indeed in this limit the two multiplication operators for the wave-vectors defined by the biorthogonal quasipolynomials become commuting functions in the shift operator here replaced by the variable λ ([7] and references therein) and they corresponds to the Lax operators of the dToda hierarchy:

$$Q(\lambda) = \gamma\lambda + \sum_{k=0}^{\infty} \alpha_j \lambda^{-j} \quad (2-1)$$

$$P(\lambda) = \frac{\gamma}{\lambda} + \sum_{j=0}^{\infty} \beta_j \lambda^j . \quad (2-2)$$

The original noncommutativity of the operators now translates to the following Poisson-bracket which is nothing but the dispersionless form of the string equation (1-7) [23, 24, 25]

$$\{P, Q\} := \lambda \frac{\partial P}{\partial \lambda} \frac{\partial Q}{\partial t} - \lambda \frac{\partial Q}{\partial \lambda} \frac{\partial P}{\partial t} = 1 , \quad (2-3)$$

where $t = \hbar N$ in the large limit. The parameter t is the (scaled) number of eigenvalues of the matrix model and enters the relations

$$\frac{1}{2i\pi} \oint P dQ = t ; \quad \frac{1}{2i\pi} \oint Q dP = t ; \quad (2-4)$$

The contour chosen is a contour on the physical sheet of the Q (P respectively) plane around infinity (i.e. around $\lambda = \infty$, $\lambda = 0$ respectively).

The *deformation equations* describe the infinitesimal variations of the operators P, Q under variations of the parameters of the potentials; they are known in the finite N regime as well ([5, 7] and references therein) whereas in the dispersionless limit are given by the evolution equations [23, 24, 25, 27]

$$(\partial_{u_K} Q)_\lambda = \{Q, (Q^K)_{+0}\} ; \quad (\partial_{v_J} Q)_\lambda = \{Q, (P^J)_{-0}\} \quad (2-5)$$

$$(\partial_{u_K} P)_\lambda = \{P, (Q^K)_{+0}\} ; \quad (\partial_{v_J} P)_\lambda = \{P, (P^J)_{-0}\} \quad (2-6)$$

where the subscript 0_{\pm} denotes the positive (negative) part of the Laurent polynomial plus half the part constant in λ , viz, e.g.

$$(Q^K)_{+0}(\lambda) = (Q^K)_+(\lambda) + \frac{1}{2}(Q^K)_0 . \quad (2-7)$$

If the potentials V_i are polynomials of degrees $d_i + 1$, $i = 1, 2$ then both P and Q are finite Laurent polynomials

$$Q(\lambda) = \gamma\lambda + \sum_{k=0}^{d_2} \alpha_j \lambda^{-j} \quad (2-8)$$

$$P(\lambda) = \frac{\gamma}{\lambda} + \sum_{j=0}^{d_1} \beta_j \lambda^j . \quad (2-9)$$

In what follows we restrict to this case so as to avoid complication of convergence; however one may replace the contour integrals that will follow with formal residues of formal Laurent series and carry out the same computations. Another reason why we prefer the truncated setting is that then the two functions P, Q define a (singular) spectral curve of genus $g = 0$ which is given by the polynomial locus (resultant-like) obtained from the determinant of the

following Sylvester matrix ([6], thanks to a remark by J. Hurtubise)

$$0 = E(P, Q) = \frac{1}{\gamma^{d_1+d_2}} \det \begin{bmatrix} \gamma & \beta_0-P & \beta_1 & \cdots & \cdots & \beta_{d_1} & 0 & 0 & 0 \\ 0 & \gamma & \beta_0-P & \beta_1 & \cdots & \cdots & \beta_{d_1} & 0 & 0 \\ 0 & 0 & \gamma & \beta_0-P & \beta_1 & \cdots & \cdots & \beta_{d_1} & 0 \\ 0 & 0 & 0 & \gamma & \beta_0-P & \beta_1 & \cdots & \cdots & \beta_{d_1} \\ \hline \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0-Q & \gamma & 0 & 0 & 0 & 0 \\ 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0-Q & \gamma & 0 & 0 & 0 \\ 0 & 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0-Q & \gamma & 0 & 0 \\ 0 & 0 & 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0-Q & \gamma & 0 \\ 0 & 0 & 0 & 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0-Q & \gamma \end{bmatrix}. \quad (2-10)$$

This spectral curve is precisely the (limit of the) spectral curve of the four finite-dimensional folded differential systems for the quasipolynomials [5, 7]. If we worked with formal power series it is not known whether a spectral curve in this sense can be defined.

Taking this viewpoint $\lambda \in \mathbb{CP}^1$ is the uniformizing parameter of the spectral curve $E(P, Q)$. The two potentials are related to the parameters $\gamma, \alpha_j, \beta_j$ by the relations

$$P = V_1'(Q) - \frac{t}{Q} + \mathcal{O}(Q^{-2}), \quad \text{near } \infty_Q \quad (2-11)$$

$$Q = V_2'(P) - \frac{t}{P} + \mathcal{O}(P^{-2}), \quad \text{near } \infty_P \quad (2-12)$$

where the point ∞_Q (∞_P) is the point on the spectral curve where Q (P respectively) has a simple pole; in the uniformization provided by the coordinate λ it corresponds to $\lambda = \infty$ ($\lambda = 0$ respectively). By expanding both sides in powers of λ and matching the coefficients one can realize that the coefficients of the two potentials are rational functions of the parameters $\gamma, \alpha_j, \beta_j$. More explicitly we have

$$V_1(q) = \sum_{K=1}^{d_1+1} \frac{u_K}{K} q^K; \quad V_2(p) = \sum_{J=1}^{d_2+1} \frac{v_J}{J} p^J, \quad (2-13)$$

$$u_K = -\frac{1}{2i\pi} \oint \frac{P}{Q^K} dQ, \quad v_J = -\frac{1}{2i\pi} \oint \frac{Q}{P^J} dP. \quad (2-14)$$

The leading term of the free energy of the model is then defined by the differential equations

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial u_K} &= U_K := \frac{1}{K2i\pi} \oint P Q^K dQ = \frac{1}{K} \operatorname{res}_{\infty_Q} P Q^K dQ \\ \frac{\partial \mathcal{F}}{\partial v_J} &= V_J := \frac{1}{J2i\pi} \oint Q P^J dP = \frac{1}{J} \operatorname{res}_{\infty_P} Q P^J dP. \end{aligned} \quad (2-15)$$

These equations are precisely the same that define the τ -function of the dToda hierarchy. In the relevant literature [23, 24, 25, 27] the functions P, Q are the Lax operators denoted by L, \tilde{L} or $\mathcal{L}, \tilde{\mathcal{L}}$ and the normalization is slightly different.

