Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions[1]

M. Bertola[†‡2] , B. Eynard[†⋆3] and J. Harnad[†‡4]

[†] *Centre de recherches mathématiques, Université de Montréal*
C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7

[‡] *Department of Mathematics and Statistics, Concordia University*
7141 Sherbrooke W., Montréal, Québec, Canada H4B 1R6

[⋆] *Service de Physique Théorique, CEA/Saclay*
Orme des Merisiers F-91191 Gif-sur-Yvette Cedex, FRANCE

Abstract

The differential systems satisfied by orthogonal polynomials with arbitrary semiclassical measures supported on contours in the complex plane are derived, as well as the compatible systems of deformation equations obtained from varying such measures. These are shown to preserve the generalized monodromy of the associated rank-2 rational covariant derivative operators. The corresponding matrix models, consisting of unitarily diagonalizable matrices with spectra supported on these contours are analyzed, and it is shown that all coefficients of the associated spectral curves are given by logarithmic derivatives of the partition function or, more generally, the gap probabilities. The associated isomonodromic tau functions are shown to coincide, within an explicitly computed factor, with these partition functions.

Contents

1 Introduction

The partition function for Hermitian random matrix models with measures that are exponentials of a polynomial potential was shown in [4] to be equal, within a multiplicative factor independent of the deformation parameters, to the Jimbo-Miwa–Ueno isomonodromic tau function [11] for the rank 2 linear differential system satisfied by the corresponding set of orthogonal polynomials. The results of [4] were in fact more general, in that polynomials orthogonal with respect to complex measures supported along certain contours in the complex plane were considered. These may be viewed as corresponding to unitarily diagonalizable matrix models in which the spectrum is constrained to lie on these contours.

The purpose of the present work is to extend these considerations to the more general setting of complex measures whose logarithmic derivatives are arbitrary *rational* functions, the associated *semiclassical* orthogonal polynomials and generalized matrix models. By also including contours with endpoints, the latter viewed as further deformation parameters, the gap probability densities are included as special cases of partition functions.

To place the results in context, we first briefly recall the main points of [4], restricting to the more standard case of Hermitian matrices and real measures. Consider orthogonal polynomials $\pi_n(x) \in L^2(\mathbb{R}, \mathrm{e}^{-\hbar^{-1}V(x)}dx)$ supported on the real line, with the measure defined by exponentiating a real polynomial potential

$$V(x) = \sum_{J=1}^{d} \frac{t_J}{J} x^J \ . \tag{1-1}$$

(Here we assume $V(x)$ is of even degree and with positive leading coefficient, although these restrictions are unnecessary in the more general setting of [4].) The small parameter $\hbar$ is usually taken as $\mathcal{O}(N^{-1})$ when considering the limit $N \to \infty$.

Any two consecutive polynomials satisfy a first order system of ODE's

$$\hbar\frac{d}{dx} \begin{pmatrix} \pi_{n-1}(x) \\ \pi_n(x) \end{pmatrix} = \mathcal{D}_n(x) \begin{pmatrix} \pi_{n-1}(x) \\ \pi_n(x) \end{pmatrix}, \tag{1-2}$$

where $\mathcal{D}_n(x)$ is a 2×2 matrix with polynomial coefficients of degree at most $d - 1 = \deg(V'(x))$. The infinitesimal deformations corresponding to changes in the coefficients $\{t_J\}$ result in a sequence of Frobenius compatible, overdetermined systems of PDE's

$$\hbar\frac{\partial}{\partial t_J} \begin{pmatrix} \pi_{n-1}(x) \\ \pi_n(x) \end{pmatrix} = T_{n,J}(x) \begin{pmatrix} \pi_{n-1}(x) \\ \pi_n(x) \end{pmatrix} \quad J = 1, \ldots, d, \tag{1-3}$$

2

where the matrices $T_{n,J}(x)$ are polynomials in x of degree J which satisfy the compatibility conditions

$$\left[\hbar \frac{\partial}{\partial t_J} - T_{n,J}(x), \hbar \frac{d}{dx} - \mathcal{D}_n(x) \right] = 0 \ . \tag{1-4}$$

It follows that the generalized monodromy data of the sequence of rational covariant derivative operators $\hbar \frac{d}{dx} - \mathcal{D}_n(x)$ are invariant under these deformations, and independent of the integer n.

This is a particular case of the general problem of rational isomonodromic deformation systems [11]. An important rôle is played in this theory by the *isomonodromic tau function* τ_n^{IM} associated with any solution of an isomonodromic deformation system. This function on the space of deformation parameters is obtained by integrating a closed differential whose coefficients are given by residues involving the fundamental solutions of the system. The main results of [4] were the following. First, the coefficients of the associated *spectral curve*, given by the characteristic equation of the matrix $\mathcal{D}_n(x)$, can be obtained by applying certain first order differential operators with respect to the deformation parameters (Virasoro generators) to $\ln(\mathcal{Z}_n(V))$, where the partition function $\mathcal{Z}_n(V)$ of the associated $n \times n$ matrix model is

$$\mathcal{Z}_n(V) := \int_{\mathcal{H}_n} dM \ \exp\left(-\frac{1}{\hbar} \mathrm{Tr} V(M) \right) \ . \tag{1-5}$$

Second, this partition function is equal to the isomonodromic tau function up to a multiplicative factor that does not depend on the deformation parameters

$$\tau_n^{IM} = \mathcal{Z}_n(V) \, \mathcal{F}_n. \tag{1-6}$$

The present work generalizes these results to the case of measures whose logarithmic derivatives are arbitrary rational functions, including those supported on curve segments in which the endpoints may play the rôle of further deformation parameters. The latter are of importance in the calculation of gap probabilities in matrix models [17] since these may, in this way, be put on the same footing as partition functions using measures supported on such segments [6]. A Frobenius compatible system of first order differential and deformation equations satisfied by the corresponding orthogonal polynomials is derived (Propositions 3.2, 3.4) and the coefficients of the associated spectral curve are again shown to be obtained by applying suitable Virasoro generators to $\ln(\mathcal{Z}_N(V))$ (Theorems 4.1–4.2). A formula that generalizes (1-6) is also derived (Theorem 5.1):

$$\tau_n^{IM} = \mathcal{Z}_n(V) \mathcal{F}_n(V) \ , \tag{1-7}$$

where the factor $\mathcal{F}_n(V)$ is an explicitly computed function of the deformation parameters determining V, which can in fact be eliminated by a making a suitable scalar gauge transformation.

The results of Theorems 4.1, 4.2 give a precise meaning, for finite n, to formulæ that are usually derived in the asymptotic limit $n \to \infty$ ($\hbar n \sim \mathcal{O}(1)$) through saddle point computations, relating the free energy to the asymptotic spectral curve. It is well known [9] that the free energy in the large n limit is given by solving a minimization problem (in the Hermitian matrix model)

$$F_0 := - \lim_{n \to \infty} \hbar^2 \ln \mathcal{Z}_n = \min_{\rho(x) \geq 0} \left[\int V(x)\rho(x)dx - \int\int \rho(x)\rho(x') \ln|x - x'| \right] \ , \tag{1-8}$$

giving the equilibrium density ρ_{eq} for the eigenvalue distribution. If, for instance, the potential is a real polynomial bounded from below, it is known [8] that the support of the equilibrium density is a union of finite segments $I \subset \mathbb{R}$. The density ρ_{eq} is obtained from the variational equation

$$2\mathcal{P} \int \frac{\rho_{\mathrm{eq}}(x)dx}{x - x'} = V'(x) \tag{1-9}$$

(for $x \in I$), and is related to the *resolvent* by

$$\omega(z) := \hbar \lim_{n \to \infty} \left\langle \mathrm{Tr} \frac{1}{M - z} \right\rangle = \int_I dx \frac{\rho_{\mathrm{eq}}(x)}{x - z} \ , \quad z \in \mathbb{C} \setminus I \ . \tag{1-10}$$

In terms of this, the spectral density may be recovered as the jump-discontinuity of $\omega(z)$ across I, and all its moments are given by

$$\partial_{t_J} F_0 = \frac{1}{J} \lim_{n \to \infty} \hbar \left\langle \mathrm{Tr} M^J \right\rangle = \int_I dx \frac{x^J}{J} \rho_{\mathrm{eq}}(x) = - \operatorname*{res}_{z = \infty} \frac{z^J}{J} \omega(z) dz \ . \tag{1-11}$$

The function $y = -\omega(x)$ satisfies an algebraic relation given by

$$y^2 = y V'(x) + R(x) \ , \tag{1-12}$$

where $R(x)$ is a polynomial of degree less than $V'(x)$ that is uniquely determined by the consistency of (1-9) and (1-12).

The point to be stressed here is that this asymptotic spectral curve should be compared with the spectral curve of Theorem 4.1, given by the characteristic equation (4-28), which also contains all the relevant information about the finite n case. In the $n \to \infty$ limit, logarithmic derivatives of the partition function are expressed in (1-11) as residues of the meromorphic differentials $z^k y dz$ on the curve. The *same* formulæ are shown in Theorem 4.2 to hold as *exact* relations for the finite n case if we replace the "asymptotic" spectral curve by the spectral curve given by the characteristic equation of the matrix $\mathcal{D}_n(x)$.

The paper is organized as follows. In section 2 the problem is defined in terms of polynomials orthogonal with respect to an arbitrary semiclassical measure supported on complex contours, and the corresponding generalized matrix model partition functions. In section 2.2 the recursion relations, differential systems and deformation equations which these satisfy are expressed in terms of the semi-infinite "wave vector" formed from the orthogonal polynomials. In section 3 the notion of "folding" is introduced and used (Propositions 3.2–3.4) to express the preceding equations as an infinite sequence of compatible overdetermined 2×2 systems of linear differential equations and recursion relations satisfied by pairs of consecutive orthogonal polynomials. In section 4 the results of folding are used to express the spectral curve in terms of logarithmic derivatives of the partition function and it is shown that the $n \to \infty$ relation between the free energy and the spectral curve is also valid as an exact result for finite n. In section 5 the definition of the isomonodromic tau function [11] is recalled and it is computed by relating it to the spectral invariants of the rational matrix generalizing $\mathcal{D}_n(x)$ in (1-2). These invariants are shown to give the logarithmic derivatives of the tau functions in terms of residues of meromorphic differentials on the spectral curves through formulæ that are nearly identical to those for the partition function, This leads to the main result, Theorem 5.1, which gives the explicit relation between $\mathcal{Z}_n$ and τ_n^{IM}.

2 Generalized orthogonal polynomials and partition functions

2.1 Orthogonality measures and integration contours

Given a measure on the real line, the associated orthogonal polynomials are those that diagonalize the quadratic form associated to the corresponding (complex) *moment functional*; i.e., the linear form over the space of polynomials obtained by integration with respect to the measure.

$$\begin{aligned} \mathcal{L} : \mathbb{C}[x] &\to \mathbb{C} \ . \\ p(x) &\mapsto \mathcal{L}(p(x)) = \int_{\mathbb{R}} p(x) d\mu(x) \ . \end{aligned} \tag{2-1}$$

A natural generalization consists of including moment functionals that are expressed by integration along more general contours in the complex x plan, with respect to a complex measure defined by locally analytic weight functions that may have isolated essential singular points and complex power-like branch points.

We thus consider linear forms on polynomials given by integrals of the form

$$\mathcal{L}(p(x)) = \int_{\varkappa} p(x)\mu(x)dx$$
$$\mu(x) = e^{-\frac{1}{\hbar}V(x)}$$
$$V(x) := \sum_{r=0}^{K} T_r(x) \ , \tag{2-2}$$

where

$$T_0(x) := t_{0,0} + \sum_{J=1}^{d_0} \frac{t_{0,J}}{J}x^J$$

$$T_r(x) := \sum_{J=1}^{d_r} \frac{t_{r,J}}{J(x-c_r)^J} - t_{r,0}\ln(x-c_r)$$

$$-\hbar\partial_x \ln \mu(x) = V'(x) = \sum_{J=1}^{d_0-1} t_{0,J}x^{J-1} - \sum_{r=1}^{K}\sum_{J=1}^{d_r+1} \frac{t_{r,J-1}}{(x-c_r)^J} \ , \tag{2-3}$$

and the symbol $\int_{\varkappa}$ denotes integration over linear combinations of contours on which the integrals are convergent, as explained below. This class of linear functionals is sometimes referred to as *semiclassical* moment functionals [3, 14, 15]. We consider the corresponding monic generalized orthogonal polynomials $p_n(x)$, which satisfy

$$\int_{\varkappa} p_n(x)p_m(x)\mu(x)dx = h_n\delta_{nm} \ . \qquad , h_n \in \mathbb{C}\setminus\{0\} \ . \tag{2-4}$$

If all the contours are contained in the real axis and the weight is real and positive, we reduce to the usual notion of semiclassical orthogonal polynomials. The small parameter $\hbar$ introduced in (2-2) is not of essential importance here; it is only retained in the formulæ below to recall that, when taking the large n limit, it plays the rôle of small parameter for which $\hbar n$ remains finite as $n \to \infty$.

To describe the contours of integration, we first define sectors $S_r^{(j)}$, $r = 0, \ldots K$, $k = 1, \ldots d_r$. around the points c_r for which $d_r > 1$ ($c_0 := \infty$) in such a way that

$$\Re(V(x)) \underset{\substack{x \to c_r, \\ x \in S_r^{(j)}}}{\longrightarrow} +\infty \ . \tag{2-5}$$

The number of sectors for each pole in V is equal to the degree of that pole; that is, d_0 for the pole at infinity and d_r for the pole at c_r. Explicitly

$$S_k^{(0)} := \left\{ x \in \mathbb{C}; \ \frac{2k\pi - \arg(t_{0,d_0}) - \frac{\pi}{2}}{d_0} < \arg(x) < \frac{2k\pi - \arg(t_{0,d_0}) + \frac{\pi}{2}}{d_0} \right\}, \tag{2-6}$$

$$k = 0 \ldots d_0 - 1 \ ;$$

$$S_k^{(r)} := \left\{ x \in \mathbb{C}; \ \frac{2k\pi + \arg(t_{r,d_r}) - \frac{\pi}{2}}{d_r} < \arg(x - c_r) < \frac{2k\pi + \arg(t_{r,d_r}) + \frac{\pi}{2}}{d_r} \right\}, \tag{2-7}$$

$$k = 0, \ldots, d_r - 1, \quad r = 1, \ldots, K \ .$$

These sectors are defined precisely so that approaching within them any of the essential singularities of $\mu(x)$ (i.e. a c_r such that $d_r > 0$), the function $\mu(x)$ tends to zero faster than any power of the local parameter.

2.1.1 Definition of the boundary-free contours

The definition of the contours follows [16] (see fig. 1).

1. For any c_r for which there is *no essential singularity* in the measure (i.e. $d_r = 0$), there are two subcases:

   (a) For the c_r's that are branch points or poles in μ (i.e., $t_{r,0} \notin \mathbb{N}$), we take a loop starting at infinity in some fixed sector $S_k^{(0)}$ encircling the singularity and going back to infinity in the same sector. (Note that if c_r is just a pole; i.e., $t_{r,0} \in -\mathbb{N}^+$, the contour could equivalently be taken as a circle around c_r.)

   (b) For the c_r's that are regular points ($t_{r,0} \in \mathbb{N}$), we take a line joining c_r to infinity, approaching ∞ in a sector $S_k^{(0)}$ as before.

