Midterm Exam Emat 213

February 2006

Instructor: Dr. Marco Bertola

Time allowed: 1h15min.

Material allowed: calculators.

Recommendations: **use only blue or black ink**

Solve four problems: the top score is 40 points
Clearly indicate which problems you wish marked

[10 points] Problem 1.

Find the general solution of the following separable ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(y^2+1\right)^{\frac{2}{3}}}{y} x^2 \ .$$

Solution

$$\int \frac{y}{(y^2+1)^{\frac{2}{3}}} \mathrm{d}y = \int x^2 dx$$

$$\frac{3}{2}(y^2+1)^{\frac{1}{3}} = \frac{x^3}{3} + C$$

[10 points] Problem 2.

Determine which of the following ODE's is exact and then solve it (in implicit form).

$$(a) \quad (x^2+y^2+\mathrm{e}^{y+x})\mathrm{d}x - \left(\mathrm{e}^{y+x}+y^3\right)\mathrm{d}y = 0$$

$$(b) \quad \left(2x - 4\mathrm{e}^{2x+3y}\right)\mathrm{d}x + \left(3y^2 - 6\mathrm{e}^{2x+3y}\right)\mathrm{d}y = 0$$

Solution (just answer)

The second is exact (because the cross derivatives match). The solution reads (implicitly)

$$x^2 - 2\mathrm{e}^{2x+3y} + y^3 = C$$

[10 points] Problem 3.

(i) Which of the following two first order equations is linear? **Explain** why the other is not linear

$$(a) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 + \frac{2}{x}\right) y = \frac{\mathrm{e}^{-x}}{x^2}$$

$$(b) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} - x = \cos(y)$$

(ii) Find the general solution of the linear equation that you have found above.

Solution

The second is nonlinear because the unknown function y is precomposed with cos.

The solution of the linear one is

$$y = \frac{C}{x^2}\mathrm{e}^{-x} + \frac{\mathrm{e}^{-x}}{x}$$

[10 points] Problem 4.

Perform the substitution $u = y^{-1}$ in the following Bernoulli ODE so as to obtain a new linear ODE in the dependent variable u and find the general solution in term of y

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} - xy = y^2 \ .$$

Solution

$$\begin{aligned} y = \frac{1}{u} \qquad &; \quad y' = -\frac{u'}{u^2} \\ -x^2\frac{u'}{u^2} - \frac{x}{u} &= \frac{1}{u^2} \\ u' + \frac{1}{x}u &= -\frac{1}{x^2} \\ u_c &= \frac{C}{x} \\ u_p = -\frac{1}{x}\int x\frac{1}{x^2}dx &= -\frac{\ln|x|}{x} \\ u &= \frac{C}{x} - \frac{\ln|x|}{x} \\ y = \frac{1}{\frac{C}{x} - \frac{\ln|x|}{x}} &= \frac{x}{C - \ln|x|} \end{aligned}$$

[10 points] Problem 5.

A cake is removed from an oven at a temperature of 150 degrees (Celsius) and left to cool down on a countertop in a room at a temperature of 20 degrees. After 2 minutes the temperature of the cake is 120 degrees. After how many minutes (approximately) the cake's temperature will have dropped to 40 degrees?

Solution

The equation of Newton says

$$T(t) = C\mathrm{e}^{-kt} + T_{env}$$

We have

$$\begin{aligned} T_{env} = 20 \ ; \quad T(0) = 150 \quad &\Longrightarrow \quad C + 20 = 150 \ ; C = 130; \\ T(2) = 120 \quad &\Longrightarrow \quad 130\mathrm{e}^{-2k} = 100 \\ &\qquad k = -\frac{1}{2}\ln\left(\frac{10}{13}\right) \end{aligned}$$

The solution is obtained by solving for t the following equation

$$T(t) = 40$$

$$130\exp\left(\frac{1}{2}\ln\left(\frac{10}{13}\right)t\right)+20=40$$
$$t=2\frac{\ln\left(\frac{2}{13}\right)}{\ln\left(\frac{10}{13}\right)}\simeq 14.25\ mins$$

[10 points] Problem 6.

Solve the following equation by using a substitution of the form $u = Ax + By + C$

$$y' = (-2x+y)^2 - 7$$

Solution

$$\begin{aligned}
u &= -2x + y \\
y &= u + 2x \\
y' &= u' + 2 \\
u' + 2 &= u^2 - 7 \\
u' &= u^2 - 9 \\
\int \frac{du}{u^2-9} &= \int dx = x + C \\
\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| &= x + C \\
\text{(by partial fraction expansion ....)} \\
\frac{u-3}{u+3} &= \tilde{C}e^{6x} \\
\frac{y-2x-3}{y-2x+3} &= \tilde{C}e^{6x}
\end{aligned}$$