We should also remark the following link to the works [20, 19, 30, 31, 26] in that if we take $V_1 = V = \overline{V_2}$ we then have $\gamma \in \mathbb{R}, \alpha_j = \overline{\beta_j}$. The function $Q(\lambda)$ is then the uniformizing map of a Jordan curve Γ in the Q -plane (at least for suitable ranges of the parameters) which is defined by either of the following relations

$$P(1/\lambda) = Q(\overline{\lambda}); \quad |\lambda| = 1. \quad (2-16)$$

In the setting of [30, 19, 20] the function Q is denoted by z (and λ by w) so that then P is nothing but the Schwartz function of the curve Γ , defined by

$$\overline{z} = S(z), \quad z \in \Gamma. \quad (2-17)$$

The coefficients of the potential $V(x) = \sum_k \frac{t^k}{k} x^k$ are the so-called “exterior harmonic moments” of the region \mathcal{D} enclosed by the curve Γ

$$t_K = \frac{1}{2i\pi} \oint_{\Gamma} \bar{z} z^{-K} dz \quad (2-18)$$

$$t = t_0 = \frac{1}{2i\pi} \iint_{\mathcal{D}} dz \wedge d\bar{z} = \frac{1}{2i\pi} \oint_{\Gamma} \bar{z} dz = \frac{Area(\mathcal{D})}{\pi} . \quad (2-19)$$

By writing $\bar{z} = S(z(w))$ these integrals become residues in the w -plane. In our general situation the representation of the coefficients u_K, v_J (2-14) cannot be translated to a surface integral, hence the need for a separate analysis. For conformal maps (i.e. Jordan curves) the τ -function has been defined in [26] and given an appealing interpretation as (exponential of the Legendre transform of) the electrostatic potential of a uniform 2-dimensional distribution of charge in \mathcal{D} [19]

$$\ln(\tau_{\Gamma}) = -\frac{1}{\pi} \iint_{\mathcal{D}} \ln \left| \frac{1}{z} - \frac{1}{z'} \right| d^2 z d^2 z' . \quad (2-20)$$

It can be rewritten as a (formal) series in the exterior and interior moments as

$$2 \ln(\tau_{\Gamma}) = -\frac{1}{4\pi} \iint_{\mathcal{D}} d^2 z |z|^2 + t_0 w_0 + \sum_{K>0} (t_K w_K + \bar{t}_K \bar{w}_K) \quad (2-21)$$

where the interior moments are defined by (the normalization here differs slightly from [30])

$$w_K = \frac{1}{\pi K} \iint_{\mathcal{D}} z^K d^2 z , \quad K > 0 \quad (2-22)$$

$$w_0 = \frac{1}{\pi} \iint_{\mathcal{D}} \ln |z|^2 d^2 z . \quad (2-23)$$

Unfortunately this formula is not exportable to our more general setting (in particular the logarithmic moment above) and to the general setting of the dispersionless Toda hierarchy; our principal objective is to fill this gap.

2.1 Free energy of the 2-matrix model in the planar limit

Let us focus on the function P as a (multivalued) function of Q (similar argument can be reversed for Q as a function of P); on the physical sheet $P(Q)$ will have in general some branch-cuts with square-root singularities at the endpoint of each cut. These cuts are bounded in the physical sheet because $P(Q)$ is analytic in a neighborhood of ∞_Q . Note that

$$V_1(q) = \frac{1}{2i\pi} \oint \ln \left(1 - \frac{q}{Q} \right) P dQ \quad (2-24)$$

$$V_2(p) = \frac{1}{2i\pi} \oint \ln \left(1 - \frac{p}{P} \right) Q dP , \quad (2-25)$$

as one can immediately realize by expanding in powers of q, p . The integrals are well defined provided that q (p respectively) are kept inside the contour of integration. In the following we will develop all the necessary arguments only for $V_1(q)$ and related objects, where the reader can obtain the relevant proofs for $V_2(p)$ by interchanging the rôle of P and Q .

Let us now introduce the *exterior potentials*²

$$\Phi_1(q_{out}) = -\frac{1}{2i\pi} \oint \ln \left(1 - \frac{Q}{q_{out}} \right) P dQ \quad (2-26)$$

² It has been pointed out to me by B. Eynard that they are sometimes referred to as *effective potentials* because they can be thought of as the effective potential felt by one eigenvalue in the field of the others.

In this formula the contour of integration is such as to leave the point q_{out} in the outside region (whence the subscript). In general we will distinguish the choice of the contours by a subscript q_{out} or q_{in} in what follows. The coefficients of $\Phi_1(q_{out})$ in inverse powers of q_{out} are precisely (minus) the U_K defined in (2-15). Note that expanding eq. (2-26) in inverse powers of q_{out} the first $d_1 + 1$ coefficients are the U_K coefficients as defined in (2-15).

The first objective is to analytically continue Φ_1 to the physical sheet so as to obtain a different representation of it. For this purpose we compute

$$\Phi_1'(q_{out}) = \frac{1}{2i\pi} \oint \frac{Q}{q_{out}(Q - q_{out})} PdQ = \frac{\oint PdQ}{q_{out}2i\pi} + \frac{1}{2i\pi} \oint \frac{1}{(Q - q_{out})} PdQ = \quad (2-27)$$

$$= \frac{t}{q} + \frac{1}{2i\pi} \oint \frac{1}{Q - q_{in}} PdQ + P(q) = \frac{t}{q} - V_1'(q) + P(q) , \quad (2-28)$$

where we have dropped the subscript outside of the integral as those terms are analytic functions in the whole physical sheet. Integrating once we obtain

$$\Phi_1(Q) = -V_1(Q) + t \ln(Q) + \int_{X_q} PdQ , \quad (2-29)$$

where X_q is a point defined implicitly by the requirement $\Phi_1 = \mathcal{O}(Q^{-1})$ near $\infty_Q (\equiv (\lambda = \infty))$. Note that, in spite of the $\ln(Q)$ term, this is an analytic function around ∞_Q .

By similar reasoning we get

$$\Phi_2(P) = -\frac{1}{2i\pi} \oint \ln \left(1 - \frac{\tilde{P}}{P} \right) \tilde{Q} d\tilde{P} = -V_2(P) + t \ln(P) + \int_{X_p} Q dP \quad (2-30)$$

We need to introduce two more points (beside X_q and X_p) on the spectral curve and a lemma: those are the points Λ_q, Λ_p defined implicitly by the relations

$$\int_{\Lambda_q} PdQ = V_1(Q(\lambda))_{>0} + t \ln(\lambda) + \mathcal{O}(\lambda^{-1}) , \quad \text{near } \infty_Q \quad (2-31)$$

$$\int_{\Lambda_p} Q dP = V_2(P(\lambda))_{<0} + t \ln(\lambda) + \mathcal{O}(\lambda) , \quad \text{near } \infty_P . \quad (2-32)$$

By a simple inspection of the λ^0 -coefficient in the LHS and RHS one finds that

$$\int_{X_q} PdQ = \int_{\Lambda_q} PdQ + \int_{X_q}^{\Lambda_q} PdQ = \int_{\Lambda_q} PdQ + (V_1(Q))_0 - t \ln(\gamma) \quad (2-33)$$

$$\int_{X_p} Q dP = \int_{\Lambda_p} Q dP + \int_{X_p}^{\Lambda_p} Q dP = \int_{\Lambda_p} Q dP + (V_2(P))_0 - t \ln(\gamma) , \quad (2-34)$$

where the subscript $_0$ denotes the constant part in λ (which can be written as a residue).

We now have the

Lemma 2.1 *The following relation holds*

$$\mu := Q(X_p)P(X_p) + \int_{X_p}^{X_q} \tilde{P} d\tilde{Q} = Q(X_q)P(X_q) + \int_{X_q}^{X_p} Q dP = \left(PQ - V_1(Q) - V_2(P) \right)_0 + 2t \ln(\gamma) . \quad (2-35)$$

Remark 2.1 The quantity μ will be proved in Corollary 2.2 to be the derivative of the free energy w.r.t t , i.e. the chemical potential. It therefore corresponds to the logarithmic moment in the setting of [30, 20, 19, 31].