2. For any c_r for which there is an essential singularity in μ (i.e. , $d_r > 0$) we define d_r contours starting from c_r in the sector $S_k^{(r)}$ and returning to it in the next sector $S_{k+1}^{(r)}$. Also, if $t_{r,0} \notin \mathbb{Z}$, we join the singularity c_r to ∞ by a path approaching ∞ within one fixed sector $S_k^{(0)}$.

3. For $c_0 := \infty$, we take $d_0 - 1$ contours starting at c_0 in the sector $S_k^{(0)}$ and returning at c_0 in the next sector $S_{k+1}^{(0)}$.

Note that, with these definitions, the integrals involved are convergent and we can perform integration by parts. Moreover, any contour in the complex plane for which the integral of $\mu(x)p(x)dx$ is convergent for all polynomials $p(x)$ is equivalent to a linear combination of the contours defined above, no two of which are, in this sense, equivalent.

2.1.2 Definition of the hard-edge contours

We also include some additional contours in the complex plane $\{m_j\}_{j=1,\ldots,L}$, starting at some points a_j, $j = 1 \ldots L$ and going to ∞ within one of the sectors $S_k^{(0)}$. These could be viewed as corresponding to additional points in 1(b) for which both $d_r = 0$ and $t_{r,0} = 0$, but we prefer to deal with them separately since integration by parts on these contours does give a contribution.

In total there are $S := d_0 + \sum_{r=1}^{K}(d_r + 1)$ boundary-free contours σ_ℓ, $, \ell = 1, \ldots, S$ and L hard-edge contours m_h, $h = 1, \ldots, L$. The moment functional is an arbitrary linear combination of integrals taken along these contours

$$\int_\varkappa := \sum_{j=1}^{L} \varkappa_j \int_{m_j} + \sum_{j=1}^{S} \varkappa_{L+j} \int_{\sigma_j} . \tag{2-8}$$

Note that, by taking appropriate linear combinations of the contours, we could alternatively have had contours consisting of finite segments joining the points a_j.
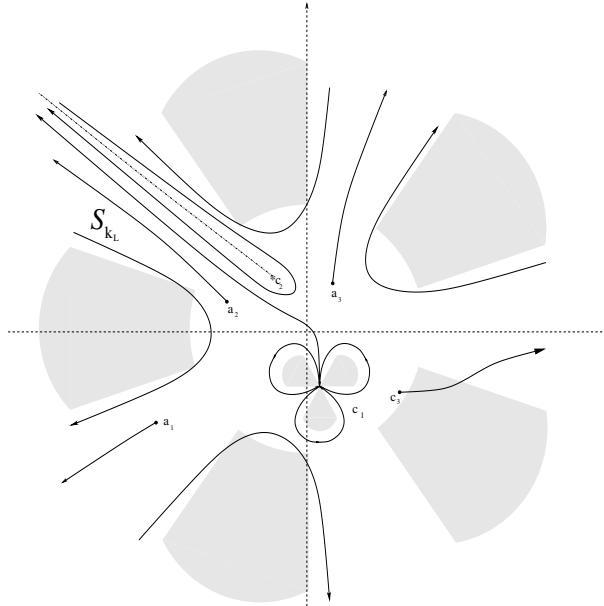
Figure 1: The types of contours considered in the x Riemann sphere $\mathbb{P}^1$. Here we have c_1 with $d_1 = 3$ and c_2 with $d_2 = 0, t_{2,0} \notin \mathbb{Z}$ (logarithmic singularity in the potential), c_3 with $d_3 = 0$, $t_{3,0} \in \mathbb{N}$ and the degree of the potential at infinity $c_0 = \infty$ is $d_0 = 5$. The essential singularity in μ at c_1 is of the form $\exp{(x - c_1)^{-3}}$ and there is also a cut extending from c_1 to ∞ if $t_{1,0} \notin \mathbb{Z}$. The point c_2 is a branch point of $\mu(x)$ since $t_{2,0} \notin \mathbb{Z}$, and the cut extends to infinity "inside" the contour (as shown here). If it were a pole ($t_{2,0} \in -\mathbb{N}^+$), the contour would be replaced by a circle around it. The point c_3 is a regular point with $t_{3,0} \in \mathbb{N}^\times$, and the contour extending from it to infinity is no different from the ones starting at the regular points a_1, a_2, a_3. The latter are the "hard-edge" segments joining the points a_1, a_2 and a_3 to ∞ within one of the sectors $S_k^{(0)}$.

2.2 Recursion relations, derivatives and deformations equations

2.2.1 Existence of orthogonal polynomials and relation to random matrices

Recall [7] that orthogonal polynomials satisfying (2-4) exist provided all the Hankel determinants formed from the moments are nonzero:

$$\Delta_n(\varkappa) := \det \left[\int_\varkappa x^{i+j} \mu(x) \mathrm{d}x \right]_{0 \le i,j \le n-1} \ne 0 \ , \quad \forall n \in \mathbb{N}. \tag{2-9}$$

Since the $\Delta_n(\varkappa)$'s are homogeneous polynomials in the coefficients $\varkappa_j$, the zero locus excluded by (2-9) is of zero measure (in the space of $\varkappa_j$'s), and hence "generically" the conditions (2-9) are fulfilled. The development to follow will in fact only involve orthogonal polynomials up to some arbitrarily large fixed degree, say N, and hence the conditions $\Delta_n(\varkappa) = 0$, $n \le N-1$ determine a Zariski closed set in $\{\varkappa_j\}$, (and a closed set of measure zero in the space of coefficients of V).

The orthogonal polynomials considered here are related to models of unitarily diagonalizable random matrices $M \in gl(n, \mathbb{C})$ with spectra supported on the contours defined above. More specifically we have the partition function

$$\mathcal{Z}_n := C_n \int_{spec(M) \in \varkappa} dM \mathrm{e}^{-\frac{1}{\hbar} \mathrm{Tr} V(M)}$$

7

$$= \int_{\varkappa} \mathrm{d}x_1 \cdots \int_{\varkappa} dx_n \, \Delta(\underline{x})^2 \mathrm{e}^{-\frac{1}{\hbar}\sum_{j=1}^{n} V(x_j)} \tag{2-10}$$

$$= n!\Delta_n(\varkappa, V) = n! \prod_{j=0}^{n-1} h_j \ , \tag{2-11}$$

where

$$C_N := \frac{1}{\left(\int_{U(n)} dU\right)} \tag{2-12}$$

is the inverse of the $U(n)$ group volume, and

$$\Delta(\underline{x}) := \prod_{i<j}(x_i - x_j) \tag{2-13}$$

is the usual Vandermonde determinant. The notation $spec(M) \in \varkappa$ in the first integral just means that M is unitarily diagonalizable

$$M = U\,D\,U^\dagger \ , \qquad D := \mathrm{diag}(x_1, \ldots x_n) \ , \qquad U \in U(n) \ , \tag{2-14}$$

and the eigenvalues $\{x_1, \ldots x_n\}$ of M are constrained to lie on the contours entering in $\int_{\varkappa}$.

In particular, as in the standard case, the orthogonal polynomials may be shown to be equal to the expectation values of the characteristic polynomials in such models

$$p_n(x) = \langle \det(x\mathbf{I} - M) \rangle$$
$$= \frac{1}{Z_n} \int_{\varkappa} \mathrm{d}x_1 \cdots \int_{\varkappa} dx_n \prod_{i=1}^{n}(x - x_i)\Delta(\underline{x})^2 \mathrm{e}^{-\frac{1}{\hbar}\sum_{j=1}^{n} V(x_j)} \ , \tag{2-15}$$

and all correlation functions between the eigenvalues may be expressed as determinants in terms of the standard Christoffel-Darboux kernel formed from them

$$K_n(x,y) := \sum_{j=0}^{n-1} \frac{1}{h_j} p_j(x) p_j(y) \mathrm{e}^{-\frac{1}{2\hbar}(V(x)+V(y))} \ . \tag{2-16}$$

More precisely, this is valid when there are no "hard-edge" contours present. Inclusion of the latter however allows one to interpret these determinants as certain conditional correlators, known as "Janossy distribution" correlators [6], giving the probability densities for a certain number of eigenvalues to lie at given locations within the complementary part of the support, while the remaining ones lie within it. The partition function Z_n in this case can be reinterpreted as the corresponding gap probability [6, 17].

2.2.2 Wave vector equations

We now define the normalized orthogonal polynomials

$$\pi_n(x) := \frac{1}{\sqrt{h_n}} p_n(x) \tag{2-17}$$

and what will be referred to as the " orthonormal quasi-polynomials"

$$\psi_n(x) := \pi_n(x)\mathrm{e}^{-\frac{1}{2\hbar}V(x)}, \tag{2-18}$$

satisfying

$$\int_{\varkappa} \psi_n(x)\psi_m(x)dx = \delta_{mn}. \tag{2-19}$$

8

We now form the semi-infinite "wave vectors"

$$\mathbf{\Pi}(x) := [\pi_0(x), \pi_1(x), \ldots, \pi_n(x), \ldots]^t \ . \tag{2-20}$$

As in the theory of ordinary orthogonal polynomials, we have

$$x\mathbf{\Pi}(x) = Q\mathbf{\Pi}(x) \ , \tag{2-21}$$

where Q is a symmetric tridiagonal semi-infinite matrix with components

$$Q_{ij} = \gamma_j \delta_{i,j-1} + \beta_i \delta_{ij} + \gamma_i \delta_{i,j+1}, \quad i, j \in \mathbb{N} \ , \tag{2-22}$$

defining a three term recursion relation of the form

$$x\pi_j(x) = \gamma_{j+1}\pi_{j+1}(x) + \beta_j\pi_j(x) + \gamma_j\pi_{j-1}(x) \ . \tag{2-23}$$

Now introduce semi-infinite matrices $P, A_i, C_r, T_{r,J}$ such that

$$\hbar \partial_x \mathbf{\Pi}(x) = P\mathbf{\Pi}(x) \tag{2-24}$$

$$\hbar \partial_{a_i} \mathbf{\Pi}(x) = A_i\mathbf{\Pi}(x) \ , \qquad i = 1, \ldots, L \tag{2-25}$$

$$\hbar \partial_{c_r} \mathbf{\Pi}(x) = C_r\mathbf{\Pi}(x) \ , \qquad r = 1, \ldots, K \tag{2-26}$$

$$\hbar \partial_{t_{r,J}} \mathbf{\Pi}(x) = T_{r,J}\mathbf{\Pi}(x) \ , \qquad r = 0, \ldots, K, \ J = 0, \ldots, d_r. \tag{2-27}$$

Their matrix elements are determined simply by integration

$$X_{nm} = \int_\varkappa (\hbar \partial \pi_n(x)) \, \pi_m(x) \mu(x) dx \ , \tag{2-28}$$

where ∂ denotes any of the derivatives $\partial_x, \partial_{a_i}, \partial_{c_r}, \partial_{t_{r,J}}$ above for which X becomes the corresponding matrices P, A_i, C_r or $T_{r,J}$ on the RHS of (2-24) - (2-27).

Remark 2.1 Such wave vectors and associated deformation equations have been studied in many previous works relating orthogonal polynomials, matrix models and integrable systems (see, e.g. [2], [18]). However, considerations of the deformation theory have mainly been within the formal setting, with the potential $V(x)$ replaced by some initial value, $V_0(x)$, plus a perturbation consisting of an infinite power series with arbitrary coefficients, without regard to domains of convergence. Results obtained in this formal setting cannot be directly applied to the study of isomonodromic deformations, where the local analytic structure in the neighborhood of a number of isolated singular points is of primary interest.

For any such semi-infinite square matrix X, let X_0, X_+, X_- denote the diagonal, upper and lower triangular parts, respectively, and let

$$X_{-0} := \frac{1}{2}X_0 + X_- \ . \tag{2-29}$$

Proposition 2.1 *The matrices P, A_i, C_r and $T_{r,J}$ are all lower semi-triangular (with P strictly lower triangular) , and are given by*

$$P = V'(Q)_{-0} - \sum_{i=1}^{L} A_i = V'(Q)_- - \sum_{i=1}^{L} (A_i)_- \tag{2-30}$$

$$A_i = \hbar\varkappa_i(\mathbf{\Pi}(a_i)\mathbf{\Pi}^t(a_i))_{-0}\mu(a_i) \tag{2-31}$$

$$C_r = \sum_{J=0}^{d_r} t_{r,J}(Q - c_r)_{-0}^{-J-1} \ , \quad r = 1, \ldots, K \tag{2-32}$$

$$T_{0,0} = \frac{1}{2}\mathbf{1} \tag{2-33}$$

$$T_{0,J} = \frac{1}{J}Q^J_{-0} , \quad J = 1, \ldots, d_0 \tag{2-34}$$

$$T_{r,J} = \frac{1}{J}(Q - c_r)^{-J}_{-0}, \quad r = 1, \ldots, K, \quad J = 1, \ldots, d_r \tag{2-35}$$

$$T_{r,0} = -\ln(Q - c_r)_{-0}, \quad r = 1, \ldots, K . \tag{2-36}$$

where $(Q - c_r)^{-J}$ and $\ln(Q - c_r)$ are defined by the formulæ

$$(Q - c_r)^{-J}_{nm} := \int_\varkappa \frac{\pi_n(z)\pi_m(z)}{(z - c_r)^J}\mu(z)dz \tag{2-37}$$

$$\ln(Q - c_r)_{nm} := \int_\varkappa \ln(z - c_r)\pi_n(x)\pi_m(z)\mu(z)dz . \tag{2-38}$$

The diagonal matrix elements for each of the above is given by the formula

$$X_{jj} = -\frac{\hbar}{2}\partial(\ln h_j) , \tag{2-39}$$

where $\partial = \partial_x$, ∂_{a_i}, ∂_{c_r} and $\partial_{t_{r,J}}$, respectively, for $X = P$, A_i, C_r and $T_{r,J}$. In particular, they vanish for P, which is strictly lower triangular, and hence

$$V'(Q)_{jj} = \hbar\sum_{i=1}^L \varkappa_i\psi_j(a_i)^2 . \tag{2-40}$$

Proof. We make use of the orthogonality relations

$$\int_\varkappa \mathbf{\Pi}(x)\mathbf{\Pi}^t(x)\mu(x)dx = \mathbf{1} . \tag{2-41}$$

Eqs. (2-32) - (2-36) are obtained as follows. Consider a deformation ∂ with respect to any of the above c_r's or $t_{r,J}$'s and denote by X the corresponding matrix; then

$$\hbar\partial\pi_n(x) = \hbar\partial\frac{p_n(x)}{\sqrt{h_n}} = -\frac{1}{2}\left(\hbar\partial\ln(h_n)\right)\pi_n(x) + \frac{1}{\sqrt{h_n}}\hbar\partial p_n(x) \tag{2-42}$$

$$= -\frac{1}{2}\left(\hbar\partial\ln(h_n)\right)\pi_n(x) + \text{ lower degree polynomials }, \tag{2-43}$$

since the polynomials π_n are monic. It follows that the deformation matrix X is lower semi-triangular. On the other hand, differentiating eq. (2-41) gives

$$0 = \int_\varkappa \left(\hbar\partial\mathbf{\Pi}\,\mathbf{\Pi}^t + \mathbf{\Pi}\,\hbar\partial\mathbf{\Pi}^t\right)\mu(x)dx + \int_\varkappa \mathbf{\Pi}\,\mathbf{\Pi}^t\hbar\partial\mu(x)dx = X + X^t + \int_\varkappa \mathbf{\Pi}\,\mathbf{\Pi}^t\hbar\partial\mu(x)dx \tag{2-44}$$

Applying the operators for each case to μ as defined in (2-2) and using eq. (2-21) then gives the result.

