Midterm Exam Emat 213

October 2005

Instructor: ____________

Time allowed: 1h15min **sharp**.

Material allowed:no calculators.

[10 points] Problem 1.

Consider the two following first-order ODE's:

$$(a) \quad \frac{dy}{dx} - 2y = 2x^2$$
$$(b) \quad \frac{dy}{dx} - 2x = 2y^2 \ .$$

Find the linear one and **explain** why the other is not linear.
Find the general solution for the linear equation and solve also the IVP, $y(0) = 2$.

Solution The second equation is not linear because the dependent variable y is raised to the second power.

The first equation is linear.

$$y_c = C\mathrm{e}^{2x}$$

For the particular solution we use the Ansatz $y_p = Ax^2 + Bx + C$; this gives

$$2Ax + B - 2Ax^2 - 2Bx - 2C = -2Ax^2 + 2(A - B)x + B - 2C = 2x^2$$

From which we read $A = -1$, $B = A = -1$, $C = \frac{1}{2}B = -\frac{1}{2}$. The general solution is

$$y = C\mathrm{e}^{2x} - x^2 - x - \frac{1}{2} \ .$$

The IVP solution is

$$y(0) = C - \frac{1}{2} = 2 \ \Rightarrow \ C = \frac{5}{2}$$
$$y_{IVP} = \frac{5}{2}\mathrm{e}^{2x} - x^2 - x - \frac{1}{2}$$

[10 points] Problem 2.

(i) Determine which of the following ODE's is exact.

$$\begin{aligned}
(a) &\quad y^4 \mathrm{d}x - (x^2\cos(y))\mathrm{d}y = 0\\
(b) &\quad (\mathrm{e}^{-xy} + \cos(x))\mathrm{d}x + (x\mathrm{e}^{-xy} - 1)\mathrm{d}y = 0\\
(c) &\quad (2\,xy - ye^{xy})\mathrm{d}x + (x^2 + 4\,y^3 - xe^{xy})\mathrm{d}y = 0
\end{aligned}$$

(ii) Solve the exact ODE in point (i) by expressing (as always) the solution in implicit form and also find the solution of the IVP

$$y(0) = 3\ .$$

Solution We check for exactness

$$\begin{aligned}
&(\mathbf{a}):\ \partial_y(y^4) = 4y^3\ , \qquad \partial_x(-x^2\cos(y)) = -2x\cos(y) \text{ not equal, hence not exact}\\
&(\mathbf{b}):\ \partial_y(\mathrm{e}^{-xy} + \cos(x)) = -x\mathrm{e}^{xy}\ , \qquad \partial_x(x\mathrm{e}^{-xy} - 1) = \mathrm{e}^{-xy} - xy\mathrm{e}^{xy} \text{ not equal, not exact}\\
&(\mathbf{c}):\ \partial_y(2\,xy - ye^{xy}) = 2x - \mathrm{e}^{xy} - xy\mathrm{e}^{xy}\ , \qquad \partial_x(x^2 + 4\,y^3 - xe^{xy}) = 2x - \mathrm{e}^{xy} - xy\mathrm{e}^{xy} \text{ exact!}
\end{aligned}$$

So we solve (**c**). We have

$$\begin{aligned}
F(x,y) &= \int (2\,xy - ye^{xy})\mathrm{d}x = x^2y - \mathrm{e}^{xy} + h(y)\\
&\partial_y F(x,y) = x^2 - x\mathrm{e}^{xy} + h'(y) = x^2 + 4\,y^3 - xe^{xy} \quad \Rightarrow \quad h'(y) = 4y^3\\
&h(y) = y^4\\
F(x,y) &= x^2y - \mathrm{e}^{xy} + y^4
\end{aligned}$$

The general solution is thus

$$F(x,y) = x^2y - \mathrm{e}^{xy} + y^4 = C$$

The solution of the IVP is

$$\begin{aligned}
F(0,3) = C = 81 - 1 = 80\\
x^2y - \mathrm{e}^{xy} + y^4 = 80.
\end{aligned}$$

[10 points] Problem 3.

(i) Perform the substitution $y = u^{-1}$ in the following Bernoulli equation and reduce it to a linear ODE.

$$\frac{dy}{dx} - y = \mathrm{e}^x y^2\ .$$

(ii) Perform the substitution $y = x\,u$ in the ODE which is homogeneous among the following two and reduce it to a separable equation for the unknown function u (**it is not required that you solve it**).

$$\begin{aligned} &(a) \quad (y^2 - 3xy)\mathrm{d}x - x^2\mathrm{d}y = 0 \\ &(b) \quad (2yx - x^3)\mathrm{d}x - y\mathrm{d}y = 0 \; . \end{aligned}$$

Solution

(i)

$$\begin{aligned} &y' = -\frac{1}{u^2}u' \\ &-\frac{u'}{u^2} - \frac{1}{u} = \mathrm{e}^x \frac{1}{u^2} \\ &u' + u = -\mathrm{e}^x \\ &u = C\mathrm{e}^{-x} - \mathrm{e}^{-x}\int \mathrm{e}^x\mathrm{e}^x\mathrm{d}x = C\mathrm{e}^{-x} - \mathrm{e}^{-x}\int \mathrm{e}^{2x}\mathrm{d}x = \\ &= C\mathrm{e}^{-x} - \frac{1}{2}\mathrm{e}^{-x}\mathrm{e}^{2x} = C\mathrm{e}^{-x} - \frac{1}{2}\mathrm{e}^x \end{aligned}$$

So we have

$$y = \frac{1}{u} = \frac{1}{C\mathrm{e}^{-x} - \frac{1}{2}\mathrm{e}^x}$$

(ii) The first equation is homogeneous of degree 2; the second is not homogeneous.

$$\begin{aligned} &y = xu \\ &dy = x\,du + u\,dx \\ &(y^2 - 3xy)\mathrm{d}x - x^2\mathrm{d}y = \\ &= (u^2x^2 - 3x^2u)\mathrm{d}x - x^2(x\,du + u\,dx) = x^2(u^2 - 3u - u)\mathrm{d}x - x^3\mathrm{d}u \end{aligned}$$

Hence the ODE becomes

$$\frac{1}{x}\mathrm{d}x - \frac{1}{u^2 - 2u}\mathrm{d}u = 0$$
$$\int \frac{1}{u^2 - 2u}\mathrm{d}u = \int \frac{\mathrm{d}x}{x}$$

[10 points] Problem 4.

Find the orthogonal trajectories to the family of curves

$$x^4 + 2y^4 = C \;, \qquad (C > 0)$$

Solution
The implicit derivative gives

$$\begin{aligned} 4x^3\mathrm{d}x + 4y^3\mathrm{d}y &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x^3}{y^3} \end{aligned}$$

The orthogonal family has ODE (separable)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^3}{x^3} \\ \int \frac{\mathrm{d}y}{y^3} &= \int \frac{\mathrm{d}x}{x^3} \\ -\frac{1}{3y^2} &= -\frac{1}{3x^2} + C \\ \frac{1}{x^3} - \frac{1}{y^3} &= \tilde{C} \end{aligned}$$