Isomonodromic deformation of resonant rational connections

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Abstract

We analyze isomonodromic deformations of rational connections on the Riemann sphere with Fuchsian and irregular singularities. The Fuchsian singularities are allowed to be of arbitrary resonant index; the irregular singularities are also allowed to be resonant in the sense that the leading coefficient matrix at each singularity may have arbitrary Jordan canonical form, with a genericity condition on the Lidskii submatrix of the subleading term. We also give the relevant notion of isomonodromic tau function extending the one given for non-resonant deformations by Miwa-Jimbo-Ueno. The tau function is expressed purely in terms of spectral invariants of the matrix of the connection.

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1 Introduction

The history of isomonodromic deformations dates back to Schlesinger [17] who studied rational connections of size $r \times r$ with simple poles (Fuchsian singularities)

$$A(x) = \sum_{\gamma_i} \frac{A_i}{x - \gamma_i} \tag{1-1}$$

and defined a system of differential equations describing the dependence of the residue matrices A_i on the position of the poles γ_i under the condition that the monodromy representation induced by the kernel of the connection is independent of the position of the poles.

Later on the authors of [11] described an extended system of equations where *irregular* singularities were considered;

$$A(x) = \sum_{\gamma_i} \sum_{k=0}^{r_i} \frac{A_{i.k}}{(x - \gamma_i)^{k+1}} \tag{1-2}$$

in this case the differential equations are with respect to certain exponents in the formal asymptotic behavior of the kernel solution and the "monodromic" data are extended to include the parameters appearing in Stokes' phenomenon.

In both settings the study was limited to **non-resonant** connections. The notion of resonance for a rational connection is as follows; for a Fuchsian singularity we say that the connection is resonant if the residual spectrum at a singularity (the spectrum of the residue matrix) contains eigenvalues differing by a nonzero integer. An irregular singularity is called resonant if the eigenvalues of the leading coefficient are repeated.

In the past years significant classes of examples have appeared in which one is forced to consider resonant isomonodromic deformations. Probably one of the most relevant is in the analysis of quantum cohomology using the notion of Frobenius manifolds [8], where a Fuchsian resonant singularity appears in the relevant isomonodromic deformation.

Lately, the analysis of the Riemann–Hilbert problem associated to biorthogonal polynomials for the multi-matrix models has lead to a system with an irregular resonant singularity [3, 4].

The general structure of an isomonodromic deformation of a resonant Fuchsian connection was addressed in [7] whereas it seems that no attempt is being made in the literature to address isomonodromic deformations of irregular resonant singularities in general. It is the purpose of this paper to analyze these issues. The completely general classification of Fuchsian resonant singularities and their isomonodromic deformations are essentially contained in [7] and we are rephrasing it in Section 4 for the reader's convenience. The main feature which distinguishes these deformations from the usual Schlesinger equations is that in a monodromy-preserving deformation of a resonant Fuchsian connection the deformation matrices may in fact have higher order poles, of degree at most equal to the maximal integral difference between two residual eigenvalues at each pole (the **resonance index**); these higher order poles -however- are the result of the bigger "local" gauge freedom that arises due to the resonant character of the singularity and can in fact always be gauged to zero by a rational gauge equivalence (without changing the position of the poles and the Fuchsian character of the connection, see Sect. 4).

The situation is not dissimilar from the nonresonant case, in which -however- the gauge freedom is restricted to a point where the only arbitrariness is global constant gauge transformations.

The case of isomonodromic deformations of irregular resonant singularities (in the generalized sense of [11]) is quite unexplored, mainly because of the difficulty in analyzing the normal asymptotic form near any such singularity.

The class of resonant singularities which we analyze here may well be considered "minimally" resonant in the sense to follow: we will consider rational connections $A(x)$ such that near an irregular singularity $x = \gamma$ the leading coefficient matrix $A_{r,\gamma}$ may have an arbitrary Jordan canonical form. However we impose a (completely explicit) genericity assumption on the second-leading coefficient matrix as explained in Section 2.

The meaning of our genericity assumption has a clear interpretation in terms of the spectral curve of the connection $A(x)$, i.e. the algebraic curve satisfied by the eigenvalues $y(x)$ of $A(x)$. It ensures that the branching structure of the (desingularized) spectral curve above $x = \gamma$ is in agreement with the dimension of the Jordan cells of the leading coefficient matrix.