Proof. The equivalence of the two integrals is immediate by an integration by parts. And integration by parts is in fact the key to prove also the last part:

$$\int_{\Lambda_q} P dQ = PQ - P(\Lambda_q)Q(\Lambda_q) - \int_{\Lambda_q} Q dP = PQ - P(\Lambda_q)Q(\Lambda_q) - \int_{\Lambda_q}^{\Lambda_p} Q dP - \int_{\Lambda_p} Q dP. \quad (2-36)$$

Looking at the constant part in λ in both sides of the above equation we conclude that

$$(QP)_0 = P(\Lambda_q)Q(\Lambda_q) + \int_{\Lambda_q}^{\Lambda_p} Q dP. \quad (2-37)$$

Therefore we have

$$(QP)_0 = P(\Lambda_q)Q(\Lambda_q) + \int_{\Lambda_q}^{\Lambda_p} Q dP = P(\Lambda_q)Q(\Lambda_q) + \int_{X_p}^{\Lambda_p} Q dP + \int_{\Lambda_q}^{X_p} Q dP = \quad (2-38)$$

$$= P(\Lambda_q)Q(\Lambda_q) + \int_{X_p}^{\Lambda_p} Q dP + Q(X_p)P(X_p) - Q(\Lambda_q)P(\Lambda_q) - \int_{\Lambda_q}^{X_p} P dQ = \quad (2-39)$$

$$= \int_{X_p}^{\Lambda_p} Q dP + Q(X_p)P(X_p) - \int_{X_q}^{X_p} P dQ - \int_{\Lambda_q}^{X_q} P dQ = \quad (2-40)$$

$$= (V_1(Q) + V_2(P))_0 - 2t \ln(\gamma) + Q(X_p)P(X_p) + \int_{X_p}^{X_q} P dQ. \quad (2-41)$$

This concludes the proof of the Lemma. Q.E.D.

We are now in a position to formulate the first main result of this paper.

Theorem 2.1 The Free energy is given by the formula (up to constant)

$$2\mathcal{F} = \operatorname{res}_{\infty_Q} (P\Phi_1 dQ) + \operatorname{res}_{\infty_P} (Q\Phi_2 dP) + \frac{1}{2} \operatorname{res}_{\infty_Q} (P^2 Q dQ) + t \operatorname{res}_{\lambda=\infty} \left(\frac{V_1(Q) + V_2(P) - PQ}{\lambda} d\lambda \right) + 2t^2 \ln(\gamma) \quad (2-42)$$

$$= \sum_K u_K U_K + \sum_J v_J V_J + \frac{1}{2} \operatorname{res}_{\infty_Q} P^2 Q dQ + t \operatorname{res}_{\lambda=\infty} \left(\frac{V_1(Q) + V_2(P) - PQ}{\lambda} d\lambda \right) + 2t^2 \ln(\gamma). \quad (2-43)$$

Before proceeding to the proof a corollary and some remarks are in order. First off from the expressions in Thm. 2.1 we find the well-known scaling property of the free energy [19, 31, 10]

Corollary 2.1 The free energy defined in Thm. 2.1 satisfies the scaling equation

$$4\mathcal{F} = -t^2 + \left(2t\partial_t + \sum_K (2-K)u_K \partial_{u_K} + \sum_J (2-J)v_J \partial_{v_J} \right) \mathcal{F}. \quad (2-44)$$

(More general scaling equations will be introduced later in Corollary 3.2).

Proof. The proof in the context of the normal matrix model can be found in the references quoted above and, in view of the formal equivalence of the normal matrix model with the two-matrix model [19], the statement would follow also in our case.

In order to be self contained we rederive this property in the present context; one way of proving formula (2-44) is from the expression (2-43) for \mathcal{F} given in Thm. 2.1, by computing the residues involved after rewriting symmetrically the term $\operatorname{res}_{\infty_Q} P^2 Q dQ$ as $\frac{1}{2} \left(\operatorname{res}_{\infty_Q} P^2 Q dQ + \operatorname{res}_{\infty_P} Q^2 P dP \right)$ and computing it. A second, possibly instructive way to derive it also directly from eq. (2-42) is by using the scaling properties of the various quantities involved. To this purpose we introduce the rescaling according to the formulæ

$$Q = \delta \tilde{Q}, \quad P = \delta \tilde{P}. \quad (2-45)$$

Under this change of the functions Q, P we have (using formulas (2-4, 2-15))

$$u_K = \delta^{2-K} \tilde{u}_K ; \quad v_J = \delta^{2-J} \tilde{v}_J ; \quad t = \delta^2 \tilde{t} . \quad (2-46)$$

Moreover, computing explicitly the residue $\operatorname{res}_{\lambda=\infty} PQ' d\lambda$ we find that

$$\gamma^2 = -t + \sum_j j \alpha_j \beta_j , \quad (2-47)$$

from which one immediately obtains that $\gamma = \delta \tilde{\gamma}$. The exterior potentials are also conformally invariant, indeed

$$\Phi_1 = -V_1(Q) + t \ln(Q) + \int_{X_q} P dQ = -\delta^2 \tilde{V}_1(\tilde{Q}) + \delta^2 \tilde{t} \ln(\tilde{Q}) + \int_{\tilde{X}_q} \tilde{P} d\tilde{Q} + \delta^2 \tilde{t} \ln(\delta) + \int_{X_q} \tilde{P} d\tilde{Q} \quad (2-48)$$

$$= \delta^2 \tilde{\Phi}_1 + \delta^2 \tilde{t} \ln(\delta) + \int_{X_q} \tilde{P} d\tilde{Q} \quad (2-49)$$

The last two terms cancel each other out because both the LHS and RHS must be $\mathcal{O}(Q^{-1}) = \mathcal{O}(\tilde{Q}^{-1})$. Repeating the argument for Φ_2 we finally get

$$\Phi_1 = \delta^2 \tilde{\Phi}_1 ; \quad \Phi_2 = \delta^2 \tilde{\Phi}_2 . \quad (2-50)$$

Plugging the relations (2-45, 2-46, 2-47, 2-50) into the expression (2-43) for the free energy we obtain

$$\mathcal{F} = \delta^4 \tilde{\mathcal{F}} + \delta^4 \tilde{t}^2 \ln \delta = \delta^4 \tilde{\mathcal{F}} + t^2 \ln(\delta) \quad (2-51)$$

Applying the operator $\delta \partial_\delta |_{\delta=1}$ to both sides we obtain

$$0 = 4\mathcal{F} - \left[\sum_K (2-K) u_K \partial_{u_K} + \sum_J (2-J) v_J \partial_{v_J} + 2t \partial_t \right] \mathcal{F} + t^2 , \quad (2-52)$$

whence the statement of the corollary. Q.E.D.

We also add a few remarks before moving on with the proof of Thm. 2.1.

Remark 2.2 Formulas (2-42, 2-43) are symmetric in the rôles of P and Q : the only non-immediately symmetric term is $\operatorname{res}_{\infty Q} P^2 Q dQ$ but an integration by parts restores the symmetry.

Remark 2.3 The formula is derived for polynomial potentials but it could possibly be extended to convergent or formal series.

Remark 2.4 The genus 1 correction to the above formula has been computed in [14] and is given by

$$\mathcal{F}^{(1)} = -\frac{1}{24} \ln(\gamma^4 D) \quad (2-53)$$

$$D = \frac{1}{\gamma^{d_1+d_2+2}} \det \begin{bmatrix} -\gamma & 0 & \beta_1 & \cdots & \cdots & d_1 \beta_{d_1} & 0 & 0 & 0 \\ 0 & -\gamma & 0 & \beta_1 & \cdots & \cdots & d_1 \beta_{d_1} & 0 & 0 \\ 0 & 0 & -\gamma & 0 & \beta_1 & \cdots & \cdots & d_1 \beta_{d_1} & 0 \\ 0 & 0 & 0 & -\gamma & 0 & \beta_1 & \cdots & \cdots & d_1 \beta_{d_1} \\ \hline d_2 \alpha_{d_2} & \cdots & \alpha_1 & 0 & -\gamma & 0 & 0 & 0 & 0 \\ 0 & d_2 \alpha_{d_2} & \cdots & \alpha_1 & 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & d_2 \alpha_{d_2} & \cdots & \alpha_1 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & d_2 \alpha_{d_2} & \cdots & \alpha_1 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 & d_2 \alpha_{d_2} & \cdots & \alpha_1 & 0 & -\gamma \end{bmatrix} \quad (2-54)$$

Proof of Theorem 2.1. The equivalence of lines (2-42) and (2-43) follows from the definition of the exterior potentials and Lemma 2.1.