Now consider the deformations of the endpoints a_i of the "hard-edge" contours. Differentiating (2-41) gives

$$0 = \hbar\partial_{a_i}\int_\varkappa \mathbf{\Pi}\mathbf{\Pi}^t\mu(x)dx = -\varkappa_i\hbar\mathbf{\Pi}(a_i)\mathbf{\Pi}^t(a_i)\mu(a_i) + \int_\varkappa \left[(\hbar\partial_{a_i}\mathbf{\Pi})\mathbf{\Pi}^t + \mathbf{\Pi}\hbar\partial_{a_i}\mathbf{\Pi}^t\right]\mu(x)dx$$
$$= -\hbar\varkappa_i\mathbf{\Pi}(a_i)\mathbf{\Pi}^t(a_i)\mu(a_i) + A_i + A_i^t, \tag{2-45}$$

where

$$A_i := \hbar \int_{\varkappa} \partial_{a_i} \mathbf{\Pi} \, \mathbf{\Pi}^t \mu(x) dx \; , \tag{2-46}$$

It follows that

$$A_i = \hbar \varkappa_i (\mathbf{\Pi} \mathbf{\Pi}^t)_{-0}\big|_{x=a_i} \mu(a_i) \; , \tag{2-47}$$

proving eq. (2-31), and also that

$$(A_i)_{nn} = -\frac{\hbar}{2} \partial_{a_i} \ln(h_n) = \frac{\hbar}{2} \varkappa_i \psi_n^2(a_i) \; . \tag{2-48}$$

To determine the matrix P, note that it is *strictly lower triangular* and

$$\hbar \mu(a_i) \mathbf{\Pi} \mathbf{\Pi}^t \big|_{\partial \varkappa} = -\hbar \sum_{i=1}^{L} \left(A_i + A_i^t \right) = \hbar \int_{\varkappa} \left(\mathbf{\Pi}' \mathbf{\Pi}^t + \mathbf{\Pi} \mathbf{\Pi}'^t + \mathbf{\Pi} \mathbf{\Pi}^t \partial_x \ln \mu(x) \right) \mu(x) dx$$

$$= \int_{\varkappa} \left(\hbar \mathbf{\Pi}' \mathbf{\Pi}^t + \hbar \mathbf{\Pi} \mathbf{\Pi}'^t - V'(x) \mathbf{\Pi} \mathbf{\Pi}^t \right) \mu(x) dx = P + P^t - V'(Q). \tag{2-49}$$

This implies that

$$P = V'(Q)_{-0} - \sum_{i=1}^{L} A_i = V'(Q)_- - \sum_{i=1}^{L} (A_i)_- \; . \tag{2-50}$$

This last equality follows from (2-40), which, in turn, follows from integration by parts in the definition of $V'(Q)_{nn}$. It may be seen as a consequence of the invariance of the partition function under an infinitesimal change in the integration variables $x_j \to x_j + \epsilon$ in (2-11); i.e., translational invariance.

From (2-39) and (2-11) follows a relation between the diagonal elements of the deformation matrices and the logarithmic derivatives of the partition function that will be very important in what follows. Define the truncated trace of a semi-infinite matrix X to be

$$\mathrm{Tr}_n X := \sum_{j=0}^{n-1} X_{jj} \; . \tag{2-51}$$

Corollary 2.1 *For $\partial = \partial_{a_j}$, ∂_{c_r}, and $\partial_{t_{r,J}}$,*

$$\hbar \partial \ln \mathcal{Z}_n = -2\mathrm{Tr}_n X \; , \tag{2-52}$$

with $X = A_j$, C_r and $T_{r,J}$, respectively. For the cases ∂_{c_r} and $\partial_{t_{r,J}}$,

$$\hbar \partial \ln \mathcal{Z}_n = \mathrm{Tr}_n \partial V(Q) \; , \tag{2-53}$$

while for the ∂_{a_i}'s we have

$$\hbar \sum_{i=1}^{L} \partial_{a_i} \ln \mathcal{Z}_n = -V'(Q)_{nn} \tag{2-54}$$

Proof. The first of these relations follows from (2-39) and (2-11) directly, the second from the explicit expressions for the deformation matrices (2-32)–(2-36) and of the potential $V(x)$, and the third is a restatement of the (2-40) (translational invariance).

Corollary 2.2 *The compatibility conditions*

$$[\mathcal{G}, \mathcal{H}] = 0 \ , \tag{2-55}$$

are satisfied, where $\mathcal{G}$, $\mathcal{H}$ *are any of the following operators*

$$\hbar\partial_{a_i} - A_i, \quad \hbar\partial_{t_{r,J}} - T_{r,J}, \quad \hbar\partial_{c_r} - C_r, \quad \hbar\partial_x - P, \quad x - Q \tag{2-56}$$

and $r = 0, \ldots K$, $J = 0, \ldots d_r$.

Proof. This follows immediately from the fact that the orthogonal polynomials entering in eqs. (2-24)-(2-27) are linearly independent.

Remark 2.2 Note that

$$[\hbar\partial_x - P, x - Q] = 0 \tag{2-57}$$

is just the string equation, while the other compatibility conditions involving $x - Q$ imply the Lax equations:

$$\hbar\partial_{a_i} Q = [A_i, Q], \qquad \hbar\partial_{t_{r,J}} Q = [T_{r,J}, Q], \qquad \hbar\partial_{c_r} Q = [C_r, Q] \ , \tag{2-58}$$

showing that the spectrum of the matrix Q is invariant under these deformations.

2.2.3   Wave vector of the second kind

We now consider solutions of the second kind,

$$\phi_n(x) := e^{\frac{1}{\hbar} V(x)} \int_\varkappa \frac{e^{-\frac{1}{\hbar} V(z)} \pi_(z)}{x - z} dz. \ , \tag{2-59}$$

which may be combined to form the components of a wave vector of the second kind

$$\mathbf{\Phi}(x) := [\phi_0(x), \phi_1(x), \ldots, \phi_n(x), \ldots]^t \ . \tag{2-60}$$

Denote by

$$\nabla_Q V'(x) := \frac{V'(x) - V'(Q)}{x - Q} \tag{2-61}$$

the semi-infinite square matrix with elements

$$\left(\frac{V'(x) - V'(Q)}{x - Q}\right)_{nm} = \int_\varkappa dz e^{-\frac{1}{\hbar} V(z)} \pi_n(z) \pi_m(z) \frac{V'(x) - V'(z)}{x - z} \ , \tag{2-62}$$

and define $\mathbf{U}(x)$ to be the semi-infinite column vector (with only its zeroth component nonvanishing) given by

$$(\mathbf{U}(x))_n := \sqrt{h_0} e^{\frac{1}{\hbar} V(x)} \delta_{n,0}. \tag{2-63}$$

The following lemma gives the effect of multiplication of $\mathbf{\Pi}(x)$ by x and of application of $\hbar\partial_x$ to it. It may be deduced immediately from eqns. (2-21) and (2-24), applied inside the integral, together with integration by parts.

Lemma 2.1

$$x\mathbf{\Phi}(x) = Q\mathbf{\Phi}(x) + \mathbf{U}(x) \tag{2-64}$$

$$\hbar\partial_x \mathbf{\Phi}(x) = P\mathbf{\Phi}(x) + \nabla_Q V'(x)\mathbf{U}(x) + \hbar \sum_{i=1}^L \varkappa_i \frac{e^{\frac{1}{\hbar}(V(x) - V(a_i))}}{x - a_i} \mathbf{\Pi}(a_i) \ , \tag{2-65}$$

12

The next proposition, which is similarly verified, gives the effects of the above deformations on the wave vector of the second kind.

Proposition 2.2

$$\hbar\partial_{a_i}\boldsymbol{\Phi}(x) = A_i\boldsymbol{\Phi}(x) - \hbar\varkappa_i\frac{\mathrm{e}^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x-a_i}\boldsymbol{\Pi}(a_i), \quad i=1,\ldots,L, \tag{2-66}$$

$$\hbar\partial_{c_r}\boldsymbol{\Phi}(x) = C_r\boldsymbol{\Phi}(x) + \sum_{J=0}^{d_r} t_{r,J}\frac{(Q-c_r)^{-J-1}-(x-c_r)^{-J-1}}{Q-x}\boldsymbol{U}(x), \quad r=1,\ldots,K, \tag{2-67}$$

$$\hbar\partial_{t_{0,J}}\boldsymbol{\Phi}(x) = T_{0,J}\boldsymbol{\Phi}(x) + \frac{Q^J-x^J}{Q-x}\boldsymbol{U}(x), \quad J=1,\ldots,d_0, \tag{2-68}$$

$$\hbar\partial_{t_{r,J}}\boldsymbol{\Phi}(x) = T_{r,J}\boldsymbol{\Phi}(x) + \frac{(Q-c_r)^{-J}-(x-c_r)^{-J}}{Q-x}\boldsymbol{U}(x), \quad J=1,\ldots,d_r, \quad r=1,\ldots,K, \tag{2-69}$$

$$\hbar\partial_{t_{r,0}}\boldsymbol{\Phi}(x) = T_{r,0}\boldsymbol{\Phi}(x) + \frac{\ln(Q-c_r)-\ln(x-c_r)}{Q-x}\boldsymbol{U}(x), \quad r=1,\ldots,K. \tag{2-70}$$

The content of eqs. (2-26)–(2-27) and (2-67)–(2-70) may be summarized uniformly as follows. Let $v(x)$ be any function that is analytic at each point of the contours except, possibly, the points c_r, and for which the following integrals are convergent:

$$v(Q)_{nm} := \int_\varkappa v(z)\pi_n(z)\pi_m(z)\mathrm{e}^{-\frac{1}{\hbar}V(z)}dz = \int_\varkappa v(z)\psi_n(z)\psi_m(z)dz$$

$$(\nabla_Q v(x))_{nm} := \left(\frac{v(x)-v(Q)}{x-Q}\right)_{nm} := \int_\varkappa \frac{v(x)-v(z)}{x-z}\psi_n(z)\psi_m(z)dz. \tag{2-71}$$

Define the deformation matrix under the infinitesimal variation of the potential $V(x)\mapsto V(x)+v(x)$ to be

$$X_v := v(Q)_{-0} . \tag{2-72}$$

Then the two infinite systems

$$\hbar\delta_v\boldsymbol{\Pi}(x) := X_v\boldsymbol{\Pi}(x) \tag{2-73}$$

$$\hbar\delta_v\boldsymbol{\Phi}(x) := X_v\boldsymbol{\Phi}(x) + \nabla_Q v(x)\boldsymbol{U}(x) \tag{2-74}$$

describe the infinitesimal deformation of the orthogonal polynomials and the second-kind solutions under such infinitesimal variations of the potential.

Equivalently, define the $2\times\infty$ matrix

$$\boldsymbol{\Gamma}(x) := [\boldsymbol{\Pi}(x), \boldsymbol{\Phi}(x)]. \tag{2-75}$$

In terms of $\boldsymbol{\Gamma}(x)$, all the recursion, differential and deformation equations (2-21), (2-24)–(2-27) and (2-64)–(2-70) may be expressed simultaneously as

$$x\boldsymbol{\Gamma} = Q\boldsymbol{\Gamma} + (0,\boldsymbol{U}) \tag{2-76}$$

$$\hbar\partial_x\boldsymbol{\Gamma} = P\boldsymbol{\Gamma} + \left(0, \nabla_Q V'\boldsymbol{U} + \hbar\sum_{i=1}^K \frac{\mathrm{e}^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x-a_i}\boldsymbol{\Pi}(a_i)\right) \tag{2-77}$$

$$\hbar\delta_v\boldsymbol{\Gamma} = X_v\boldsymbol{\Gamma} + (0, \nabla_Q v\boldsymbol{U}) \tag{2-78}$$

$$\hbar\partial_{a_i}\boldsymbol{\Gamma} = A_i\boldsymbol{\Gamma} - \left(0, \hbar\varkappa_i\frac{\mathrm{e}^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x-a_i}\boldsymbol{\Pi}(a_i)\right), \tag{2-79}$$

where v signifies any of the infinitesimal deformations of the potential $\hbar\partial_{c_i}, \hbar\partial_{t_{r,J}}V$ ($i=1,\ldots L$, $r=0,\ldots K$, $J=1,\ldots,d_r$).

13

3 Folding

3.1 n-windows and Christoffel-Darboux formula

Let i_n be the $\infty \times 2$ matrix that represents the injection of the 2-dimensional subspace spanned by the $(n-1, n)$ basis elements into the (semi-)infinite space corresponding to the components of Ψ or Φ. Its matrix elements are thus:

$$(i_n)_{jk} = \delta_{k,1}\delta_{j,n-1} + \delta_{k,2}\delta_{j,n}, \quad j = 0, 1, 2, \ldots, \quad k = 1, 2. \tag{3-1}$$

Let i_n^T denote its transpose, which is the corresponding projection operator. The n-th 2×2 block (or "**window**") of $\mathbf{\Gamma}$ is then given by:

$$\mathbf{\Gamma}_n(x) := i_n^T \mathbf{\Gamma} = \left[\begin{array}{cc} \pi_{n-1}(x) & \phi_{n-1}(x) \\ \pi_n(x) & \phi_n(x) \end{array} \right]. \tag{3-2}$$

By "folding" the infinite recursion and differential-deformation equations (2-21), (2-24)–(2-27), (2-64)–(2-70), (2-76)–(2-79), we mean the corresponding sequence of recursion relations, ODEs and PDEs satisfied by the $\mathbf{\Gamma}_n(x)$'s. To derive these, a form of the Christoffel–Darboux identity for orthogonal polynomials will repeatedly be used. Let $\mathbf{\Pi}_n^0$ denote the semi-infinite square matrix whose only nonvanishing entries are 1's on the diagonal in positions 0 to n (i.e. the projection onto the first n+1 components)

$$(\mathbf{\Pi}_n^0)_{ij} := \left\{ \begin{array}{cc} \delta_{ij} & \text{if} \quad 0 \leq i, j \leq n \\ 0 & \text{otherwise .} \end{array} \right. \tag{3-3}$$

Let

$$\sigma := \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \tag{3-4}$$

be the standard 2×2 symplectic matrix, and let

$$\mathbf{\Sigma}_n := i_n \sigma i_n^T \tag{3-5}$$

denote its projection onto the 2×2 subspace in position $(n-1, n)$.