This allows us to obtain a canonical form for the asymptotic behavior of a kernel solution $\Psi' = A\Psi$ (Thm. 3.1), which is the crucial tool in order to construct isomonodromic deformations and prove their compatibility.

Tau Function. Another main point of our construction is that we define -following [11]- the notion of "isomonodromic tau function" for the deformations we have defined. Here our approach differs radically from that in [11] inasmuch as our definition of tau function is not obtained in terms of the formal asymptotic data. We rather use the spectral curve itself, thus showing explicitly the spectral nature of the tau function. The approach is along the same lines of [6], where it was shown that Miwa-Jimbo-Ueno's tau function is in fact a spectral invariant expression.

Let us briefly comment on the necessity and interest in a definition of tau function in this context; it has been shown repeatedly [2, 5, 9] that JMU's tau function coincides with Töplitz/Hänkel determinants of moments of measures. This allows to establish a direct connection between the partition function of certain matrix models and the tau function of a (naturally) associated isomonodromic deformation.

In the context of multimatrix models we have many of the same features, namely a partition function and isomonodromic deformations of a certain ODE [3, 4] and it is natural to imagine an analogous relationship with an isomonodromic tau-function. The main obstacle is the absence of a general definition of tau function for resonant irregular singular ODEs.

Although the resonance of the ODE appearing in the two-matrix model is not quite of the class which we analyze in this paper, on the other hand it has many of the same features. In particular the tools developed here are sufficient to analyze that situation (although in a *ad hoc* way). This analysis is contained in a separate paper.

2 Preamble: perturbations of spectra and Lidskii coefficients

In this section we introduce some necessary notations and definitions that will be used in the analysis of the formal asymptotics of singular ODEs. The philosophy inspiring these considerations is that solving a singular ODE by formal series is -to high order- the "same" as finding perturbative eigenvectors of an analytic (formal) perturbation. We recall some relevant facts about analytic (or formal) perturbations of the spectrum of matrices. From our point of view the issue of convergence of the series that will appear is irrelevant since that in our applications only a finite number of coefficients will appear; for this reason we will limit the discussion to the formal aspects of the problem, with the understanding that under mild additional assumptions the considerations to come could be set in an analytic framework.

By *perturbation* we mean a (formal) power series in a small parameter ϵ of the form

$$M(\epsilon) := M_0 + \epsilon M_1 + \mathcal{O}(\epsilon^2) \ . \tag{2-1}$$

One of the main questions in perturbation theory is to understand the behavior of the spectrum of $M(\epsilon)$ and its relation to the "unperturbed" spectrum of $M(0)$. A related question is that of describing the perturbation of the corresponding eigenvectors.

The generic case is very well understood and studied and corresponds to the case where all unperturbed eigenvalues are distinct and simple (i.e. with algebraic multiplicity one). In this case it is not hard to show that each perturbed eigenvalue admits a power series expansion (possibly formal) [12]

$$\lambda_j(\epsilon) = \lambda_j(0) + \mathcal{O}(\epsilon). \tag{2-2}$$

Complications arise when the unperturbed spectrum consists of eigenvalues with algebraic multiplicity higher than one, namely when the Jordan canonical form of M_0 is allowed to be the most general. Without loss of generality we may assume that M_0 is in its Jordan canonical form. Adopting Arnold's [1] notation we denote a matrix in Jordan canonical form by the product of the determinants of its blocks. For example $\alpha^3\alpha^2\beta^4$ denotes a matrix with two Jordan blocks with eigenvalue α of dimension 3×3 and 2×2, and another Jordan block of size 4×4 with eigenvalue β. In this example then the algebraic multiplicity of α is $3+2=5$ but the geometrical multiplicity (the rank of the eigenspace) is 2.

As a general rule a multiple eigenvalue like the α in the example, will split under perturbation in 5 distinct eigenvalues; in fact under certain genericity assumption on the first jet of the perturbation (i.e. M_1) the splitting that will occur is in a triplet and a doublet according to the size of the two Jordan blocks. The eigenvalues of the triplet will be Puiseux series in $\epsilon^{\frac{1}{3}}$ and will be cyclically permuted after a loop around the origin in the ϵ plane; similarly the eigenvalues of the doublet will be Puiseux series in $\epsilon^{1/2}$ enjoying a similar cyclicity.