Let us consider the derivative $\partial_K := \frac{\partial}{\partial u_K}$ done at argument (P or Q) fixed, where the value being kept fixed under differentiation is denoted by the corresponding subscript

$$(\partial_K \Phi_1)_Q = -\frac{Q^K}{K} + \int_{X_q} (\partial_K P)_Q dQ - P(X_q) \partial_K(Q(X_q)) \quad (2-55)$$

$$(\partial_K \Phi_2)_P = \int_{X_p} (\partial_K Q)_P dP - Q(X_p) \partial_K(P(X_p)) \quad (2-56)$$

Let us set

$$4i\pi \mathcal{F}_0 := \oint_{\infty_Q} P \Phi_1(Q) dQ + \oint_{\infty_P} Q \Phi_2(P) dP + \frac{1}{2} \oint_{\infty_Q} P^2 Q dQ, \quad (2-57)$$

and study the variation of this functional. First off we have the formulas

$$P = V_1'(Q) - \frac{t}{Q} + \sum_{K=1}^{\infty} K U_K Q^{-K-1} \quad (2-58)$$

$$\Phi_1 = - \sum_{K=1}^{\infty} U_K Q^{-K} \quad (2-59)$$

$$Q = V_2'(P) - \frac{t}{P} + \sum_{J=1}^{\infty} J V_J P^{-J-1} \quad (2-60)$$

$$\Phi_2 = - \sum_{J=1}^{\infty} V_J P^{-J}, \quad (2-61)$$

where the U_K, V_J 's have been defined in (2-15). We also need the variations of the functions P, Q given here below

$$(\partial_K P)_Q = Q^{K-1} + \mathcal{O}(Q^{-2}); \quad (\partial_K Q)_P = \mathcal{O}(P^{-2}). \quad (2-62)$$

With these formulæ we can now compute the variation of \mathcal{F}_0 (the subscript on the loop integrals that follow specify the points around which we circulate):

$$4i\pi \partial_K \mathcal{F}_0 = \overbrace{\oint_{\infty_Q} (\partial_K P)_Q \Phi_1 dQ}^{=2i\pi U_K} + \overbrace{\oint_{\infty_Q} P (\partial_K \Phi_1)_Q dQ}^{=\star} + \overbrace{\oint_{\infty_P} (\partial_K Q)_P \Phi_2 dP}^{=0} + \oint_{\infty_P} Q (\partial_K \Phi_2)_P dP + \oint_{\infty_Q} P (\partial_K P)_Q Q dQ \quad (2-63)$$

$$= 4i\pi U_k + \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ + \oint_{\infty_P} Q \left[\int_{X_p} (\partial_k Q)_P dP \right] dP + \oint_{\infty_Q} P (\partial_K P)_Q Q dQ + \quad (2-64)$$

$$- 2i\pi t [P(X_q) \partial_K(Q(X_q)) + Q(X_p) \partial_K(P(X_p))] = \quad (2-65)$$

$$= 4i\pi U_k + \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ + \oint_{\infty_P} Q \left[\int_{X_q} (\partial_k Q)_P dP \right] dP + \quad (2-66)$$

$$- \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ - \oint_{\infty_Q} Q \left[\int_{X_q} (\partial_k P)_Q dQ \right] dP + \quad (2-67)$$

$$- 2i\pi t \left[P(X_q) \partial_K(Q(X_q)) + Q(X_p) \partial_K(P(X_p)) - \int_{X_p}^{X_q} (\partial_K Q)_P dP \right] \quad (2-68)$$

where

$$\star = 2i\pi U_k + \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ - 2i\pi t P(X_q) \partial_K(Q(X_q)). \quad (2-69)$$

We now use the ‘‘thermodynamic identity’’

$$(\partial_K P)_Q dQ = -(\partial_K Q)_P dP, \quad (2-70)$$

so that the last term on line (2-67) reads (notice the double change of sign due to the definition of the circle around ∞_Q , which is (homologically) minus the circle around ∞_P)

$$\oint_{\infty_Q} Q \left[\int_{X_q} (\partial_k P)_Q dQ \right] dP = \oint_{\infty_P} Q \left[\int_{X_q} (\partial_k Q)_P dP \right] dP \quad (2-71)$$

Plugging into the variation of \mathcal{F}_0 we get

$$4i\pi \partial_K \mathcal{F}_0 = 4i\pi U_K - 2i\pi t \left[P(X_q) \partial_K(Q(X_q)) + Q(X_p) \partial_K(P(X_p)) - \int_{X_p}^{X_q} (\partial_K Q)_P dP \right] \quad (2-72)$$

Finally we claim that the term in the square brackets in (2-72) is

$$P(X_q) \partial(Q(X_q)) + Q(X_p) \partial(P(X_p)) - \int_{X_p}^{X_q} (\partial Q)_P dP = \partial \left[Q(X_p) P(X_p) + \int_{X_p}^{X_q} P dQ \right] = \quad (2-73)$$

$$= \partial \left[\left(PQ - V_1(Q) - V_2(P) \right)_0 + 2t \ln(\gamma) \right], \quad (2-74)$$

where ∂ denotes any vector field in the space of parameters u_K, v_J (or even t). Indeed

$$\partial \left[Q(X_p) P(X_p) + \int_{X_p}^{X_q} P dQ \right] = \partial \left[Q(X_p) P(X_p) + \int_{X_p}^{X_q} P(\lambda) Q'(\lambda) d\lambda \right] = \quad (2-75)$$

$$= \partial(Q(X_p)) P(X_p) + Q(X_p) \partial(P(X_p)) + \quad (2-76)$$

$$+ \int_{X_p}^{X_q} [(\partial P)_\lambda(\lambda) Q'(\lambda) + P(\lambda) (\partial Q')_\lambda(\lambda)] d\lambda - P(X_p) Q'(X_p) \partial X_p + P(X_q) Q'(X_q) \partial X_q = \quad (2-77)$$

$$= \partial(Q(X_p)) P(X_p) + Q(X_p) \partial(P(X_p)) + \quad (2-78)$$

$$+ \int_{X_p}^{X_q} [(\partial P)_\lambda(\lambda) Q'(\lambda) - P'(\lambda) (\partial Q)_\lambda(\lambda)] d\lambda + P(X_q) \partial(Q)_\lambda(X_q) - P(X_p) \partial(Q)_\lambda(X_p) + \quad (2-79)$$

$$- P(X_p) Q'(X_p) \partial X_p + P(X_q) Q'(X_q) \partial X_q. \quad (2-80)$$

In order to proceed we note that if X is a point depending on the parameters, we have

$$\partial(Q(X)) = (\partial Q)_\lambda(X) + Q'(X) \partial X \quad (2-81)$$

Using this formula we get

$$(2-80) = Q(X_p) \partial(P(X_p)) + \partial(Q(X_p)) P(X_p) + \int_{X_p}^{X_q} [(\partial P)_\lambda(\lambda) Q'(\lambda) - P'(\lambda) (\partial Q)_\lambda(\lambda)] d\lambda = \quad (2-82)$$

$$= Q(X_p) \partial(P(X_p)) + \partial(Q(X_p)) P(X_p) + \int_{X_p}^{X_q} (\partial P)_Q dQ, \quad (2-83)$$

which is the term in square brackets in (2-72). Using then Lemma 2.1 we have proven that

$$2\partial_K \mathcal{F}_0 = 2U_K - t\partial_K \{[-V_1(Q) - V_2(P) + QP]_0 + 2t \ln(\gamma)\} = 2U_K - t\partial_K \mu. \quad (2-84)$$

The second term is exactly the opposite of the variation of the last two terms in formula (2-43), and this concludes the proof (the derivatives w.r.t. v_J are treated by interchanging the roles of P and Q). Q.E.D.

The derivative w.r.t the number operator t has to be treated in a separate way.