Proposition 3.1 *The following* **extended Christoffel-Darboux** *formulæ are satisfied:*

$$(x - x')\mathbf{\Gamma}^T(x)\mathbf{\Pi}_{n-1}^0\mathbf{\Gamma}(x') \tag{3-6}$$

$$= \gamma_n \mathbf{\Gamma}_n^T(x)\sigma\mathbf{\Gamma}_n(x') + \left(\begin{array}{cc} 0 & -e^{\frac{1}{\hbar}V(x')} \\ e^{\frac{1}{\hbar}V(x)} & e^{\frac{1}{\hbar}(V(x)+V(x'))} \int_\varkappa e^{-\frac{1}{2}V(z)} \left(\frac{1}{x-z} - \frac{1}{x'-z} \right) dz \end{array} \right) \tag{3-7}$$

$$= \gamma_n \mathbf{\Gamma}^T(x)\mathbf{\Sigma}_n\mathbf{\Gamma}(x') + \left(\begin{array}{cc} 0 & -e^{\frac{1}{\hbar}V(x')} \\ e^{\frac{1}{\hbar}V(x)} & e^{\frac{1}{\hbar}(V(x)+V(x'))} \int_\varkappa e^{-\frac{1}{2}V(z)} \left(\frac{1}{x-z} - \frac{1}{x'-z} \right) dz \end{array} \right). \tag{3-8}$$

Equivalently, in components,

$$(x - x') \sum_{j=0}^{n-1} \pi_j(x)\pi_j(x') = \gamma_n \left[\pi_n(x)\pi_{n-1}(x') - \pi_{n-1}(x)\pi_n(x') \right] \tag{3-9}$$

$$(x - x') \sum_{j=0}^{n-1} \pi_j(x)\phi_j(x') = \gamma_n \left[\pi_n(x)\phi_{n-1}(x') - \pi_{n-1}(x)\phi_n(x') \right] - e^{\frac{1}{\hbar}V(x')} \tag{3-10}$$

$$(x - x') \sum_{j=0}^{n-1} \phi_j(x)\phi_j(x') = \gamma_n \left[\phi_n(x)\phi_{n-1}(x') - \phi_{n-1}(x)\phi_n(x') \right]$$

$$+ \, \mathrm{e}^{\frac{1}{\hbar}(V(x)+V(x'))} \int_{\varkappa} \mathrm{e}^{-\frac{1}{2}V(z)} \left(\frac{1}{x-z} - \frac{1}{x'-z} \right) dz, \tag{3-11}$$

and evaluating at $x = x'$ gives

$$\det \mathbf{\Gamma}_n(x) = \pi_{n-1}(x)\phi_n(x) - \phi_{n-1}(x)\pi_n(x) = -\frac{1}{\gamma_n} \mathrm{e}^{\frac{1}{\hbar}V(x)} . \tag{3-12}$$

Proof. Eq. (3-9) is the standard Christoffel-Darboux relation for orthogonal polynomials. The extended system may be derived as follows. Multiplying the expression $\mathbf{\Gamma}(x)\mathbf{\Pi}_{n-1}^0\mathbf{\Gamma}(x')$ by $(x-x')$, and applying the relation (2-76) with respect to both x and x' gives

$$(x-x')\mathbf{\Gamma}^T(x)\mathbf{\Pi}_{n-1}^0\mathbf{\Gamma}(x') = \mathbf{\Gamma}^T(x)\left(Q\mathbf{\Pi}_{n-1}^0 - \pi_{n-1}^0 Q\right)\mathbf{\Gamma}(x') \tag{3-13}$$

$$+ \begin{pmatrix} 0 & -\mathrm{e}^{\frac{1}{\hbar}V(x')} \\ \mathrm{e}^{\frac{1}{\hbar}V(x)} & \mathrm{e}^{\frac{1}{\hbar}(V(x)+V(x'))} \int_{\varkappa} \mathrm{e}^{-\frac{1}{2}V(z)} \left(\frac{1}{x-z} - \frac{1}{x'-z} \right) dz \end{pmatrix} . \tag{3-14}$$

The result (3-8) is obtained by substituting the following identity, which holds for any tridiagonal symmetric matrix of the form Q

$$Q\mathbf{\Pi}_{n-1}^0 - \mathbf{\Pi}_{n-1}^0 Q = \gamma_n i_n \sigma i_n^T . \tag{3-15}$$

3.2 Folded version of the deformation equations for changes in the potential

Under infinitesimal changes of the parameters in the potential V and the end-points of the "hard-edge" contours, the wave vectors $\mathbf{\Pi}(x)$ and $\mathbf{\Phi}(x)$ and the combined system $\mathbf{\Gamma}(x)$ undergo changes determined by equations (2-25)-(2-27), (2-66)-(2-70)) and (2-79)-(2-78). Besides the deformations induced by infinitesimal changes of the endpoints $\{a_j\}$, all these deformations have the same general form, depending only on the function $v(x) = \delta V(x)$ that gives the infinitesimal deformation of the potential. We deal with them all on the same footing in the following proposition, which expresses the explicit form they take on the window $\mathbf{\Gamma}_n(x)$.

Proposition 3.2 *The deformation equations (2-73), (2-74) (2-78) are equivalent to the infinite sequence of 2×2 equations*

$$\hbar\delta_v\mathbf{\Gamma}_n(x) = \mathcal{V}_n(x)\mathbf{\Gamma}_n(x), \tag{3-16}$$

where the folded matrix of the deformation is defined by

$$\mathcal{V}_n(x) = \begin{bmatrix} v(x) - \frac{1}{2}v(Q)_{n-1,n-1} & 0 \\ 0 & \frac{1}{2}v(Q)_{nn} \end{bmatrix} + \gamma_n \begin{bmatrix} \nabla_Q v(x)_{n-1,n-1} & \nabla_Q v(x)_{n-1,n} \\ \nabla_Q v(x)_{n,n-1} & \nabla_Q v(x)_{nn} \end{bmatrix} \sigma . \tag{3-17}$$

For the deformations in (2-24)–(2-27) and (2-66)–(2-70), this gives the following equations corresponding to changes in the potential.

$$\hbar\partial_{c_r}\mathbf{\Gamma}_n(x) = \mathcal{C}_{r;n}(x)\mathbf{\Gamma}_n(x) \tag{3-18}$$

$$\hbar\partial_{t_{r,J}}\mathbf{\Gamma}_n(x) = \mathcal{T}_{r,J;n}(x)\mathbf{\Gamma}_n(x), \tag{3-19}$$

where the sequence of 2×2 matrices $\mathcal{C}_{r;n}$ and $\mathcal{T}_{r,J;n}(x)$ are rational in x, with poles at the points $\{c_r\}$, obtained by making the following substitutions in eq. ((3-17).

$$\begin{aligned} C_r : \quad v(x) &\rightarrow \sum_{J=0}^{d_r} t_{r,J}(x-c_r)^{-J-1} \\ T_{r,J} : \quad v(x) &\rightarrow \frac{1}{J}(x-c_r)^{-J} \\ T_{0,J} : \quad v(x) &\rightarrow \frac{1}{J}x^J \\ T_{r,0} : \quad v(x) &\rightarrow -\ln(x-c_r) . \end{aligned} \tag{3-20}$$

Proof. Using the definition (2-71) of $\nabla_Q v(x)$ and the extended Christoffel-Darboux relation (3-8), we have

$$\gamma_n \nabla_Q v(x) \mathbf{\Sigma}_n \mathbf{\Gamma} = \gamma_n \int_\varkappa dy \, e^{-\frac{1}{\hbar}V(y)} \left(v(y) - v(x)\right) \mathbf{\Pi}(y) \frac{\mathbf{\Pi}^T(y) \mathbf{\Sigma}_n^T \mathbf{\Gamma}(x)}{y - x} \tag{3-21}$$

$$= \int_\varkappa dy e^{-\frac{1}{\hbar}V(y)} \left(v(y) - v(x)\right)) \mathbf{\Pi}(y) \mathbf{\Pi}^T(y) \mathbf{\Pi}_{n-1}^0 \mathbf{\Gamma}(x) \tag{3-22}$$

$$+ \left(0, \; -e^{\frac{1}{\hbar}V(x)} \int_\varkappa dy e^{-\frac{1}{\hbar}V(y)} \left(\frac{v(y) - v(x)}{y - x}\right) \mathbf{\Pi}(y)\right) \tag{3-23}$$

$$= v(Q)\mathbf{\Pi}_{n-1}^0 \mathbf{\Gamma}(x) - v(x)\mathbf{\Pi}_{n-1}^0 \mathbf{\Gamma}(x) - (0, \nabla_Q v(x)\boldsymbol{U}(x)) \,. \tag{3-24}$$

Applying the projector i_n^T and noting that

$$i_n^T v(Q)\mathbf{\Pi}_{n-1}^0 \mathbf{\Gamma} = i_n^T v(Q)_{-0} \mathbf{\Gamma} + \frac{1}{2} \begin{pmatrix} v(Q)_{n-1,n-1} & 0 \\ 0 & -v(Q)_{n,n} \end{pmatrix} \mathbf{\Gamma}_n, \tag{3-25}$$

we obtain

$$\hbar\delta_v \mathbf{\Gamma}_n(x) = i_n^T \hbar\delta_v \mathbf{\Gamma}(x) = i_n^T \left(X_v \mathbf{\Gamma}(x) + (0, \nabla_Q v \boldsymbol{U}(x)\right) \tag{3-26}$$

$$= i_n^T v(Q)_{-0} \mathbf{\Gamma}(x) + (0, i_n^T \nabla_Q v(x)\boldsymbol{U}(x)) \tag{3-27}$$

$$= i_n^T v(Q)\mathbf{\Pi}_{n-1}^0 \mathbf{\Gamma}(x) + \begin{pmatrix} -\frac{1}{2}v(Q)_{n-1,n-1} & 0 \\ 0 & \frac{1}{2}v(Q)_{n,n} \end{pmatrix} \mathbf{\Gamma}_n + (0, i_n^T \nabla_Q v(x)\boldsymbol{U}(x)) \tag{3-28}$$

$$= \gamma_n i_n^T \nabla_Q v(x) i_n^T \sigma \mathbf{\Gamma}_n(x) + \begin{pmatrix} v(x) - \frac{1}{2}v(Q)_{n-1,n-1} & 0 \\ 0 & \frac{1}{2}v(Q)_{n,n} \end{pmatrix} \mathbf{\Gamma}_n(x), \tag{3-29}$$

proving the relation (3-17).

Remark 3.1 Note that formula (3-17) for the deformation of the measure in Proposition 3.2, as well as those below, (3-36), (3-37), which are obtained through folding of the $\hbar\partial_x$ operator, could also be derived for arbitrary locally analytic potentials $V(x)$, provided all the integrals involved are convergent [13]. However applicability of the subsequent isomonodromic analysis would be lost if the derivatives were not rational, since the resulting deformation equations would then have essential singularities.

3.3   Folding of the endpoint deformations

The case (2-25) and (2-66) involving deformations of the locations of the "hard edge" end–points must be considered separately.

Proposition 3.3 *The following gives a closed system for the n-th window of eqs. (2-25) and (2-66)*

$$\hbar\partial_{a_i} \mathbf{\Gamma}_n(x) = \mathcal{A}_{i,n}(x)\mathbf{\Gamma}_n(x) \tag{3-30}$$

where

$$\mathcal{A}_{i,n} := \frac{\hbar\varkappa_i \gamma_n}{a_i - x} \begin{bmatrix} \psi_{n-1}(a_i)\psi_n(a_i) & -\psi_{n-1}^2(a_i) \\ \psi_n^2(a_i) & -\psi_{n-1}(a_i)\psi_n(a_i) \end{bmatrix} + \frac{\hbar\varkappa_i}{2} \begin{bmatrix} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{bmatrix}$$

$$= \frac{\hbar\varkappa_i \gamma_n}{a_i - x} \begin{bmatrix} \psi_{n-1}^2(a_i) & \psi_{n-1}(a_i)\psi_n(a_i) \\ \psi_{n-1}(a_i)\psi_n(a_i) & \psi_n^2(a_i) \end{bmatrix} \sigma + \frac{\hbar\varkappa_i}{2} \begin{bmatrix} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{bmatrix} \tag{3-31}$$

Proof.   This is very similar to the proof of Prop. 3.2. Using the definition (2-31) of the matrices A_i and the extended Christoffel–Darboux relation (3-8) we have

$$\hbar\partial_{a_i} \mathbf{\Gamma}_n(x) = \hbar i_n^T \partial_{a_i} \mathbf{\Gamma}(x) = i_n^T \left[\left(\hbar\varkappa_i \Psi(a_i)\Psi^T(a_i)\right)_{-0} \mathbf{\Gamma}(x) - \left(0, \hbar\varkappa_i \frac{e^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x - a_i} \mathbf{\Pi}(a_i)\right) \right]$$

16

$$= i_n^T \left[\left(\hbar \varkappa_i \Psi(a_i) \Psi^T(a_i) \right) \mathbf{\Pi}_{n-1}^0 \mathbf{\Gamma}(x) - \left(0, \hbar \varkappa_i \frac{e^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x - a_i} \mathbf{\Pi}(a_i) \right) \right]$$
$$+ \frac{\hbar \varkappa_i}{2} \left[\begin{array}{cc} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{array} \right] \mathbf{\Gamma}_n(x)$$
$$= i_n^T \frac{\gamma_n \hbar \varkappa_i}{a_i - x} \Psi(a_i) \Psi^T(a_i) \mathbf{\Sigma}_n \mathbf{\Gamma}(x) + \frac{\hbar \varkappa_i}{2} \left[\begin{array}{cc} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{array} \right] \mathbf{\Gamma}_n(x) \ . \tag{3-32}$$

Recalling the definition (3-5) of $\mathbf{\Sigma}_n$ and computing the matrix product yields the result in the statement. Q.E.D.

3.4 Folded version of the recursion relations and $\hbar \partial_x$ relations

We now consider the recursion relations (2-21), (2-64) and (2-76) and the action of the $\hbar \partial_x$ operator in (2-24), (2-65) and (2-77) which, in their folded form are given by the following.