In [14] were studied sufficient conditions for the splitting of an eigenvalue to occur in cyclic k-tuplets according to the sizes of the Jordan blocks of the unperturbed matrix M_0. More recently the approach of Lidskii has been refined (see [10] and references therein) to handle the cases in which the splitting of eigenvalues does not necessarily respect the Jordan decomposition of M_0. For the purposes of the present paper we restrict ourselves to the "generic" stratum within this class.

In order to describe this theory we start by observing that we can limit ourselves to the case where M_0 has a single eigenvalue which we can set to zero by shifting with the identity matrix. Indeed blocks with distinct eigenvalues will have eigenvalues which will not "mix" under a small perturbation.
Suppose thus that $M_0 = 0^{n_1}0^{n_2}\cdots 0^{n_K}$; we arrange these blocks in weakly decreasing order

$$n_1 \geq n_2 \geq \ldots \geq n_K. \tag{2-3}$$

We next partition the first jet of the perturbation (M_1) in $K\times K$ blocks according to the same block decomposition of M_0; from each of these blocks of M_1 we extract the lower-left entry and form a $K\times K$ matrix; we will call such matrix $L(M_1)_{\{n_1,n_2,\ldots,n_K\}}$ the **Lidskii matrix** of M_1 subordinated to the block decomposition of M_0.

$$M_0 = \left[\begin{array}{cccc|cc|cc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{2-4}$$

$$
M_1 = \left[\begin{array}{c|c|c|c}
 & & & \\
L_{11} & L_{12} & L_{13} & L_{14} \\ \hline
L_{21} & L_{22} & L_{23} & L_{24} \\ \hline
L_{31} & L_{32} & L_{33} & L_{34} \\ \hline
L_{41} & L_{42} & L_{43} & L_{44}
\end{array}\right] \tag{2-5}
$$

Let ℓ_1 be the number of blocks of the same size as n_1: the next block of strictly smaller dimension will be then the $\ell_1 + 1$. Let ℓ_2 be the number of blocks of the same size as the n_{ℓ_1+1}, and so on and so forth. At the end of this procedure the K diagonal blocks are grouped together according to the dimensions

$$
\underbrace{n_1, n_2, \ldots, n_{\ell_1}}_{\ell_1}, \underbrace{n_{\ell_1+1}, \ldots, n_{\ell_1+\ell_2}}_{\ell_2}, \ldots \ . \tag{2-6}
$$

This grouping induces a partitioning of the Lidskii matrix L into blocks; for the example in the figure above we have $(\ell_1 = 1, \ell_2 = 2, \ell_3 = 1)$

$$
L = \left[\begin{array}{c|cc|c}
L_{11} & L_{12} & L_{13} & L_{14} \\ \hline
L_{21} & L_{22} & L_{23} & L_{24} \\
L_{31} & L_{32} & L_{33} & L_{34} \\ \hline
L_{41} & L_{42} & L_{43} & L_{44}
\end{array}\right] \tag{2-7}
$$

We now consider the principal block-submatrix of L according to this decomposition: namely the first principal block submatrix is of size $\ell_1 \times \ell_1$, the next is of size $(\ell_1+\ell_2)^2$, etc. For each of these submatrices we construct the pseudo-characteristic polynomial, namely the determinant of the submatrix minus λ times the projector onto the lower right sub-matrix. At each step we have a submatrix of size $(\ell_1 + \ldots + \ell_j)^2$ and the corresponding pseudo charpoly is of degree ℓ_j.