Corollary 2.2 The derivative of the free energy w.r.t t is given by the formula

$$\partial_t \mathcal{F} = Q(X_p)P(X_p) + \int_{X_p}^{X_q} Q dP = \mu . \quad (2-85)$$

Proof. We start with the following observation

$$(\partial_t P)_Q dQ = -(\partial_t Q)_P dP = -\frac{d\lambda}{\lambda} . \quad (2-86)$$

In other words they are differential of the third kind with poles at ∞_Q and ∞_P and opposite residues, to wit (using the thermodynamical identity (2-70))

$$-\frac{1}{P}dP + \mathcal{O}(P^{-2}) = (\partial_t Q)_P dP = -(\partial_t P)_Q dQ = \frac{1}{Q}dQ + \mathcal{O}(Q^{-2}) \quad (2-87)$$

and the differentials have no other singularities. We then proceed as in the proof of Thm. 2.1

$$(\partial_t \Phi_1)_Q = \ln(Q) + \int_{X_q} (\partial_t P)_Q dQ - P(X_q)\partial_t(Q(X_q)) \quad (2-88)$$

$$(\partial_t \Phi_2)_P = \ln(P) + \int_{X_p} (\partial_t Q)_P dP - Q(X_p)\partial_t(P(X_p)) = \quad (2-89)$$

$$= \ln(P) + \int_{X_q} (\partial_t Q)_P dP + \int_{X_p}^{X_q} (\partial_t Q)_P dP - Q(X_p)\partial_t(P(X_p)) = \quad (2-90)$$

$$= \ln(P) + \int_{X_q} (\partial_t Q)_P dP - \int_{X_p}^{X_q} (\partial_t P)_Q dQ - Q(X_p)\partial_t(P(X_p)) \quad (2-91)$$

$$(\partial_t P)_Q = -\frac{1}{Q} + \mathcal{O}(Q^{-2}) \quad (2-92)$$

$$(\partial_t Q)_P = -\frac{1}{P} + \mathcal{O}(P^{-2}) . \quad (2-93)$$

We then find that

$$\begin{aligned} 4i\pi\partial_t \mathcal{F}_0 &= \oint_{\infty_Q} P \left[\ln(Q) + \int_{X_q} \overbrace{(\partial_t P)_Q dQ}^{=-d\lambda/\lambda} \right] dQ + \oint_{\infty_P} Q \left[\ln(P) + \int_{X_p} \overbrace{(\partial_t Q)_P dP}^{=d\lambda/\lambda} \right] dP + \\ &\quad -2i\pi t \partial_t \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} Q dP \right] + \oint_{\infty_Q} PQ \overbrace{(\partial_t P)_Q dQ}^{=d\lambda/\lambda} = \quad (2-94) \\ &= -2i\pi t \partial_t \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} Q dP \right] - 2i\pi(PQ)_0 + \oint_{\infty_Q} P [\ln(Q) - \ln(\lambda)] dQ + \oint_{\infty_P} Q [\ln(P) + \ln(\lambda)] dP \end{aligned}$$

We now claim that

$$\oint_{\infty_Q} P [\ln(Q) - \ln(\lambda)] dQ = V_1(Q)_0 - t \ln(\gamma) , \quad (2-96)$$

and the symmetric formula for the other term. Indeed

$$\ln\left(\frac{Q}{\lambda}\right) = \ln \gamma + \mathcal{O}(Q^{-1}) , \quad \text{near } \infty_Q \quad (2-97)$$

and thus

$$\oint_{\infty_Q} P [\ln(Q) - \ln(\lambda)] dQ = \oint_{\infty_Q} \left(V_1'(Q) - \frac{t}{Q} + \mathcal{O}(Q^{-2}) \right) [\ln(Q) - \ln(\lambda)] dQ = \quad (2-98)$$

$$= \oint_{\infty_Q} (V_1'(Q) + \mathcal{O}(Q^{-2})) [\ln(Q) - \ln(\lambda)] dQ + 2i\pi t \ln(\gamma) = \quad (2-99)$$

$$= - \oint_{\infty_Q} (V_1(Q) + \mathcal{O}(Q^{-1})) \left[\frac{dQ}{Q} - \frac{d\lambda}{\lambda} \right] + 2i\pi t \ln(\gamma) = \quad (2-100)$$

$$= -2i\pi (V_1(Q))_0 + 2i\pi t \ln(\gamma) . \quad (2-101)$$

Plugging this into (2-95) we obtain

$$2\partial_t \mathcal{F}_0 = -t\partial_t \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} Q dP \right] + \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} Q dP \right] \quad (2-102)$$

so that

$$2\partial_t \mathcal{F} = 2 \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} Q dP \right] . \quad (2-103)$$

Q.E.D.

2.2 Canonical transformations

In the Poisson structure (2-3) the coordinates $\ln(\lambda)$ and t are canonically conjugate, as well as the functions P, Q . Therefore it makes sense to find the generating function of these transformations. This was accomplished in the context of conformal maps in [30] but it is probably a new result in this context, since now we can express explicitly the generating function as integrals of the dToda operators P, Q . I recall that we are looking for a function $\Omega(Q, t)$ with the property that

$$d_{Q,t}\Omega = PdQ + \ln(\lambda)dt . \quad (2-104)$$

The following proposition gives a representation of such function

Proposition 2.1 The generating function of the canonical change of coordinates $(\ln(\lambda), t) \mapsto (P, Q)$ is given by

$$\Omega = \int_{X_q} PdQ - \frac{\mu}{2} = \int_{X_q} PdQ - \frac{1}{2} \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} PdQ \right] = \quad (2-105)$$

$$= \int_{X_q} PdQ + \frac{1}{2} (V_1(Q) + V_2(P) - PQ)_0 - t \ln(\gamma) \quad (2-106)$$

is the generating function of the canonical transformation $(\ln(\lambda), t) \rightarrow (P, Q)$, or,

$$d_{Q,t}\Omega = \partial_Q \Omega_1 dQ + (\partial_t \Omega)_Q dt = PdQ + \ln(\lambda)dt . \quad (2-107)$$

Remark 2.5 The function that we get by interchanging the rôle of P and Q would generate the transformation $(-\ln(\lambda), t) \mapsto (Q, P)$ (or the anti canonical one).

Proof. The first part is obvious

$$\partial_Q \Omega = P \quad \text{by definition.} \quad (2-108)$$

As for the second we compute

$$(\partial_t \Omega)_Q = \partial_t \left(V_1(Q) + t \ln(Q) - \frac{1}{2} \mu + \mathcal{O}(Q^{-1}) \right)_Q = \quad (2-109)$$

$$= \ln(Q) - \frac{1}{2} \partial_t \mu + \mathcal{O}(Q^{-1}) = \quad (2-110)$$

$$= \ln(\lambda) + \ln(\gamma) - \frac{1}{2} \partial_t \mu + \mathcal{O}(\lambda^{-1}) . \quad (2-111)$$

It is well known in the theory of the dToda tau-function that

$$\partial_t^2 \mathcal{F} = \partial_t \mu = 2 \ln(\gamma) , \quad (2-112)$$

and hence the constant term in the above expression vanishes. On the other hand

$$(\partial_t \Omega)_Q = \int_{X_q} (\partial_t P)_Q dQ - P(X_q) \partial_t(Q(X_q)) = \int_{X_q} \frac{d\lambda}{\lambda} - P(X_q) \partial_t(Q(X_q)) = \quad (2-113)$$

$$= \ln(\lambda) - \ln(\lambda(X_q)) - P(X_q) \partial_t(Q(X_q)) . \quad (2-114)$$

By comparison we conclude not only that $\partial_t \Omega = \ln(\lambda)$ but also

$$\ln(\lambda(X_q)) = -P(X_q) \partial_t(Q(X_q)) . \quad (2-115)$$

The proof of the Lemma is complete. Q.E.D.