Proposition 3.4 *The folded forms of the relations (2-76) and (2-77) are*

$$\mathbf{\Gamma}_{n+1}(x) = R_n(x) \mathbf{\Gamma}_n(x) \ , \quad n \geq 1, \tag{3-33}$$
$$\partial_x \mathbf{\Gamma}_n(x) = \mathcal{D}_n(x) \mathbf{\Gamma}_n(x) \tag{3-34}$$

where

$$R_n := \left[\begin{array}{cc} 0 & 1 \\ -\frac{\gamma_n}{\gamma_{n+1}} & \frac{x - \beta_n}{\gamma_{n+1}} \end{array} \right] \ . \tag{3-35}$$

and

$$\mathcal{D}_n(x) = \mathcal{D}_n^{(0)}(x) + \sum_{i=1}^{L} \frac{\hbar \varkappa_i \gamma_n}{x - a_i} \left[\begin{array}{cc} \psi_{n-1}(a_i)\psi_n(a_i) & -\psi_{n-1}^2(a_i) \\ \psi_n^2(a_i) & -\psi_{n-1}(a_i)\psi_n(a_i) \end{array} \right] \tag{3-36}$$

with

$$\mathcal{D}_n^{(0)}(x) = \left[\begin{array}{cc} V'(x) & 0 \\ 0 & 0 \end{array} \right] + \left[\begin{array}{cc} (\nabla_Q V'(x))_{n-1,n-1} & (\nabla_Q V'(x))_{n-1,n} \\ (\nabla_Q V'(x))_{n,n-1} & (\nabla_Q V'(x))_{nn} \end{array} \right] \left[\begin{array}{cc} 0 & -\gamma_n \\ \gamma_n & 0 \end{array} \right]$$
$$= \left[\begin{array}{cc} V'(x) & 0 \\ 0 & 0 \end{array} \right] + \gamma_n \left[\begin{array}{cc} (\nabla_Q V'(x))_{n-1,n} & -(\nabla_Q V'(x))_{n-1,n-1} \\ (\nabla_Q V'(x))_{nn} & -(\nabla_Q V'(x))_{n,n-1} \end{array} \right] \tag{3-37}$$

Remark 3.2 Note that formula (3-36) implies that

$$\mathrm{Tr}(\mathcal{D}_n(x)) = V'(x) \ . \tag{3-38}$$

Proof: The folded form (3-33) of the recursion relations follows directly from eqs. (2-21) and (2-64))

$$x \pi_n(x) = \gamma_{n+1} \pi_{n+1}(x) + \beta_n \pi_n(x) + \gamma_n \pi_{n-1}(x) \tag{3-39}$$
$$x \phi_n(x) = \gamma_{n+1} \phi_{n+1}(x) + \beta_n \phi_n(x) + \gamma_n \phi_{n-1}(x) + \delta_{n0} \sqrt{h_0} e^{\frac{1}{\hbar} V(x)} \ . \tag{3-40}$$

To prove (3-34), note that the folding relations (3-16) may be expressed

$$i_n^T \delta_v \mathbf{\Gamma}_n = \mathcal{V}_n \mathbf{\Gamma}_n \tag{3-41}$$

for any infinitesimal variation $v = \delta V$ in the potential. Choosing

$$\delta := -\sum_{r=1}^{K} \partial_{c_r} + \sum_{J=1}^{d_0 - 1} J t_{0,J+1} \partial_{t_{0,J}} + t_{0,1} \partial_{t_{0,0}}, \tag{3-42}$$

17

we have

$$V'(x) \equiv \delta V. \tag{3-43}$$

Using (2-30) and (2-65), we have

$$
\begin{aligned}
\hbar \partial_x \boldsymbol{\Gamma}_n &= i_n^T \left[P\boldsymbol{\Gamma} + \left(0 \ , \ \nabla_Q V'(x) \boldsymbol{U} - \hbar \sum_{i=1}^{L} \varkappa_i \frac{\mathrm{e}^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x-a_i} \boldsymbol{\Pi}(a_i) \right) \right] \\
&= i_n^T \left[(V'(Q)_{-0} - \sum_{i=1}^{L} A_i)\boldsymbol{\Gamma} + \left(0 \ , \ \nabla_Q (\delta V)(x)\boldsymbol{U} - \hbar \sum_{i=1}^{L} \varkappa_i \frac{\mathrm{e}^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x-a_i} \boldsymbol{\Pi}(a_i) \right) \right] \\
&= i_n^T \left[((\delta V)(Q)_{-0} - \sum_{i=1}^{L} A_i)\boldsymbol{\Gamma} + \left(0 \ , \ \nabla_Q (\delta V)(x)\boldsymbol{U} - \hbar \sum_{i=1}^{L} \varkappa_i \frac{\mathrm{e}^{\frac{1}{\hbar}(V(x)-V(a_i))}}{x-a_i} \boldsymbol{\Pi}(a_i) \right) \right] \\
&= i_n^T \left[\delta - \sum_{i=1}^{L} \partial_{a_i} \right] \boldsymbol{\Gamma}, \tag{3-44}
\end{aligned}
$$

where we have used the deformation equations (2-25)–(2-27), (2-66)–(2-70). Applying the folded relations (3-16), (3-17) and (3-30), this gives

$$\hbar \partial_x \boldsymbol{\Gamma}_n = \left[\mathcal{V}_n - \sum_{i=1}^{K} \hat{\mathcal{A}}_{i,n} - \sum_{i=1}^{K} \bar{\mathcal{A}}_{i,n} \right] \boldsymbol{\Gamma}_n, \tag{3-45}$$

where

$$\hat{\mathcal{A}}_{i,n} := \frac{\hbar \varkappa_i \gamma_n}{a_i - x} \begin{bmatrix} \psi_{n-1}(a_i)\psi_n(a_i) & -\psi_{n-1}^2(a_i) \\ \psi_n^2(a_i) & -\psi_{n-1}(a_i)\psi_n(a_i) \end{bmatrix} \tag{3-46}$$

$$\bar{\mathcal{A}}_{i,n} := \frac{\hbar \varkappa_i}{2} \begin{bmatrix} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{bmatrix}, \tag{3-47}$$

and

$$
\begin{aligned}
\mathcal{V}_n(x) = & \begin{bmatrix} V'(x)-\frac{1}{2}V'(Q)_{n-1,n-1} & 0 \\ 0 & \frac{1}{2}V'(Q)_{nn} \end{bmatrix} \\
& + \begin{bmatrix} \nabla_Q V'(x)_{n-1,n-1} & \nabla_Q V'(x)_{n-1,n} \\ \nabla_Q V'(x)_{n,n-1} & \nabla_Q V'(x)_{nn} \end{bmatrix} \begin{bmatrix} 0 & -\gamma_n \\ \gamma_n & 0 \end{bmatrix}. \tag{3-48}
\end{aligned}
$$

It follows from (2-40) that the diagonal $V'(Q)$ terms in $\mathcal{V}_n(x)$ are canceled by the sum in the last term of (3-45), giving the stated result (3-36), (3-37).

Combining the differential, recursion and deformations relations (3-34), (3-33), (3-18), (3-19) and (3-30), the fact that the invertible matrices $\boldsymbol{\Gamma}_n$ are simultaneous fundamental systems for all these equation implies the compatibility of the cross-derivatives; i.e., the corresponding set of zero-curvature equations.

Corollary 3.1 *For $n \geq 0$ the set of PDE's and recursion equations*

$$
\begin{aligned}
\hbar \partial_x \boldsymbol{\Gamma}_n(x) &= \mathcal{D}_n(x)\boldsymbol{\Gamma}_n(x), \qquad \hbar \partial_{a_i} \boldsymbol{\Gamma}_n(x) = \mathcal{A}_{i;n}(x)\boldsymbol{\Gamma}_n(x) \\
\hbar \partial_{c_r} \boldsymbol{\Gamma}_n(x) &= \mathcal{C}_{r;n}(x)\boldsymbol{\Gamma}_n(x), \qquad \hbar \partial_{t_{r,J}} \boldsymbol{\Gamma}_n(x) = \mathcal{T}_{r,J;n}(x)\boldsymbol{\Gamma}_n(x) \\
\boldsymbol{\Gamma}_{n+1}(x) &= R_n(x)\boldsymbol{\Gamma}_n(x)
\end{aligned} \tag{3-49}
$$

are simultaneously satisfied by the invertible matrices $\boldsymbol{\Gamma}_n(x)$, and hence the zero-curvature equations

$$[\hbar \partial_x - \mathcal{D}_n, \ \hbar \partial_{a_i} - \mathcal{A}_{i;n}] = 0, \quad [\hbar \partial_x - \mathcal{D}_n, \ \hbar \partial_{a_i} - \mathcal{C}_{r;n}] = 0,$$

$$[\hbar\partial_x - \mathcal{D}_n, \ \hbar\partial_{a_i} - \mathcal{T}_{r,J;n}] = 0 \ , \quad [\hbar\partial_{a_i} = \mathcal{A}_{i;n}, \ \hbar\partial_{a_i} - \mathcal{C}_{r;n}] = 0,$$
$$[\hbar\partial_{a_i} - \mathcal{A}_{i;n}, \hbar\partial_{a_i} - \mathcal{T}_{r,J;n}] = 0 \ , \quad [\hbar\partial_{a_i} - \mathcal{C}_{r;n}, \hbar\partial_{a_i} - \mathcal{T}_{r,J;n}] = 0,$$
$$\hbar\partial_{a_i} R_n = \mathcal{A}_{i;n+1} R_n - R_n \mathcal{A}_{i;n},$$
$$\hbar\partial_{c_r} R_n = \mathcal{C}_{r;n+1} R_n - R_n \mathcal{C}_{r;n},$$
$$\hbar\partial_{t_{r,J}} R_n = \mathcal{T}_{r,J;n+1} R_n - R_n \mathcal{T}_{r,J;n} \tag{3-50}$$

are satisfied.

Remark 3.3 (The Riemann–Hilbert method.)

The Riemann-Hilbert method for characterizing orthogonal polynomials [10, 8] provides an alternative approach to deriving the results of this section. This is a well-established approach, and will not be developed in detail here, except to indicate briefly how it could be applied to deducing the differential and deformation equations satisfied by the fundamental systems.

The fundamental system $\mathbf{\Gamma}_n(x)$ has, by construction, a jump-discontinuity across any of the contours defining the orthogonality measure. Denoting the limiting values when approaching any of these contours from the left or the right by $\mathbf{\Gamma}_{n,\pm}$ we have the jump discontinuity conditions

$$\mathbf{\Gamma}_{n,+}(x) = \mathbf{\Gamma}_{n,-}(x) \begin{bmatrix} 1 & 2i\pi\varkappa_j \\ 0 & 1 \end{bmatrix} \ , \qquad x \in \gamma_j \tag{3-51}$$

Furthermore, the local asymptotic behavior near the singularities at ∞ are specified as in section 5.2. To be more precise the function $\Gamma_n(x)$ has local formal asymptotic form, within any of the Stokes sectors,

$$e^{-\frac{1}{2\hbar} V(x)} \mathbf{\Gamma}_n(x) \sim \begin{cases} C_r \left(\mathbf{1} + \mathcal{O}(x - c_r) \right) e^{-\frac{1}{2\hbar} T_r(x)\sigma_3} & x \to c_r \\[1em] A_j \left(\mathbf{1} + \mathcal{O}(x - a_j) \right) e^{-\varkappa_j \ln(x - a_j)\sigma_+} & x \to a_j \\[1em] C_0 \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{x}\right) \right) e^{-\frac{1}{2\hbar} T_0(x)\sigma_3 + (n - \sum_r t_{0,r})\sigma_3 \ln(x)} & x \to \infty \end{cases} \tag{3-52}$$

It follows from the usual argument based on Liouville's theorem that any two fundamental solutions (with the same Stokes matrices, given in fact by the same matrices (3-51)) satisfying the above Riemann–Hilbert conditions are equal, within a constant scalar multiple. Also, from Liouville's theorem it follows that the first column of $\Psi(x)$ consists of polynomials (the orthogonal polynomials). Using similar arguments one can show that the following matrix is rational with poles, of the correct order, at the singular points c_r, a_j, ∞:

$$\mathcal{D}_n(x) := \hbar\partial_x \mathbf{\Gamma}_n(x) \mathbf{\Gamma}_n(x)^{-1} + \frac{1}{2} V'(x)\mathbf{1}. \tag{3-53}$$

By comparing the local singular behavior of the logarithmic (matrix) derivatives of any two solutions and applying Liouville's theorem, it follows again that these globally combine to define rational matrix functions which give the deformation matrices with respect to the various parameters at the poles.

4 Spectral curve and spectral invariants

The aim of this section is to express the spectral curve of the ODE (3-34) (i.e., the characteristic equation of $\mathcal{D}_n(x)$) in terms of the partition function. In fact we will prove an exact finite n analog (Thm. 4.2) of the formulæ that are obtained by variational methods in the $n \to \infty$ limit [9]. We start by expressing the explicit relation between the partition function and the spectral curve of the isomonodromic system.

4.1 Virasoro generators and the spectral curve

To express the result in a compact form, introduce the following **local Virasoro** generators

$$\mathbb{V}^{(r)}_{-J} := \sum_{M=1}^{d_r-J} M t_{r,M+J} \frac{\partial}{\partial t_{r,M}} , \qquad J = 0, \dots, d_r - 1 , \ r = 0, \dots, K \tag{4-1}$$

in terms of which we define the following differential operator with coefficients that are rational functions of x

$$\mathbb{D}(x) := \sum_{i=1}^{L} \frac{1}{x - a_i} \frac{\partial}{\partial a_i} - \sum_{J=0}^{d_0-3} x^J \mathbb{V}^{(0)}_{-J-2} - \sum_{r=1}^{K} \sum_{J=2}^{d_r+1} \frac{1}{(x - c_r)^J} \mathbb{V}^{(r)}_{2-J} - \sum_{r=1}^{K} \frac{1}{x - c_r} \frac{\partial}{\partial c_r} . \tag{4-2}$$

Theorem 4.1 *The characteristic polynomial of the matrix $\mathcal{D}_n(x)$ in the differential system (3-34) is given by*

$$\det \left(y - \mathcal{D}_n(x) \right) = y^2 - y V'(x) + \hbar \left\langle \mathrm{Tr} \frac{V'(M) - V'(x)}{M - x} \right\rangle - \sum_{i=1}^{L} \frac{\hbar^2}{x - a_i} \partial_{a_i} \ln(\mathcal{Z}_n) \tag{4-3}$$

$$= y^2 - y V'(x) + \hbar \mathrm{Tr}_n \left(\frac{V'(Q) - V'(x)}{Q - x} \right) - \sum_{i=1}^{L} \frac{\hbar^2}{x - a_i} \partial_{a_i} \ln(\mathcal{Z}_n) \tag{4-4}$$

$$= y^2 - y V'(x) + n \sum_{J=1}^{d_0-1} t_{0,J+1} x^{J-1} - \hbar^2 \mathbb{D}(x) \ln \mathcal{Z}_n, \tag{4-5}$$

and the quadratic trace invariant is

$$\mathrm{Tr} \mathcal{D}_n(x)^2 = V'(x)^2 - 2n \sum_{J=1}^{d_0-1} t_{0,J+1} x^{J-1} + 2\hbar^2 \mathbb{D}(x) \ln \mathcal{Z}_n. \tag{4-6}$$

Proof: The equivalence of (4-3) and (4-4) follows from the well–known relation

$$\langle \mathrm{Tr}(f(M)) \rangle = \mathrm{Tr}_n(f(Q)) \tag{4-7}$$

for any scalar function $f(x)$ for which the $\langle \mathrm{Tr}(f(M)) \rangle$ is a convergent integral. The equivalence of (4-5) and (4-6) follows from (3-38).