Definition 2.1 *The roots of these polynomials will be called the* **Lidskii pseudovalues**.
We will say that they are **generic** *if none of them is zero and the discriminant of each pseudocharpoly is nonzero.*

We think that the description of the procedure is sufficiently involved to require an example: continuing with the above one, the Lidskii pseudo-charpoly's are

$$
P_1 = \det[L_{11} - \lambda] \tag{2-8}
$$

$$
P_2 = \det \left[\begin{array}{c|cc}
L_{11} & L_{12} & L_{13} \\ \hline
L_{21} & L_{22} - \lambda & L_{23} \\
L_{31} & L_{32} & L_{33} - \lambda
\end{array}\right] \tag{2-9}
$$

$$
P_3 = \det \left[\begin{array}{c|cc|c}
L_{11} & L_{12} & L_{13} & L_{14} \\ \hline
L_{21} & L_{22} & L_{23} & L_{24} \\
L_{31} & L_{32} & L_{33} & L_{34} \\ \hline
L_{41} & L_{42} & L_{43} & L_{44} - \lambda
\end{array}\right] \tag{2-10}
$$

Let us denote $\lambda_{j,\rho}$, $\rho = 1, \ldots, \ell_j$ the Lidskii pseudovalues: then the eigenvalue 0 of M_0 splits into K cyclic multiplets, the j-th of which is expandable as a Puiseux series in $\epsilon^{\frac{1}{n_j}}$ and with the leading coefficients of these series given by the Lidskii pseudovalues.
In the above example:

we have a $n_1 = 4$-tuplet of cyclic eigenvalues $\lambda_1(\epsilon) = \lambda_{1,1}\epsilon^{1/4} + \ldots$;

we have $\ell_2 = 2$ $n_2 = n_3 = 2$-tuplets (doublets) with expansion $\lambda_{2,1}(\epsilon) = \lambda_{2,1}\sqrt{\epsilon} + \ldots$ and $\lambda_{2,2}(\epsilon) = \lambda_{2,2}\sqrt{\epsilon} + \ldots$;

we have $\ell_3 = 1$ eigenvalue with the form $\lambda_{3,1} = \lambda_{3,1}\epsilon + \mathcal{O}(\epsilon^2)$.

It should be clear now that the genericity condition is such to ensure that the (germ of) the spectral curve at $\epsilon = 0$ can be minimally resolved.

For later purposes we now investigate a bit closer the structure of the perturbative expansion in the case of a single Jordan block with nonvanishing Lidskii coefficient.

Proposition 2.1 Let $M(\epsilon) = \mathcal{N} + \sum_{j=1}^{\infty} \epsilon^j M_j$ be a (formal) perturbation of the single nilpotent Jordan block $\mathcal{N}$ of size $n \times n$. Suppose that the Lidskii pseudovalue is nonzero, namely ${\lambda_1}^n := (M_1)_{n1} \neq 0$. Then:

1. There exists a similarity transformation constant in ϵ which transforms the problem in the following perturbation problem

$$\tilde{M}(\epsilon) = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \ldots & \lambda_{n-2} & \lambda_{n-1} \\ 0 & 0 & \lambda_1 & \ddots & \ddots & \lambda_{n-2} \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ & \ddots & & & & \lambda_2 \\ & & & & & \lambda_1 \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} \star & \star & \ldots & & \star \\ & & & & \\ & & & & \\ \star & & \ldots & & \\ \lambda_1 & \star & \ldots & & \end{bmatrix} + \mathcal{O}(\epsilon^2) \qquad \text{(2-11)}$$

   Such similarity is unique up to the centralizer of $\mathcal{N}$.

2. The coefficients $\lambda_j,\ j = 1, \ldots n-1$ are the first coefficients in the Puiseux expansion of one of the n cyclically permuted eigenvalues in powers of $\xi = \epsilon^{\frac{1}{n}}$. Furthermore they depend only on the coefficients of M_1, rationally in λ_1 and polynomially in the other coefficients.

Proof.
Let us define

$$G := \mathrm{diag}(0, 1, 2, \ldots, n-1) \qquad \text{(2-12)}$$

$$\xi := \epsilon^{\frac{1}{n}} \qquad \text{(2-13)}$$

$$\lambda_1 := {(M_1)_{n1}}^{\frac{1}{n}}. \qquad \text{(2-14)}$$

We first "shear" the perturbation problem

$$S = (\lambda_1\xi)^{-G} M(\lambda_1\xi)^G =$$

$$= \xi\lambda_1\mathcal{C} + \xi^2 \begin{bmatrix} & & \\ & & \\ \star & & \\ & \star & \end{bmatrix} + \xi^3 \begin{bmatrix} & & \\ \star & & \\ & \star & \\ & & \star \end{bmatrix} + \ldots + \xi^n \begin{bmatrix} \star & & & & \\ & \star & & & \\ & & \ddots & & \\ & & & \star & \\ & & & & \star \end{bmatrix} + \mathcal{O}(\xi^{n+1})$$

where only the marked entries in the expansion above can be nonzero and the matrix C is the cyclic permutation

$$\mathcal{C} := \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix} \qquad \text{(2-15)}$$