3 Planar Free energy for spectral curves of arbitrary genus

The situation in the case the spectral curve is of higher genus is only slightly different. We will work with the following data: a (smooth) curve Σ_g of genus g is assigned with two marked points ∞_Q, ∞_P . On the curve we are given two functions P and Q which have the following pole structure;

1. The function Q has a simple pole at ∞_Q and a pole of degree d_2 at ∞_P .
2. The function P has a simple pole at ∞_P and a pole of degree d_1 at ∞_Q .

From these data it would follow that P, Q satisfy a polynomial relation but we will not need it for our computations. By their definition we have

$$P = \sum_{K=1}^{d_1+1} u_K Q^{K-1} - \frac{t}{Q} + \mathcal{O}(Q^{-2}) =: V_1'(Q) - \frac{t}{Q} + \mathcal{O}(Q^{-2}) , \quad \text{near } \infty_Q \quad (3-1)$$

$$Q = \sum_{J=1}^{d_2+1} v_J P^{J-1} - \frac{t}{P} + \mathcal{O}(P^{-2}) =: V_2'(P) - \frac{t}{P} + \mathcal{O}(P^{-2}) , \quad \text{near } \infty_P . \quad (3-2)$$

The fact that the coefficient of the power Q^{-1} or P^{-1} is the same follows immediately from computing the sum of the residues of PdQ (or QdP). The formulas for the coefficients u_K, v_J, t are the same as in the genus zero case, viz

$$u_K = -\operatorname{res}_{\infty_Q} P Q^{-K} dQ , \quad v_J = -\operatorname{res}_{\infty_P} Q P^{-J} dP , \quad t = \operatorname{res}_{\infty_Q} P dQ = \operatorname{res}_{\infty_P} Q dP . \quad (3-3)$$

Note that the requirement that the curve possesses two meromorphic functions with this pole structure imposes strong constraints on the moduli of the curve itself. In fact a Riemann-Roch argument shows that the moduli space of these data is of dimension $d_1 + d_2 + 3 + g$; so far our data show only $(d_1 + 1) + (d_2 + 1) + 1$ parameters and therefore we add as parameters the following period integrals which, in the matrix-model literature are referred to as *filling fractions*

$$\epsilon_i := \frac{1}{2i\pi} \oint_{a_i} P dQ , \quad i = 1, \dots, g. \quad (3-4)$$

Here we have introduced a symplectic basis $\{a_i, b_i\}_{i=1 \dots g}$ in the homology of the curve Σ_g and the choice of the a -cycles over the b -cycles is purely conventional.

In this extended setting the free energy is defined by the equations

$$\partial_{u_K} \mathcal{F}_g = \frac{1}{K} \operatorname{res}_{\infty_Q} P Q^K dQ , \quad \partial_{v_J} \mathcal{F}_g = \frac{1}{J} \operatorname{res}_{\infty_P} Q P^J dP , \quad (3-5)$$

$$\partial_{\epsilon_i} \mathcal{F}_g = \oint_{b_j} P dQ =: \Gamma_i. \quad (3-6)$$

Once more we introduce the exterior potentials by the same requirements as in the genus 0 case;

$$\Phi_1 = -\frac{1}{2i\pi} \oint \ln \left(1 - \frac{\tilde{Q}}{Q} \right) \tilde{P} d\tilde{Q} = -V_1(Q) + t \ln(Q) + \int_{X_q} P dQ = \mathcal{O}(Q^{-1}) , \quad \text{near } \infty_Q \quad (3-7)$$

$$\Phi_2 = -\frac{1}{2i\pi} \oint \ln \left(1 - \frac{\tilde{P}}{P} \right) \tilde{Q} d\tilde{P} = -V_2(P) + t \ln(P) + \int_{X_p} Q dP = \mathcal{O}(P^{-1}) , \quad \text{near } \infty_P . \quad (3-8)$$

In the loop integral formulas the contours wind around the marked points so as to leave the point where the potentials are computed inside, i.e., for instance, $\tilde{Q}/Q \ll 1$. As in the genus 0 case the two points X_p, X_q are implicitly defined by the requirement $\mathcal{O}(\text{local parameter})$. Recall the definition of the ‘‘chemical’’ potential

$$\mu = Q(X_p)P(X_p) + \int_{X_p}^{X_q} P dQ . \quad (3-9)$$

Theorem 3.1 The free energy over the curve Σ_g is given by

$$2\mathcal{F}_g = \text{res}_{\infty_Q} P \Phi_1 dQ + \text{res}_{\infty_P} Q \Phi_2 dP + \frac{1}{2} \text{res}_{\infty_Q} P^2 Q dQ + t\mu + \sum_{i=1}^g \epsilon_i \Gamma_i . \quad (3-10)$$

Proof. It is essentially the same as in the genus zero case. We start from the same expression \mathcal{F}_0 used in the proof there (2-57) and proceed to the variation w.r.t. u_K . Following the same steps we obtain the expression

$$4i\pi \partial_K \mathcal{F}_0 = 4i\pi U_k + \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ + \oint_{\infty_P} Q \left[\int_{X_q} (\partial_k Q)_P dP \right] dP + \oint_{\infty_Q} Q P d \left[\int_{X_q} (\partial_k P)_Q dQ \right] + (3-11)$$

$$-2i\pi t \underbrace{\left[P(X_q) \partial_K(Q(X_q)) + Q(X_p) \partial_K(P(X_p)) - \int_{X_p}^{X_q} (\partial_K Q)_P dP \right]}_{= \partial_K \mu} , \quad (3-12)$$

where the subscripts in the contour integrals specify the point around which we are circulating. The integration by parts of the third term gives

$$4i\pi \partial_K \mathcal{F}_0 = 4i\pi U_k + \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ + \oint_{\infty_P} Q \left[\int_{X_q} (\partial_k Q)_P dP \right] dP + (3-13)$$

$$- \oint_{\infty_Q} P \left[\int_{X_q} (\partial_k P)_Q dQ \right] dQ - \oint_{\infty_Q} Q \left[\int_{X_q} (\partial_k P)_Q dQ \right] dP - 2i\pi t \partial_K \mu = (3-14)$$

$$= 4i\pi U_k + \oint_{\infty_P} Q \left[\int_{X_q} (\partial_k Q)_P dP \right] dP + \oint_{\infty_Q} Q \left[\int_{X_q} (\partial_k Q)_P dP \right] dP - 2i\pi t \partial_K \mu , \quad (3-15)$$

where we have used the thermodynamical identity (2-70). Due to the genus of the curve the two contours are not homologically opposite and the sum of the two integrals gives finally (using the Riemann bilinear identity which we recall in Appendix A)

$$4i\pi \partial_K \mathcal{F}_0 = 4i\pi U_k - 2i\pi \sum_{i=1}^g \left[\oint_{a_i} P dQ \oint_{b_i} (\partial_K P)_Q dQ - \oint_{b_i} P dQ \oint_{a_i} (\partial_K P)_Q dQ \right] - 2i\pi t \partial_K \mu = (3-16)$$

$$= 4i\pi U_k - 2i\pi \sum_{i=1}^g \left[\epsilon_i \partial_K \Gamma_i - \Gamma_i \underbrace{\partial_K \epsilon_i}_{=0} \right] - 2i\pi t \partial_K \mu = (3-17)$$

$$= 4i\pi U_k - 2i\pi \sum_{i=1}^g \epsilon_i \partial_K \Gamma_i - 2i\pi t \partial_K \mu . \quad (3-18)$$

This implies promptly that

$$2\partial_K \mathcal{F}_g = 2\partial_K \mathcal{F}_0 + \sum_i \epsilon_i \partial_K \Gamma_i + t \partial_K \mu = 2U_K , \quad (3-19)$$

as desired. The check for the variation w.r.t. the filling fractions is the same by noticing that

$$(\partial_{\epsilon_i} P)_Q = \mathcal{O}(Q^{-2}) ; \quad (\partial_{\epsilon_i} Q)_P = \mathcal{O}(P^{-2}) , \quad (3-20)$$

near the corresponding marked points. Moreover, by definition

$$\oint_{a_j} (\partial_{\epsilon_j} P)_Q dQ = \delta_{ij} . \quad (3-21)$$

From this, following the same formal steps and using the Riemann bilinear identity one easily finds

$$2\partial_{\epsilon_i} \mathcal{F}_0 = \Gamma_i - \sum_{j=1}^g \epsilon_j \partial_{\epsilon_i} \Gamma_j - t \partial_{\epsilon_i} \mu , \quad (3-22)$$

$$2\partial_{\epsilon_i} \mathcal{F}_g = 2\partial_{\epsilon_i} \mathcal{F}_0 + t \partial_{\epsilon_i} \mu + \Gamma_i + \sum_{j=1}^g \epsilon_j \partial_{\epsilon_i} \Gamma_j = 2\Gamma_i . \quad (3-23)$$

This concludes the proof. Q.E.D.