To prove (4-4) we use the recursion relation (3-33) and the explicit expression of $\mathcal{D}_n$, we obtain

$$\mathcal{D}_{n+1} = R_n \mathcal{D}_n R_n^{-1} + \hbar R_n' R_n^{-1} \tag{4-8}$$

$$R_n' R_n^{-1} = \begin{bmatrix} 0 & 0 \\ \frac{1}{\gamma_{n+1}} & 0 \end{bmatrix} , \qquad R_n^{-1} R_n' = \begin{bmatrix} 0 & -\frac{1}{\gamma_n} \\ 0 & 0 \end{bmatrix} . \tag{4-9}$$

Therefore,

$$\mathrm{Tr} \left(\mathcal{D}_{n+1}(x)^2 \right) = \mathrm{Tr} \left(\mathcal{D}_n(x)^2 \right) + 2\hbar \mathrm{Tr} \left(\mathcal{D}_n(x) R_n^{-1} R_n' \right) + \hbar^2 \mathrm{Tr} \left(\left(R_n' R_n^{-1} \right)^2 \right) \tag{4-10}$$

$$= \mathrm{Tr} \left(\mathcal{D}_n(x)^2 \right) - 2\hbar \left(\frac{V'(Q) - V'(x)}{Q - x} \right)_{nn} - 2\hbar^2 \sum_{i=1}^{L} \frac{\varkappa_i \psi_n^2(a_i)}{x - a_i} \tag{4-11}$$

$$= \mathrm{Tr} \left(\mathcal{D}_n(x)^2 \right) - 2\hbar \left(\frac{V'(Q) - V'(x)}{Q - x} \right)_{nn} + 2\hbar^2 \sum_{i=1}^{L} \frac{1}{x - a_i} \partial_{a_i} \ln(h_n), \tag{4-12}$$

20

where we have used eqs. (3-36), (3-37) in (4-11) and (2-48) in (4-12). These equations imply that

$$\text{Tr}(\mathcal{D}_n(x)^2) = \text{Tr}(\mathcal{D}_1(x)^2) - 2\hbar \sum_{j=1}^{n-1} \left(\frac{V'(Q) - V'(x)}{Q - x} \right)_{jj} + 2\hbar^2 \sum_{j=1}^{n-1} \sum_{i=1}^{L} \frac{1}{x - a_i} \partial_{a_i} \ln(h_j). \tag{4-13}$$

From the definition of $\mathcal{D}_1$, we have

$$\mathcal{D}_1 = \hbar \begin{bmatrix} \pi_0' & \phi_0' \\ \pi_1' & \phi_1' \end{bmatrix} \begin{bmatrix} \pi_0 & \phi_0 \\ \pi_1 & \phi_1 \end{bmatrix}^{-1}. \tag{4-14}$$

Using (3-12), this gives

$$\det(\mathcal{D}_1(x)) = \hbar^2 \gamma_1 e^{-\frac{1}{\hbar}V(x)} \phi_0' \pi_1' \tag{4-15}$$

$$= \hbar^2 \sqrt{\frac{h_1}{h_0}} e^{-\frac{1}{\hbar}V(x)} \sqrt{\frac{1}{h_0 h_1}} \frac{d}{dx} \left[e^{+\frac{1}{\hbar}V(x)} \int_{\varkappa} \frac{e^{-\frac{1}{\hbar}V(z)}}{x - z} dz \right] \tag{4-16}$$

$$= \frac{\hbar^2}{h_0} \left[\frac{1}{\hbar} V'(x) \int_{\varkappa} \frac{e^{-\frac{1}{\hbar}V(z)}}{x - z} dz - \int_{\varkappa} \frac{e^{-\frac{1}{\hbar}V(z)}}{(x - z)^2} dz \right] \tag{4-17}$$

$$= \frac{\hbar^2}{h_0} \left[\frac{1}{\hbar} V'(x) \int_{\varkappa} \frac{e^{-\frac{1}{\hbar}V(z)}}{x - z} dz - \int_{\varkappa} e^{-\frac{1}{\hbar}V(z)} \frac{\partial}{\partial z} \left(\frac{1}{x - z} \right) dz \right] \tag{4-18}$$

$$= \left[\hbar \int_{\varkappa} \frac{(V'(x) - V'(z))\psi_0^2(z)}{x - z} dz + \hbar^2 \sum_{i=1}^{K} \frac{\varkappa_i \psi_0^2(a_i)}{x - a_i} \right], \tag{4-19}$$

and hence

$$\text{Tr}(\mathcal{D}_1^2(x)) = -2\det(\mathcal{D}_1(x)) + \text{Tr}(\mathcal{D}_1(x))^2 \tag{4-20}$$

$$= ((V'(x))^2 - 2\hbar \int_{\varkappa} \frac{(V'(x) - V'(z))\psi_0^2(z)}{x - z} dz - 2\hbar^2 \sum_{i=1}^{K} \frac{\varkappa_i \psi_0^2(a_i)}{x - a_i} \tag{4-21}$$

$$= (V'(x))^2 - 2\hbar \left(\frac{V'(x) - V'(Q)}{x - Q} \right)_{00} + 2\hbar^2 \sum_{i=1}^{L} \frac{1}{x - a_i} \partial_{a_i} \ln(h_0). \tag{4-22}$$

Combining this with (4-13) gives

$$\text{Tr}(\mathcal{D}_n(x)^2) = (V'(x))^2 - 2\hbar \sum_{j=0}^{n-1} \left(\frac{V'(Q) - V'(x)}{Q - x} \right)_{jj} + 2\hbar^2 \sum_{j=0}^{n-1} \sum_{i=1}^{L} \frac{1}{x - a_i} \partial_{a_i} \ln(h_j), \tag{4-23}$$

which, taking the expression (2-11) for the partition function into account, completes the proof of eq. (4-4).
We now proceed to the proof of eq. (4-5). By expanding the third term on the right of (4-4), we obtain

$$\frac{V'(x) - V'(Q)}{x - Q} = \sum_{J=1}^{d_0-1} t_{0,J+1} \sum_{M=0}^{J-1} x^M Q^{J-M-1} + \sum_{r=1}^{K} \sum_{J=1}^{d_r+1} t_{r,J-1} \sum_{M=0}^{J-1} \frac{1}{(x - c_r)^{J-M}} (Q - c_r)^{-M-1}$$

$$= \sum_{J=1}^{d_0-1} t_{0,J+1} x^{J-1} \mathbf{1} + \sum_{J=2}^{d_0-1} \sum_{M=0}^{J-2} x^M t_{0,J+1} Q^{J-M-1} + \sum_{r=1}^{K} \sum_{J=1}^{d_r+1} t_{r,J-1} \sum_{M=0}^{J-1} \frac{1}{(x - c_r)^{J-M}} (Q - c_r)^{-M-1}$$

$$= \sum_{J=1}^{d_0-1} t_{0,J+1} x^{J-1} \mathbf{1} + \sum_{M=0}^{d_0-3} x^M \sum_{J=M+2}^{d_r} t_{0,J} Q^{J-M-1} + \sum_{r=1}^{K} \sum_{M=1}^{d_r+1} \frac{1}{(x - c_r)^M} \sum_{J=M}^{d_r+1} t_{r,J-1} (Q - c_r)^{M-J-1}$$

Now recall that for any deformation matrix X corresponding to an infinitesimal variation ∂ we have

$$\sum_{j=0}^{n-1} X_{jj} = -\frac{\hbar}{2}\partial \ln(\mathcal{Z}_n).$$

Summing the diagonal terms of (4-24) up to $n-1$ and substituting into (4-4) we therefore obtain

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{D}_n(x)^2) = {} & (V'(x))^2 - 2n\hbar \sum_{J=1}^{d_0-1} t_{0,J+1}x^{J-1} \\
& - 2\hbar^2 \sum_{M=1}^{d_0-2} x^M \sum_{J=M+1}^{d_0} (J-M-1)t_{0,J+1}\frac{\partial \ln(\mathcal{Z}_n)}{\partial t_{0,J-M-1}} \\
& - 2\hbar^2 \sum_{r=1}^{K}\sum_{M=2}^{d_r+1} \frac{1}{(x-c_r)^M}\sum_{J=M}^{d_r+1}(J-M+1)t_{r,J-1}\frac{\partial \ln(\mathcal{Z}_n)}{\partial t_{r,J-M+1}} \\
& - 2\hbar^2 \sum_{r=1}^{K}\frac{1}{x-c_r}\frac{\partial \ln(\mathcal{Z}_n)}{\partial c_r} + 2\hbar^2 \sum_{i=1}^{L}\frac{1}{x-a_i}\frac{\partial \ln(\mathcal{Z}_n)}{\partial a_i}
\end{aligned}
$$

Using eq. (4-25) we finally get

$$
\begin{aligned}
\det\big(y - \mathcal{D}_n(x)\big) = {} & y^2 - yV'(x) + n\hbar \sum_{J=1}^{d_0-1} t_{0,J+1}x^{J-1} + \hbar^2 \sum_{M=0}^{d_0-3} x^M \sum_{J=M+2}^{d_0-1}(J-M-1)t_{0,J+1}\frac{\partial \ln(\mathcal{Z}_n)}{\partial t_{0,J-M-1}} \\
& + \hbar^2 \sum_{r=1}^{K}\sum_{M=2}^{d_r+1}\frac{1}{(x-c_r)^M}\sum_{J=M}^{d_r+1}(J-M+1)t_{r,J-1}\frac{\partial \ln(\mathcal{Z}_n)}{\partial t_{r,J-M+1}} \\
& + \hbar^2 \sum_{r=1}^{K}\frac{1}{x-c_r}\frac{\partial \ln(\mathcal{Z}_n)}{\partial c_r} - \hbar^2 \sum_{i=1}^{L}\frac{1}{x-a_i}\frac{\partial \ln(\mathcal{Z}_n)}{\partial a_i},
\end{aligned}
$$

which completes the proof of eq. (4-5).

4.2 Spectral residue formulæ

Theorem 4.1, which determines all the coefficients of the spectral curve as logarithmic derivatives of the partition function, may be expressed in another form, in which the individual deformation parameters, as well as the logarithmic derivatives with respect to them, may be directly expressed as spectral invariants. The characteristic equation of $\mathcal{D}_n(x)$

$$\det\left(y(x)\mathbf{I} - \mathcal{D}_n(x)\right) = 0,$$

defines a hyperelliptic curve $\mathcal{C}_n$ as a 2–sheeted branched cover of the Riemann sphere, on which y is a meromorphic function. It follows from (3-36) and Theorem 4.1 that y, viewed as a double valued function of x, has the same pole structure and degree as $\mathcal{D}_n(x)$ at the points $\{c_0 = \infty, c_r, a_i\}$, but that the points $\{a_i\}$ are branch points.

Let $Y_{\pm}(x)$ denote the two values of $y(x)$. Defining

$$W(x) := \hbar \mathrm{Tr}_n \frac{V'(x) - V'(Q)}{x - Q} - \sum_{j=1}^{L} \frac{\hbar^2 \partial_{a_j} \ln \mathcal{Z}_n}{x - a_j},$$

it follows from the explicit expression (4-4) for the spectral curve that, near any of the poles $c_0 = \infty$, $c_1, \ldots, c_K$, the two branches have the asymptotic form

$$Y_\pm(x) = \frac{1}{2}V'(x) \pm \sqrt{\frac{1}{4}(V'(x))^2 - W}$$

$$\sim \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} V'(x) \mp \frac{1}{V'(x)}\left(W + \frac{W^2}{(V(x)')^2} + \ldots \right) + \left\{ \begin{matrix} \mathcal{O}(x^{-2d_0-1}) & x \to \infty \\ \mathcal{O}((x - c_r)^{2d_r+2}) & x \to c_r \end{matrix} \right. \qquad (4\text{-}30)$$

Before proceeding we have to point out that here we have assumed that $d_0 \geq 1$, for otherwise ∞ is a branch-point of the spectral curve.

Theorem 4.2 *The following residue formulæ express the deformation parameters and the logarithmic derivatives of $\mathcal{Z}_n$ as spectral invariants of the matrix $\mathcal{D}_n(x)$.*

$$t_{0,J} = - \operatorname*{res}_{x=\infty} \frac{Y_+(x)}{x^J} dx, \quad J = 1 \ldots d_0 \tag{4-31}$$

$$t_{r,J} = - \operatorname*{res}_{x=c_r} (x - c_r)^J Y_+(x) dx, \quad r = 1, \ldots, K, \ J = 1, \ldots, d_r \tag{4-32}$$

$$\hbar^2 \partial_{t_{0,0}} \ln \mathcal{Z}_n = \operatorname*{res}_{x=\infty} Y_-(x) dx = -n\hbar, \qquad (\text{ for } d_0 \geq 1) \tag{4-33}$$

$$\hbar^2 \partial_{t_{0,J}} \ln \mathcal{Z}_n = \frac{1}{J} \operatorname*{res}_{x=\infty} Y_-(x) x^J dx, \quad J = 1, \ldots, d_0 \tag{4-34}$$

$$\hbar^2 \partial_{t_{r,J}} \ln \mathcal{Z}_n = \frac{1}{J} \operatorname*{res}_{x=c_r} Y_-(x) \frac{1}{(x - c_r)^J} dx, \quad r = 1, \ldots, K, \ J = 1, \ldots, d_r \tag{4-35}$$

$$\hbar^2 \partial_{c_r} \ln \mathcal{Z}_n = - \operatorname*{res}_{x=c_r} Y_-(x) T'_r(x) dx, \quad r = 1, \ldots, K, \tag{4-36}$$

$$\hbar^2 \partial_{a_j} \ln \mathcal{Z}_n = \frac{1}{2} \operatorname*{res}_{x=a_j} \operatorname{Tr}(\mathcal{D}_n^2(x)) dx . \tag{4-37}$$

Note that identity (4-33) can be established only for $d_0 \geq 1$ for otherwise the residue would not make sense since ∞ becomes a branch-point.

Proof. Considering the various deformations associated to the poles and end-points we have:
At infinity

$$\frac{1}{J} \operatorname*{res}_{x=\infty} x^J Y_- dx = \operatorname*{res}_{x=\infty} \frac{x^J/J}{V'(x)} \left(\hbar \operatorname{Tr}_n \frac{V'(x) - V'(Q)}{x - Q} - \sum \frac{\hbar^2}{x - a_j} \partial_{a_j} \ln \mathcal{Z}_n \right)$$

$$= \hbar \operatorname*{res}_{x=\infty} \frac{x^J/J}{V'(x)} \operatorname{Tr}_n \frac{V'(x) - V'(Q)}{x - Q} = -\frac{\hbar}{J} \operatorname{Tr}_n Q^J = \hbar^2 \partial_{t_{0,J}} \ln \mathcal{Z}_n , \qquad (4\text{-}38)$$

$$J = 1, \ldots d_0 - 2.$$

Note that this computation does not provide the derivatives with respect to the two highest coefficients t_{0,d_0} and t_{0,d_0-1}, which will be computed below. Moreover we should remark that the last equality follows from the following interchange of order of integrals

$$\operatorname*{res}_{x=\infty} \frac{x^J/J}{V'(x)} \operatorname{Tr}_n \frac{V'(x) - V'(Q)}{x - Q} = \sum_{j=0}^{n-1} \operatorname*{res}_{x=\infty} \frac{x^J/J}{V'(x)} \int_\varkappa \frac{V'(x) - V'(z)}{x - z} \pi_j^2(z) e^{-\frac{1}{\hbar}V(z)} dz =$$

$$= \sum_{j=0}^{n-1} \int_\varkappa \operatorname*{res}_{x=\infty} \frac{x^J/J}{V'(x)} \frac{V'(x) - V'(z)}{x - z} \pi_j^2(z) e^{-\frac{1}{\hbar}V(z)} dz = -\sum_{j=0}^{n-1} \int_\varkappa \frac{z^J}{J} \pi_j^2(z) e^{-\frac{1}{\hbar}V(z)} dz = -\frac{1}{J} \operatorname{Tr}_n Q^J \ (4\text{-}39)$$

The exchange is justified by the usual arguments observing that the expression $\frac{V'(x) - V'(z)}{x - z}$ has no singularities at coinciding points $x = z$ (away from the singularities of V').