Note that the sheared perturbation problem has the following periodicity and structure

$$S(\xi) = \sum_{j=1}^{\infty} \xi^j s_j \mathcal{C}^j \tag{2-16}$$

$$s_1 := \lambda_1 \mathbf{1},\ \ s_j = \text{ diagonal matrices },j \geq 2 \tag{2-17}$$

$$S(\omega\xi) = \Omega^{-1} S(\xi)\Omega \tag{2-18}$$

$$\Omega := \omega^G, \quad \omega := \mathrm{e}^{2i\pi/n} \tag{2-19}$$

Moreover the first $s_2, \ldots, s_n$ are the stars of the expansion above and involve only the corresponding entries of M_1. In particular the first $n-j$ coefficients of s_j are zeroes (for $j \leq n$).

The reason for the shearing is that the leading coefficient of this new perturbation problem has nondegenerate spectrum $\lambda_1\omega^j,\ j = 0, \ldots n$ and hence can be dealt with usual techniques. In fact, let us introduce the following eigenvector matrix W, of $\mathcal{C}$

$$\mathcal{C}W = W\Omega^{-1}\ ,\quad W_{ij} = \omega^{-(i-1)(j-1)}\ . \tag{2-20}$$

We claim that we can find a unique perturbative eigenvalue matrix of the following form

$$P(\xi) := \sum_{j=0}^{\infty} \xi^j \mathcal{C}^j p_j W \tag{2-21}$$

$$p_0 = \mathbf{1}\ , p_j = \text{ diagonal traceless,} \tag{2-22}$$

which diagonalizes $S(\xi)$ with cyclically permuting eigenvalues

$$y_j(\xi) = y(\omega^j\xi) \tag{2-23}$$

$$y(\xi) := \sum_{j=1}^{\infty} \xi^j \lambda_j \tag{2-24}$$

$$y_{j+1}(\xi) = y_j(\omega\xi)\ . \tag{2-25}$$

In order to show this we have to solve the following formal power series identity

$$S(\xi)P(\xi) = P(\xi)\Lambda(\xi)\ , \qquad \Lambda(\xi) := y(\Omega^{-1}\xi)\ , \tag{2-26}$$

where the unknowns are the diagonal traceless matrices $p_j, j \geq 2$ and the scalars $\lambda_j\ ,\ j \geq 2$. Plugging the Ansatz and comparing the coefficient of the power ξ^{K+1} we obtain the recurrence relation

$$\sum_{j=1}^{K} s_j \mathcal{C}^K p_{K-j} W = \sum_{j=1}^{K} \lambda_j \mathcal{C}^{K-j} p_{K-j} W \Omega^{-j}\ . \tag{2-27}$$

Solving for p_{K-1} and using $W\Omega^{-j} = \mathcal{C}^j W$ we obtain

$$\lambda_1\Big(p_{K-1} - \mathcal{C}^{-1}p_{K-1}\mathcal{C}\Big) = \sum_{j=2}^{K}\Big(\lambda_j \mathcal{C}^{-j} p_{K-j}\mathcal{C}^j - \mathcal{C}^{-K} s_j \mathcal{C}^K p_{K-j}\Big)\ . \tag{2-28}$$

This recurrence relation admits a *unique* solution: indeed both sides are diagonal matrices and we must guarantee that the RHS is traceless (since the LHS is, whether p_{K-1} is traceless or not). This condition fixes λ_K uniquely to be (recalling that $p_0 = \mathbf{1}$ and all the other p's are traceless)

$$n\,\lambda_K = \mathrm{Tr}(s_K) + \sum_{j=2}^{K-2} \mathrm{Tr}\Big(\mathcal{C}^{-K} s_j \mathcal{C}^K p_{K-j}\Big) \tag{2-29}$$

Moreover this shows that $\lambda_1, \ldots, \lambda_n$ depend only on $s_1, \ldots s_n$ and hence only on the entries of M_1 in a polynomial way w.r.t all entries except $(M_1)_{n1}$, w.r.t which they depend rationally.