Remark 3.1 In the two-matrix model setting the moduli of the spectral curve are fixed uniquely in terms of the potentials V_1, V_2 by the requirement $\Gamma_i = 0$, $i = 1, \dots, g$ (which is a minimum requirement)

Remark 3.2 The case of the one matrix model is a subcase of this setting where one of the two potentials is quadratic, say V_2 ; in this case the spectral curve is hyperelliptic of genus $[(d_1 + 2)/2]$.

As for the previous case some special care must be paid for the derivative w.r.t. t , but the result is the same as in the genus zero case and it is contained in the next corollary.

Corollary 3.1 The chemical potential μ is indeed the derivative $\partial_t \mathcal{F}_g$. Moreover we have the formula

$$\mu = \operatorname{res}_{\infty_Q} [V_1(Q) - t \ln(Q/\lambda)] dS - \operatorname{res}_{\infty_P} [V_2(P) - t \ln(P\lambda)] dS - \operatorname{res}_{\infty_Q} PQ dS + \sum_{i=1}^g \epsilon_i \oint_{b_i} dS , \quad (3-24)$$

where dS is the normalized differential of the third kind with poles at $\infty_{P,Q}$ and residues ± 1 and the function λ is the following function (defined up to a multiplicative constant) on the universal covering of the curve with a simple zero at ∞_Q and a simple pole at ∞_P

$$\lambda := \exp \left(\int dS \right) . \quad (3-25)$$

Proof. The proof is quite more involved than in genus zero and requires some preparation.

We introduce the normalized differential of the third kind with simple poles at $\infty_{P,Q}$ and residues ± 1 ,

$$dS = dS_{\infty_Q, \infty_P} , \quad \oint_{a_i} dS = 0 , \quad i = 1, \dots, g ; \quad \operatorname{res}_{\infty_Q} dS = -1 = - \operatorname{res}_{\infty_P} dS . \quad (3-26)$$

This provides a coordinate on the (covering of the) curve as follows

$$\lambda := e^{\int dS} . \quad (3-27)$$

Quite clearly the parameter λ is defined up to a multiplicative constant and it is multiplicatively multivalued on the curve Σ_g (around the b -cycles). This arbitrariness and multivaluedness will not affect our computations because

formula (3-26) remains unchanged under the rescaling $\lambda \mapsto \delta\lambda$. Moreover it has no branch-points at $\infty_{P,Q}$, where instead it has a simple zero and a simple pole (and no other zeroes or singularities). By the definition then

$$dS = d \ln(\lambda) . \quad (3-28)$$

The rest mimics the genus zero case. We introduce the two points $\Lambda_{p,q}$ such that

$$\operatorname{res}_{\infty_Q} \left(-V_1(Q) + t \ln(\lambda) + \int_{\Lambda_q} P dQ \right) dS = 0 , \quad (3-29)$$

$$\operatorname{res}_{\infty_P} \left(-V_2(P) - t \ln(\lambda) + \int_{\Lambda_p} Q dP \right) dS = 0 . \quad (3-30)$$

From the definition of the points $X_{p,q}$ and $\Lambda_{p,q}$ and following the same steps that were taken in genus 0 we have

$$\int_{\Lambda_q}^{X_q} P dQ = \operatorname{res}_{\infty_Q} (V_1(Q) - t \ln(Q/\lambda)) dS \quad (3-31)$$

$$\int_{\Lambda_p}^{X_p} Q dP = - \operatorname{res}_{\infty_P} (V_2(P) - t \ln(P/\lambda)) dS . \quad (3-32)$$

The next relation is different from the genus zero case:

$$P(\Lambda_q)Q(\Lambda_q) + \int_{\Lambda_q}^{\Lambda_p} Q dP = - \operatorname{res}_{\infty_Q} P Q dS + \sum_{i=0} \epsilon_i \oint_{b_i} dS . \quad (3-33)$$

Indeed we have

$$0 = \oint_{\infty_Q} \left(\int_{\Lambda_q} P dQ - t \ln \lambda \right) dS = \oint_{\infty_Q} \left[QP - Q(\Lambda_q)P(\Lambda_q) - \int_{\Lambda_q}^{\Lambda_p} Q dP - t \ln(\lambda) - \int_{\Lambda_p} Q dP \right] dS = \quad (3-34)$$

$$= 2i\pi \left(\operatorname{res}_{\infty_Q} QP dS \right) + 2i\pi \left[Q(\Lambda_q)P(\Lambda_q) + \int_{\Lambda_q}^{\Lambda_p} Q dP \right] - \underbrace{\oint_{\infty_P} \left(\int_{\Lambda_p} Q dP - t \ln \lambda \right) dS}_{=0} + \sum_{i=1}^g \epsilon_i \oint_{b_i} dS , \quad (3-35)$$

where we have used the bilinear Riemann identity as well as the fact that $\oint_{a_i} dS = 0$ and the fact that (by definition of the point Λ_p) the term with the under-brace is residue-free at ∞_P ³. We can now proceed as in Lemma 2.1 and obtain the formula

$$\begin{aligned} \mu &= Q(X_p)P(X_p) + \int_{X_p}^{X_q} P dQ = \\ &= \operatorname{res}_{\infty_Q} [V_1(Q) - t \ln(Q/\lambda)] dS - \operatorname{res}_{\infty_P} [V_2(P) - t \ln(P/\lambda)] dS - \operatorname{res}_{\infty_Q} P Q dS + \sum_{i=1}^g \epsilon_i \oint_{b_i} dS . \end{aligned} \quad (3-36)$$

The formula (3-36) is the equivalent in higher genus of the formula in Lemma 2.1.

Coming back to the proof of the Corollary we note as in Corollary 2.2 that

$$(\partial_t P)_Q dQ = -(\partial_t Q)_P dP = dS \quad (3-37)$$

is the *normalized* Abelian differential of the third kind with poles at the marked points for we have

$$0 = \partial_t \epsilon_i = \oint_{a_i} (\partial_t P)_Q dQ . \quad (3-38)$$

³Note that $QdP - t dS$ is a meromorphic Abelian differential without residues at the poles $\infty_{P,Q}$.