At the poles c_r

$$\frac{1}{J}\operatorname*{res}_{x=c_r}(x-c_r)^{-J}Y_-\mathrm{d}x = \operatorname*{res}_{x=c_r}\frac{(x-c_r)^{-J}/J}{V'(x)}\hbar\operatorname{Tr}_n\frac{V'(x)-V'(Q)}{x-Q} = -\frac{\hbar}{J}\operatorname{Tr}_n(Q-c_r)^{-J} = \hbar^2\partial_{t_{r,J}}\ln\mathcal{Z}_n,$$
$$J=1,\dots,d_r;$$
$$-\operatorname*{res}_{x=c_r}Y_-(x)T'_r(x)dx = -\operatorname*{res}_{x=c_r}\frac{T'_r(x)}{V'(x)}\hbar\operatorname{Tr}_n\frac{V'(x)-V'(Q)}{x-Q} = \hbar\operatorname{Tr}_nT'_r(Q) = \hbar^2\partial_{c_r}\ln\mathcal{Z}_n \ . \tag{4-40}$$

The last equalities in (4-40) are obtained by a similar argument used for the deformations at $c_0 = \infty$ here above.

At the endpoints a_j

$$\frac{1}{2}\operatorname*{res}_{x=a_j}\operatorname{Tr}(\mathcal{D}_n^2(x)) = \frac{1}{2}\operatorname*{res}_{x=a_j}\left((V'(x))^2 - 2\hbar\operatorname{Tr}_n\frac{V'(x)-V'(Q)}{x-Q} + 2\sum_{j=1}^{L}\frac{\hbar^2}{x-a_j}\partial_{a_j}\ln\mathcal{Z}_n\right) = \hbar^2\partial_{a_j}\ln\mathcal{Z}_n \tag{4-41}$$

The determination Y_- has the asymptotic behavior

$$Y_-(x) \sim \frac{1}{V'(x)}\left(\hbar\operatorname{Tr}_n\frac{V'(x)-V'(Q)}{x-Q} - \sum_{j=1}^{L}\frac{\hbar^2}{x-a_j}\partial_{a_j}\ln\mathcal{Z}_n\right) + \begin{cases} -\frac{n^2\hbar^2}{x^2} + \mathcal{O}(x^{-d_0-2}) & x\to\infty \\ \mathcal{O}((x-c_r)^{d_r+1}) & x\to c_r \end{cases} \tag{4-42}$$

Identities (4-35) and (4-36) follow immediately from the expressions in (4-38), (4-40) and the asymptotic forms (4-42), as do the identities (4-34) for $J \le d_0-2$. For the remaining two values of J (d_0-1, d_0), we compute

$$-\operatorname*{res}_{x=\infty}T'_0(x)Y_-(x)dx = -\operatorname*{res}_{x=\infty}(V'(x)+\mathcal{O}(1/x))\frac{1}{V'(x)}\left(W - \frac{n^2\hbar^2}{x^2} + \mathcal{O}\left(x^{-3}\right)\right)dx$$
$$= -\sum_{r=1}^{K}\hbar\operatorname{Tr}_nT'_r(Q) + \sum_{j=1}^{L}\hbar^2\partial_{a_j}\ln\mathcal{Z}_n = \hbar\operatorname{Tr}_nT'_0(Q), \tag{4-43}$$

where the last equality follows from eq. 2-40 (translational invariance). This identity, together with eqs. (4-34) for $j \le d_0 - 2$ implies

$$\operatorname*{res}_{x=\infty}x^{d_0-1}Y_-(x)dx = \hbar^2\frac{\partial}{\partial t_{0,d_0-1}}\ln\mathcal{Z}_n \ , \tag{4-44}$$

which is the case $J = d_0 - 1$ of (4-34). Similarly

$$-\operatorname*{res}_{x=\infty}xT'_0(x)Y_-(x)dx = -\operatorname*{res}_{x=\infty}\left(xV'(x) + \sum_{r=1}^{K}t_{r,0} + \mathcal{O}(1/x)\right)\frac{1}{V'(x)}\left(W - \frac{n^2\hbar^2}{x^2} + \mathcal{O}\left(x^{-3}\right)\right)dx$$
$$= n^2\hbar^2 - n\hbar\sum_{r=1}^{K}t_{r,0} - \sum_{r=1}^{K}\hbar\operatorname{Tr}_nQT'_r(Q) + \sum_{j=1}^{L}\hbar^2a_j\partial_{a_j}\ln\mathcal{Z}_n$$
$$= \hbar\operatorname{Tr}_nQT'_0(Q) \tag{4-45}$$

where the last equality holds because of dilation invariance. This, together with the above proves (4-34) for $J = d_0$.

The formulæ (4-37) follow from equating the residues at the poles $x = a_j$ in eq. (4-6).

Finally we examine (4-33): the second equality follows from the fact that $t_{0,0}$ appears in the integral (2-11) defining $\mathcal{Z}_n$ only in the overall normalization factor $e^{-\frac{t_{0,0}}{\hbar}}$. The first equality in (4-33) for the cases $d_0 \ge 2$ follows from the asymptotic expression of Y_-. For the case $d_0 = 1$ instead one has to use (4-43) and the fact that $T'_0(x) = t_{0,1} =$ constant.

Remark 4.1 In the formulæ (4-33), (4-34), (4-36), (4-37) we may replace Y_- by $-(Y_+ - V'(x))$; this corresponds to the fact that, in the large n limit, the behavior of $Y(x)$ on the *physical sheet* (i.e. Y_+) is related to the resolvent of the model by

$$Y_+ = V'(x) + \hbar \left\langle \mathrm{Tr}(x - M)^{-1} \right\rangle . \tag{4-46}$$

5   Isomonodromic Tau function

5.1   Isomonodromic deformations and residue formula

In this section we briefly recall the definition of the isomonodromic tau-function given in [11] and compute its logarithmic derivatives in the present case in order to compare it with the partition function. This will lead to the main result of this section, Theorem 5.1, which explicitly gives this relation.

Consider a rational covariant derivative operator on a rank p vector bundle over $\mathbb{C}P^1$

$$\mathcal{D}_x = \partial_x - \mathcal{A}(x) , \tag{5-1}$$

where the connection component $\mathcal{A}(x)$ is a $p \times p$ matrix, rational in x. Deformations of such an operator that preserve its (generalized) monodromy (i.e. including the Stokes' data) are determined infinitesimally by requiring compatibility of the equations

$$\partial_x \Psi(x) = \mathcal{A}\Psi(x) \tag{5-2}$$

$$\partial_{u_i} \Psi(x) = \mathcal{U}_i(x)\Psi(x) , \quad i = 1, \dots . \tag{5-3}$$

where in the second set of equations $U_i(x)$ are also $p \times p$ matrices, rational in x, viewed as components of a connection over the extended space consisting to the product of $\mathbb{C}P^1$ with the space of deformation parameters $\{u_1, \dots\}$. The invariance of the generalized monodromy of $\mathcal{D}_x$ follows [11] from the compatibility of this overdetermined system, which is equivalent to the zero-curvature equations

$$[\partial_x - \mathcal{A}(x), \partial_{u_i} - \mathcal{U}_i(x)] = 0 , \quad [\partial_{u_i} - \mathcal{U}_i(x), \partial_{u_j} - \mathcal{U}_j(x)] = 0 \tag{5-4}$$

Near a pole $x = c_\nu$ of $\mathcal{A}(x)$ a fundamental solution can be found that has the *formal* asymptotic behavior, in a suitable sector:

$$\Psi(x) \sim C_\nu Y_\nu(x) e^{T_\nu(x)} \tag{5-5}$$

where C_ν is a constant matrix,

$$Y_\nu(x) = \mathbf{1} + \mathcal{O}(x - c_\nu) \tag{5-6}$$

is a formal power series in the local parameter $(x - c_\nu)$ (or $1/x$ for the pole at infinity) and $T_\nu(x)$ is a Laurent-polynomial matrix in the local parameter, plus a possible logarithmic term $t_0 \ln(x - c)$. In the generic case $T_\nu(x)$ is a diagonal matrix, and, more generally, may be an element of a maximal Abelian subalgebra containing an element with no multiple eigenvalues. The locations of the pols c_ν and the coefficients of the nonlogarithmic part of $T_\nu(x)$ are the independent deformation parameters. The deformation of the connection matrix $A(x)$ is determined by the requirement that the (generalized) monodromy data be independent of all these isomonodromic deformation parameters.

Given a solution of such an isomonodromic deformation problem, one is led to consider the associated isomonodromic τ-function [11], determined by integrating the following closed differential on the space of deformation parameters

$$\omega := \sum_\nu \operatorname*{res}_{x=c_\nu} \mathrm{Tr} \left(Y_\nu^{-1} Y_\nu' \cdot dT_\nu(x) \right) = d \ln \tau^{IM} , \tag{5-7}$$

where the sum is over all poles of $\mathcal{A}(x)$ (including possibly one at $x = \infty$), and the differential is over all the independent isomonodromic deformation parameters , In the present situation $\mathcal{A}(x)$ is our 2×2 matrix $\mathcal{D}_n(x)$ and the (generalized) monodromy of the operator $\partial_x - \mathcal{D}_n(x)$ is invariant under changes in the parameters $c_r,\ t_{r,J}\ , a_j$ and n.

5.2 Traceless gauge

For convenience in the computations we perform a scalar gauge transformation of the ODE by choosing *quasipolynomials* rather than polynomials. Explicitly we set

$$\Psi_n(x) := e^{-\frac{1}{2\hbar}V(x)}\mathbf{\Gamma}_n(x) = \begin{bmatrix} \psi_{n-1}(x) & \widetilde{\psi}_{n-1}(x) \\ \psi_n(x) & \widetilde{\psi}_n(x) \end{bmatrix}$$

$$\hbar\Psi_n'(x) = \mathcal{A}_n(x)\Psi_n(x)$$

$$\mathcal{A}_n(x) = \mathcal{D}_n(x) - \frac{1}{2}V'(x)\mathbf{1}, \tag{5-8}$$

where

$$\widetilde{\psi}_n := e^{-\frac{1}{2\hbar}V(x)}\phi_n \tag{5-9}$$

In this gauge the matrix of the ODE is traceless and the infinitesimal deformation matrices are transformed correspondingly by addition of the identity element multiplied by the derivatives of $-\frac{1}{2\hbar}V(x)$ with respect to the parameters $\{c_r, t_{rJ}, a_j\}$. This choice gives a consistent reduction of the general $\mathbf{gl}(p,\mathbf{C})$ isomonodromic deformation problem to $\mathbf{sl}(p,\mathbf{C})$. (To be precise, this would require a further, x–independent diagonal gauge transformation of the form $\mathrm{diag}(h_{n-1}^{\frac{-1}{2}}, h_n^{\frac{1}{2}})$ to render the infinitesimal deformation matrices also traceless.)

At each of the poles $c_0 := \infty, c_1, \ldots$ we then have the following asymptotic expansions. (To simplify notation, the index n is omitted in labeling the fundamental system Ψ and its local asymptotic form .)

$$\Psi(x) \sim C_r Y_r(x) \ \exp\left[\left(-\frac{1}{2\hbar}T_r(x) + \delta_{r0}\left(n + \frac{1}{2\hbar}\sum_{r\geq 1}t_{r,0}\right)\ln(x)\right)\sigma_3\right] . \tag{5-10}$$

Here we have set

$$Y_0(x) := \mathbf{1} + \sum_{k=1}^{\infty}\frac{Y_{0;k}}{x^k} , \qquad C_0 = \begin{bmatrix} 0 & \sqrt{h_{n-1}} \\ \frac{1}{\sqrt{h_n}} & 0 \end{bmatrix}$$

$$Y_r(x) := \mathbf{1} + \sum_{k=1}^{\infty}Y_{r;k}(x-c_r)^k ; \qquad C_r = \begin{bmatrix} \pi_{n-1}(c_r)e^{-\frac{\check{V}_r(c_r)}{2\hbar}} & (c_r - Q)_{n-1,0}^{-1}\sqrt{h_0}e^{\frac{\check{V}_r(c_r)}{2\hbar}}, \\ \pi_n(c_r)e^{-\frac{\check{V}_r(c_r)}{2\hbar}} & (c_r - Q)_{n,0}^{-1}\sqrt{h_0}e^{\frac{\check{V}_r(c_r)}{2\hbar}} \end{bmatrix}, \tag{5-11}$$

where $\check{V}_r(x) = V(x) - T_r(x)$ is the holomorphic part of the potential at c_r. The asymptotic forms given by (5-10)–(5-11) follow from the fact that, in any Stokes' sector near c_r, the second-kind solutions behave like

$$\widetilde{\psi}_n(x) \underset{x\to\infty}{\sim} e^{\frac{1}{2\hbar}V(x)}\sum_{k=n+1}^{\infty}x^{-k}\int_{\varkappa}\pi_n(z)e^{-\frac{1}{\hbar}V}z^{k-1} = e^{\frac{1}{2\hbar}V(x)}x^{-n-1}\sqrt{h_n}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \tag{5-12}$$

$$\widetilde{\psi}_n(x) \underset{x\to c_r}{\sim} e^{\frac{1}{2\hbar}V(x)}\int_{\varkappa}dz\frac{e^{-\frac{1}{\hbar}V(z)}\pi_n(z)}{c_r - z}(1 + \mathcal{O}(x - c_r))$$

$$= e^{\frac{1}{\hbar}V(x)}(c_r - Q)_{n,0}^{-1}\sqrt{h_0}(1 + \mathcal{O}(x - c_r)) \tag{5-13}$$

Near the endpoints a_j we have

$$\Psi(x) \sim A_j \cdot Y_j(x) \cdot \exp\left[-\varkappa_j\ln(x - a_j)\sigma_+\right] , \qquad \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A_j = \begin{bmatrix} \pi_{n-1}(a_j)e^{-\frac{V(a_j)}{2\hbar}} & e^{\frac{V(a_j)}{2\hbar}}\int_{\varkappa}dz\frac{e^{-\frac{1}{\hbar}V(z)}(\pi_{n-1}(z)-\pi_{n-1}(a_j))}{a_j - z} \\ \pi_n(a_j)e^{-\frac{V(a_j)}{2\hbar}} & e^{\frac{V(a_j)}{2\hbar}}\int_{\varkappa}dz\frac{e^{-\frac{1}{\hbar}V(z)}(\pi_n(z)-\pi_n(a_j))}{a_j - z}, \end{bmatrix} \tag{5-14}$$

since the matrix

$$\Psi(x) \cdot \exp\left(-\sigma_+ \int_\varkappa dz \frac{e^{-\frac{1}{\hbar}V(z)}}{x-z}\right) = \begin{bmatrix} \pi_{n-1}(x)e^{-\frac{V(x)}{2\hbar}} & e^{\frac{V(x)}{2\hbar}} \int_\varkappa dz \frac{e^{-\frac{1}{\hbar}V(z)}(\pi_{n-1}(z)-\pi_{n-1}(x))}{x-z} \\ \pi_n(x)e^{-\frac{V(x)}{2\hbar}} & e^{\frac{V(x)}{2\hbar}} \int_\varkappa dz \frac{e^{-\frac{1}{\hbar}V(z)}(\pi_n(z)-\pi_n(x))}{x-z} \end{bmatrix} \tag{5-15}$$

is *analytic* in a neighborhood of a_j and has the limiting value indicated in (5-14). The function $-\int_\varkappa dz \frac{e^{-\frac{1}{\hbar}V(z)}}{x-z}$ in the exponential of the second matrix in this formula has the same singularity as $\varkappa_j \ln(x - a_j)$. (The signs in (5-14) follow from the orientation of the contour originating at a_j.)