Computing the variation of \mathcal{F}_0 we now obtain

$$2\partial_t\mathcal{F}_0 = -t\partial_t \left[Q(X_p)P(X_p) + \int_{X_p}^{X_q} QdP \right] - \operatorname{res}_{\infty_Q} PQdS + \operatorname{res}_{\infty_Q} \ln\left(\frac{Q}{\lambda}\right) PdQ + \operatorname{res}_{\infty_P} \ln(P\lambda)QdP \quad (3-39)$$

We now have

$$\operatorname{res}_{\infty_Q} \ln\left(\frac{Q}{\lambda}\right) PdQ = \operatorname{res}_{\infty_Q} \left(V_1'(Q) - \frac{t}{Q} + \mathcal{O}(Q^{-2}) \right) \ln\left(\frac{Q}{\lambda}\right) dQ = \quad (3-40)$$

$$= \operatorname{res}_{\infty_Q} (V_1'(Q) + \mathcal{O}(Q^{-2})) \ln\left(\frac{Q}{\lambda}\right) dQ - t \operatorname{res}_{\infty_Q} \ln\left(\frac{Q}{\lambda}\right) \frac{dQ}{Q} = \quad (3-41)$$

$$= \operatorname{res}_{\infty_Q} (V_1(Q)dS - t \ln(Q/\lambda)dS) , \quad (3-42)$$

where, in the second residue, we can replace dQ/Q by dS because $\ln(Q/\lambda) = \mathcal{O}(1)$. A similar argument goes for the term involving P so that we finally have

$$2\partial_t\mathcal{F}_0 = -t\partial_t\mu - \operatorname{res}_{\infty_Q} PQdS + \operatorname{res}_{\infty_Q} (V_1(Q) - t \ln(Q/\lambda)) dS - \operatorname{res}_{\infty_P} (V_2(P) - t \ln(P\lambda)) dS . \quad (3-43)$$

Inserting this into the full expression of \mathcal{F}_g we obtain

$$2\partial_t\mathcal{F}_g = 2\partial_t\mathcal{F}_0 + \partial_t(t\mu) + \sum_{i=1}^g \epsilon_i \partial_t \Gamma_i = \quad (3-44)$$

$$= \mu - \operatorname{res}_{\infty_Q} PQdS + \operatorname{res}_{\infty_Q} (V_1(Q) - t \ln(Q/\lambda)) dS - \operatorname{res}_{\infty_P} (V_2(P) - t \ln(P\lambda)) dS + \sum_{i=1}^g \epsilon_i \oint_{b_i} dS = \quad (3-45)$$

$$= 2\mu , \quad (3-46)$$

where we have used the expression (3-36). This concludes the proof of the corollary. Q.E.D.

Remark 3.3 The formula for μ seems not symmetric in the rôles of P and Q only superficially. In fact, if we exchanged the rôles, the 3^{rd} kind differential should also change sign.

We now investigate the scaling property of this Free energy. First off we have the simple

Lemma 3.1 *Under the change of scale for the functions $Q = \delta\tilde{Q}$ and $P = \sigma\tilde{P}$ the chemical potential rescales with an anomaly as follows*

$$\mu = \delta\sigma\tilde{\mu} + \delta\sigma\tilde{t} \ln(\delta\sigma) . \quad (3-47)$$

Proof. The proof is almost immediate from the expression (3-24) considering the fact that under that rescaling we have

$$u_K = \sigma\delta^{1-K}\tilde{u}_K ; \quad v_J = \delta\sigma^{1-J}\tilde{v}_J ; \quad t = \delta\sigma\tilde{t} , \epsilon_i = \delta\sigma\tilde{\epsilon}_i . \quad (3-48)$$

On the other hand the differential dS (and hence the function λ) are invariant as follows from its expression

$$dS = (\partial_t P)_Q dQ = \partial_i \tilde{P} d\tilde{Q} . \quad (3-49)$$

The proof follows then immediately from (3-24). Q. E. D.

With the above lemma we can immediately find the scaling properties of the genus g free energy

Corollary 3.2 [Scaling properties] The Free energy of the genus g data above satisfies the scaling constraints

$$2\mathcal{F}_g = \left[\sum_K (1-K)u_K \partial_{u_K} + \sum_J v_J \partial_{v_J} + \sum_{i=1}^g \epsilon_i \partial_{\epsilon_i} + t\partial_t \right] \mathcal{F}_g - \frac{1}{2}t^2 , \quad (3-50)$$

$$2\mathcal{F}_g = \left[\sum_K u_K \partial_{u_K} + \sum_J (1-J)v_J \partial_{v_J} + \sum_{i=1}^g \epsilon_i \partial_{\epsilon_i} + t\partial_t \right] \mathcal{F}_g - \frac{1}{2}t^2 . \quad (3-51)$$

Proof. The proof is immediate from the definitions of the various objects and using the anomalous scaling of the chemical potential μ in Lemma 3.1. The constraint (3-50) is obtained by keeping $\sigma = 1$ and differentiating w.r.t. δ at $\delta = 1$, and (3-51) is obtained similarly by interchanging the rôles of δ and σ in the above procedure. Q.E.D.

The two scaling constraints (3-50, 3-51) form the “two halves” of the scaling

$$4\mathcal{F}_g = \left[\sum_K (2-K)u_K \partial_{u_K} + \sum_J (2-J)v_J \partial_{v_J} + 2 \sum_{i=1}^g \epsilon_i \partial_{\epsilon_i} + 2t \partial_t \right] \mathcal{F}_g - t^2 , \quad (3-52)$$

(obtained by adding (3-50) and (3-51)) which is the translation of the well-known property (2-44) for higher genus Free energies. Of course the same properties (3-50, 3-51) hold also for the free energy in Section 2.1 (in which case the part involving the filling fraction would be missing).

We conclude with the remark that -quite clearly- the (multivalued) function λ is playing essentially the same rôle of the uniformizing parameter in genus zero. The quantities

$$\ln(\gamma) := - \operatorname{res}_{\infty_Q} \ln(Q/\lambda) dS , \quad \ln(\tilde{\gamma}) := \operatorname{res}_{\infty_P} \ln(P\lambda) dS , \quad (3-53)$$

are the translation in this setting of the homonymous quantity in the genus zero case, except that they need not be equal. However, since now $\lambda = \exp \int dS$ is defined up to a multiplicative constant depending on the base-point of the integral, there would be a choice for the base-point which makes $\gamma = \tilde{\gamma}$.

4 Conclusion

The formulas we have presented fill a gap in both the theory of the dispersionless Toda hierarchy and the two-matrix model, where the tau-function (free energy in the planar limit) is known only through its partial derivatives but no closed formula for the tau-function itself is known.

The derivation and the technique of the proof emphasizes the importance of the spectral curve of the model, at least in the case of polynomial potentials or, in the dToda language, finite Laurent polynomials for the Lax operators.

On a slightly different perspective, we have computed the free energy of the matrix model in the case where the spectral curve is of genus $g > 0$; the computation is less explicit than in genus zero but the formula is closed.

It would be interesting to explore further the remnant of the Poisson structure (2-3) in this context. In fact it is almost immediate to verify that still

$$\{P, Q\} = \lambda \partial_\lambda P \partial_t Q - \lambda \partial_\lambda Q \partial_t P = 1. \quad (4-1)$$

The investigation of the Poisson structure will be the topic of subsequent publications.

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A The Riemann bilinear identity

It is a classical identity but it may be useful to recall it here. Let be given a curve Σ_g of genus g and a symplectic basis in the homology $\{a_i, b_i\}_{i=1\dots g}$. Let η and ω be two meromorphic Abelian differentials and ω be without residues and define

$$\Omega = \int_P \omega . \quad (A-1)$$

Let $\tilde{\Sigma}_g$ be a simply connected fundamental domain on the universal covering of the curve then the function Ω is single-valued inside $\tilde{\Sigma}_g$ with possibly poles. Let $\partial\tilde{\Sigma}_g$ be the boundary of the domain constituted by the cycles of the chosen basis. Then the bilinear Riemann identity claims

$$2i\pi \sum_{\text{residues}} \eta\Omega = \sum_{i=1}^g \left[\oint_{a_i} \eta \oint_{b_i} \omega - \oint_{a_i} \omega \oint_{b_i} \eta \right]. \quad (\text{A-2})$$

The proof is very easy and can be found in [12]

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