The differential (5-7) can now be written

$$\hbar d \ln \tau_n^{IM} = \frac{1}{2} \sum_{r=0} \operatorname*{res}_{x=c_r} dT_r(x) \operatorname{Tr}\left(Y_r^{-1}Y_r'\sigma_3\right) + \sum_j \operatorname*{res}_{x=a_j} \frac{\hbar \varkappa_j da_j}{x-a_j} \operatorname{Tr}\left(Y_j^{-1}Y_j'\sigma_+\right) \tag{5-16}$$

where the differential involves the isomonodromic parameters only

$$d := \sum_{r=0} \left(\sum_{J=1}^{d_r} dt_{r,J} \frac{\partial}{\partial t_{r,J}} + dc_r \frac{\partial}{\partial c_r}\right) + \sum_j da_j \frac{\partial}{\partial a_j} = \sum_{r=0}^K d_{(r)} + \sum_j da_j \frac{\partial}{\partial a_j} \tag{5-17}$$

We now derive residue formulæ for the deformation parameters and the logarithmic derivatives of the tau function for our rational 2×2 isomonodromic deformation problem. These essentially are the same as the formulæ of Thm. 4.2 giving the latter quantities in terms of logarithmic derivatives of the partition function of the matrix model[5].

Consider the quadratic spectral invariant near any of the singularities: by virtue of the asymptotics (5-10, 5-14) we have, near c_r and a_j respectively (setting $S := 2n\hbar + \sum_{r=1}^K t_{r,0}$)

$$\operatorname{Tr}(\mathcal{A}^2(x)) = \hbar^2 \operatorname{Tr}((\Psi'\Psi^{-1})^2) = \hbar^2 \operatorname{Tr}\left(\left(Y_r^{-1}Y_r'\right)^2\right) - \hbar \operatorname{Tr}\left(Y_r^{-1}Y_r'\sigma_3 \left(T_r' - \frac{\delta_{r0}S}{x}\right)\right)$$
$$+ \frac{1}{2}\left(T_r' - \frac{\delta_{r0}S}{x}\right)^2 \tag{5-18}$$

$$\operatorname{Tr}(\mathcal{A}^2(x)) = \hbar^2 \operatorname{Tr}((\Psi'\Psi^{-1})^2) = \hbar^2 \operatorname{Tr}\left(\left(Y_j^{-1}Y_j'\right)^2\right) + \frac{2\hbar\varkappa_j}{x-a_j} \operatorname{Tr}\left(Y_j^{-1}Y_j'\sigma_+\right) . \tag{5-19}$$

Taking the principal part at each singularity and using Liouville's theorem (since $\operatorname{Tr}\mathcal{A}^2$ is *a priori* a rational function) we find

$$\operatorname{Tr}(\mathcal{A}^2(x)) = \sum_{r=0}^K \left[\frac{1}{2}\left(T_r' - \frac{\delta_{r0}S}{x}\right)^2 + \frac{1}{\hbar}\left(\left(T_r' - \frac{\delta_{r0}S}{x}\right) \operatorname{Tr}(Y_r^{-1}Y_r'\sigma_3)\right)\right]_{r,+}$$
$$+ \frac{2}{\hbar} \sum_{j=1}^L \left[\frac{\varkappa_j \operatorname{Tr}(Y_j'(a_j)\sigma_+)}{x-a_j}\right] \tag{5-20}$$

where the subscripts $_{r,+}$ mean the singular part at the pole $x = c_r$ (including the constant for $x = c_0 = \infty$). Consider now the spectral curve of the connection $\hbar \partial_x - \mathcal{A}(x)$

$$w^2 = \frac{1}{2}\operatorname{Tr}\mathcal{A}^2(x) , \tag{5-21}$$

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[5] The case of an arbitrary rank rational, nonresonant isomonodromic deformation problem will be developed elsewhere [5], together with further properties that allows us to view these as nonautonomous Hamiltonian systems, in which the logarithmic derivatives of the τ-function computed below, are interpreted as the Hamiltonians generating the deformation dynamics.

one immediately finds

$$w_\pm(x) = \pm\sqrt{\frac{1}{4}\sum_r\left(T_r' - \frac{\delta_{r0}S}{x}\right)^2_{r,+} - \frac{1}{2\hbar}\left(\left(T_r' - \frac{\delta_{r0}S}{x}\right)\mathrm{Tr}(Y_r^{-1}Y_r'\sigma_3)\right)_{r,+} + \frac{1}{\hbar}\sum_j\left[\frac{\varkappa_j\mathrm{Tr}(Y_j'(a_j)\sigma_+)}{x - a_j}\right]}$$

(5-22)

Near any of the poles one has the asymptotic behavior

$$\pm w_\pm = \begin{cases} \dfrac{1}{2}T_r' - \dfrac{\delta_{r0}S}{2x} - \dfrac{1}{2\hbar T_r'(x)}\left(T_r'\mathrm{Tr}(Y_r^{-1}Y_r'\sigma_3)\right)_{r,+} + \begin{cases} \mathcal{O}((x - c_r)^{d_r+1}) & \text{near } x = c_r \\ \mathcal{O}(x^{-d_0-2}) & \text{near } x = \infty \end{cases} \\ \sqrt{\dfrac{\varkappa_j\mathrm{Tr}(Y_j'(a_j)\sigma_+)}{\hbar(x - a_j)}}(1 + \mathcal{O}(x - a_j)) & \text{near } x = a_j \end{cases}$$

(5-23)

This immediately implies the following identities

$$n\hbar + \frac{1}{2}\sum_r t_{r,0} = \mp\mathop{\mathrm{res}}_{x=\infty} w_\pm dx$$

$$\frac{1}{2}t_{0,J} = \mp\mathop{\mathrm{res}}_{x=\infty}\frac{1}{x^J}w_\pm dx, \quad J \geq 1$$

$$\frac{1}{2}t_{r,J} = \mp\mathop{\mathrm{res}}_{x=c_r}(x - c_r)^J w_\pm dx$$

(5-24)

and

$$\hbar^2\frac{\partial}{\partial t_{r,J}}\ln\tau_n^{IM} = \mp\mathop{\mathrm{res}}_{x=c_r}\frac{(x - c_r)^{-J}}{J}w_\pm dx$$

(5-25)

$$\hbar^2\frac{\partial}{\partial t_{0,J}}\ln\tau_n^{IM} = \mp\mathop{\mathrm{res}}_{x=\infty}\frac{x^J}{J}w_\pm dx$$

(5-26)

$$\hbar^2\frac{\partial}{\partial c_r}\ln\tau_n^{IM} = \pm\mathop{\mathrm{res}}_{x=c_r} T_r'(x)w_\pm dx$$

(5-27)

$$\hbar^2\frac{\partial}{\partial a_j}\ln\tau_n^{IM} = \mathop{\mathrm{res}}_{x=a_j}(w_\pm)^2 dx.$$

(5-28)

In order to compare with the formulæ given in Thm. (4.2) we note that the eigenvalues w of $\mathcal{A}(x)$ and Y of $\mathcal{D}_n(x)$ are related as follows due to the change of gauge (5-8)

$$Y_\pm = \frac{1}{2}V'(x) + w_\pm.$$

(5-29)

Comparing eqs. (4-34)–(4-37) with equations (5-26)–(5-28) we obtain

$$\hbar^2\partial_{t_{0,J}}\ln\frac{\mathcal{Z}_n}{\tau_n^{IM}} = \frac{1}{2J}\mathop{\mathrm{res}}_{x=\infty} x^J V'$$

$$\hbar^2\partial_{t_{r,J}}\ln\frac{\mathcal{Z}_n}{\tau_n^{IM}} = \frac{1}{2J}\mathop{\mathrm{res}}_{x=c_r}\frac{1}{(x - c_r)^j}V'(x)dx, \quad r = 1, \dots, K, \ J = 1, \dots, d_r$$

$$\hbar^2\partial_{c_r}\ln\frac{\mathcal{Z}_n}{\tau_n^{IM}} = \frac{1}{2}\mathop{\mathrm{res}}_{x=c_r} T_r'(x)V'(x)dx, \quad r = 1, \dots, K,$$

$$\hbar^2\partial_{a_j}\ln\mathcal{Z}_n = \hbar^2\partial_{a_j}\ln\tau_n^{IM}.$$

(5-30)

These relations define a closed differential

$$d\ln\left(\frac{\mathcal{Z}_n}{\tau_n^{IM}}\right) =: d\ln(\mathcal{F}_n),$$

(5-31)

where the quantity $\mathcal{F}_n$ is determined up to a multiplicative factor independent of the isomonodromic deformation parameters $\{c_r, t^a_{rJ}, a_j\}_{J \geq 1}$, but which may depend on n. This may be explicitly integrated to give

$$\ln\left(\frac{\mathcal{F}_n}{n!}\right) = -\frac{1}{2\hbar^2} \sum_{0 \leq q < r \leq K} \operatorname*{res}_{x = c_r} T'_r(x) T_q(x) , \tag{5-32}$$

where we have chosen to include the integration constant $\ln n!$ for reasons that will be explained in the remarks below. Note that $\mathcal{F}_n$ does not depend on the end-points parameters $\{a_j\}$, only those entering in the potential V. Since the definition of the isomonodromic tau function τ^{IM}_n allows normalizations depending arbitrarily on the monodromy data we obtain the following result.

Theorem 5.1 *Up to multiplicative terms that are independent of the isomonodromic deformation parameters $\{t_{r,J}, c_r, a_j\}$, the partition function $\mathcal{Z}_n = \mathcal{Z}_n(\{t_{r,J}, c_r, a_j\}|[\varkappa])$ of the generalized random matrix model and the isomonodromic tau function τ^{IM}_n for the associated ODE are related by*

$$\mathcal{Z}_n = \tau^{IM}_n \mathcal{F}_n, \tag{5-33}$$

where $\mathcal{F}_n$ is given in (5-32).

We conclude with two remarks regarding the multiplicative factor $\mathcal{F}_n$ relating $\mathcal{Z}_n$ and τ^{IM}_n.

Remark 5.1 (Change of gauge)

The factor $\mathcal{F}_n$ can be eliminated by another choice of gauge. The original traceless gauge is such that near any singularity c_r

$$\Psi(x) = C_r Y_r(x) \mathrm{e}^{-\frac{1}{2}T_r(x)\sigma_3} . \tag{5-34}$$

(The point $x = \infty$ would require slight modifications in the argument which do not, however, change the result.)

The tau function satisfies

$$\hbar^2 d_{(r)} \ln \tau^{IM}_n = -\frac{1}{2} \operatorname*{res}_{c_r} \operatorname{Tr}(Y^{-1}_r Y'_r \sigma_3 dT_r) . \tag{5-35}$$

We make a scalar gauge transformation, depending on an arbitrary parameter c, of the form

$$\widetilde{\Psi} := \Psi \mathrm{e}^{-\frac{c}{2}V(x)} = \widetilde{C}_r \widetilde{Y}_r \mathrm{e}^{-\left(\frac{1}{2}\sigma_3 + \frac{c}{2}\mathbf{1}\right)T_r}(\mathbf{1} + \mathcal{O}(x - c + r)) \tag{5-36}$$

where

$$\widetilde{Y}_r := Y_r(x) \mathrm{e}^{-\frac{c}{2}(f_r(x) - f_r(c_r))} , \quad \widetilde{C}_r = C_r \mathrm{e}^{\frac{c}{2}f_r(c_r)} \tag{5-37}$$

and $f_r(x)$ is the regular part of $V(x)$ at the point c_r

$$f_r(x) := \sum_{q \neq r} T_q(x) . \tag{5-38}$$

The tau function for the gauge transformed system satisfies

$$d_{(r)} \ln \widetilde{\tau}^{IM}_n = \operatorname*{res}_{c_r} \operatorname{Tr}\left(\widetilde{Y}^{-1}_r \widetilde{Y}'_r \left(\sigma_3 - \frac{c}{2}\mathbf{1}\right) dT_r\right) \tag{5-39}$$

$$= \operatorname*{res}_{c_r} \operatorname{Tr}\left[\left(Y^{-1}_r Y'_r - \frac{c}{2}f'_r\right)\left(\sigma_3 - \frac{c}{2}\mathbf{1}\right) dT_r\right] \tag{5-40}$$

$$= \frac{c^2}{2} \operatorname*{res}_{c_r} f'_r dT_r - \overbrace{\frac{1}{2} \operatorname*{res}_{c_r} \operatorname{Tr}(Y^{-1}_r Y'_r \sigma_3 dT_r)}^{d_{(r)} \ln \tau^{IM}_n} - \frac{c}{2} \overbrace{\operatorname*{res}_{c_r} \operatorname{Tr}(Y^{-1}_r Y'_r)}^{=0} dT_r \tag{5-41}$$

$$= \frac{c^2}{2} \operatorname*{res}_{c_r} f'_r dT_r + d_{(r)} \ln \tau^{IM}_n , \tag{5-42}$$

where we have used that fact that

$$\operatorname{Tr} Y^{-1}_r Y'_r = 0 . \tag{5-43}$$

29

Recall that

$$d_{(r)} \ln \tau_n^{IM} = d_{(r)} \ln \mathcal{Z}_n + d_{(r)} \ln \mathcal{F}_n = d_{(r)} \ln \mathcal{Z}_n + \frac{1}{2} \underset{c_r}{\mathrm{res}} \, f'_r dT_r \tag{5-44}$$

so that

$$d_{(r)} \ln \widetilde{\tau}_n^{IM} = (c^2 + 1)d_{(r)}\mathcal{F}_n + d_{(r)} \ln \mathcal{Z}_n \ . \tag{5-45}$$

It follows that if c is chosen so that $c^2 + 1 = 0$ (i.e. $c = \pm i$) we have

$$\widetilde{\tau}_n^{IM} = \mathcal{Z}_n,$$

up to a multiplicative factor independent of the isomonodromic deformation parameters.

Remark 5.2 (Schlesinger transformations) The shift $n \to n+1$ given by the ladder matrix $R_n(x)$ corresponds to an *elementary Schlesinger* transformation at infinity. (For more details on Schlesinger transformation see [12].)

In general a Schlesinger transformation corresponds to a shift by integers in the spectrum of the logarithmic terms of the matrix $T(x)$ entering the formal asymptotics. We see from formula (5-10) that the shift $n \to n+1$ amounts to increasing the first diagonal entry of the logarithmic term by 1 and decreasing the second entry by 1. As shown in [12] the two corresponding tau functions are related by

$$\frac{\tau_{n+1}^{IM}}{\tau_n^{IM}} = (Y_{0;1})_{12} \tag{5-46}$$

where the right hand side is the (1,2) matrix entry of the matrix $Y_{0;1}$ in eq. (5-11). A simple computation (using eq. (5-12)) shows that

$$(Y_{n,0;1})_{12} = h_n, \tag{5-47}$$

in agreement with Thm. 5.1, since (by eq. (2-11))

$$\frac{\mathcal{Z}_{n+1}}{\mathcal{Z}_n} = h_n(n+1) \ . \tag{5-48}$$

This explains why the integration constant $\ln n!$ has been included in eq. (5-32); it assures that the dependence on the discrete parameter n in the relation between $\mathcal{Z}_n$ and τ_n^{IM} is consistent with that implied by the Schlesinger transformations.

Remark 5.3 [Relation with Painlevé equations] Since all the Painlevé equations arise as equations of isomonodromic deformation of certain 2×2 ODEs with prescribed pole locations, it is clear that choosing appropriately the set of contours and the semiclassical measure one can generate special solutions of the Painlevé equations in terms of such orthogonal polynomials.

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