We now revert to the original problem by inverting the shearing

$$M(\epsilon) = (\lambda_1 \xi)^G S(\xi) (\lambda_1 \xi)^{-G} \ . \tag{2-30}$$

The eigenvector/eigenvalue problem for $S(\xi)$ is conjugated as well into the following equation which involves only integer powers of ϵ

$$M(\epsilon)\tilde{P}(\epsilon) = \tilde{P}(\epsilon) D(\epsilon) \tag{2-31}$$

where

$$\tilde{P}(\epsilon) \ := \ (\lambda_1 \xi)^G P(\xi) W^{-1} (\lambda_1 \xi)^{-G} = \sum_{j=0}^{\infty} \frac{1}{{\lambda_1}^j} \mathcal{F}^j(\epsilon) p_j \tag{2-32}$$

$$D(\epsilon) \ := \ (\lambda_1 \xi)^G W \Lambda(\xi) W^{-1} (\lambda_1 \xi)^{-G} = \sum_{j=1}^{\infty} \frac{\lambda_j}{(\lambda_1)^j} \mathcal{F}^j(\epsilon) \ . \tag{2-33}$$

Here we have set

$$\mathcal{F}(\epsilon) := \lambda_1 \xi \, (\lambda_1 \xi)^G \mathcal{C} (\lambda_1 \xi)^{-G} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ (\lambda_1)^n \epsilon & & & \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ (M_1)_{n1} \epsilon & & & \end{bmatrix} \tag{2-34}$$

Note now that the matrix $\tilde{P}$ is nonsingular and (formally) analytic in ϵ; the leading coefficient is an invertible constant upper triangular matrix of the form

$$\tilde{P}(\epsilon) = \mathbf{1} + \sum_{j=1}^{n-1} \mathcal{N}^j \frac{p_j}{(\lambda_1)^j} + \mathcal{O}(\epsilon) = T + \mathcal{O}(\epsilon). \tag{2-35}$$

The similarity we are looking for is $(\lambda_1)^{-G} T (\lambda_1)^G$, where T is defined here above. In fact we have the leading coefficient

$$(\lambda_1)^G \, (T^{-1} M(\epsilon) T)_0 \, (\lambda_1)^{-G} = (\lambda_1)^G \, (\tilde{P}^{-1} M \tilde{P})_0 \, (\lambda_1)^{-G} = (\lambda_1)^G \, D_0 \, (\lambda_1)^{-G} = \tag{2-36}$$

$$= (\lambda_1)^G \begin{bmatrix} 0 & 1 & \frac{\lambda_2}{\lambda_1^2} & \frac{\lambda_3}{\lambda_1^3} & \cdots & \frac{\lambda n-1}{\lambda_1^{n-1}} \\ & 0 & 1 & \frac{\lambda_2}{\lambda_1^2} & \ddots & \\ & & & \ddots & \ddots & \frac{\lambda_3}{\lambda_1^3} \\ & & & 0 & 1 & \frac{\lambda_2}{\lambda_1^2} \\ & & & & & 1 \\ & & & & & 0 \end{bmatrix} (\lambda_1)^{-G} = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \ldots & \lambda_{n-2} & \lambda_{n-1} \\ 0 & 0 & \lambda_1 & \ddots & \ddots & \lambda_{n-2} \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ & \ddots & & & & \lambda_2 \\ & & & & & \lambda_1 \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \ . \tag{2-37}$$

As for the subleading coefficient it is easy to see that the $(n, 1)$ entry is transformed to λ_1, thus completing the proof. **Q.E.D.**

Before proceeding we observe that once the perturbation matrix is in the form advocated in Prop. 2.1 the "pseudo-eigenvalue" problem can be recast as in the following corollary

Corollary 2.1 *Suppose we have a formal perturbation problem in the form guaranteed by Prop. 2.1; then there exists a pseudo-eigenvector matrix of the form*

$$P(\epsilon) = \mathbf{1} + \mathcal{O}(\epsilon) \tag{2-38}$$

such that

$$P^{-1}\, M\, P = \sum_{j=1}^{\infty} \lambda_j \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \epsilon & & & \end{bmatrix}^j \tag{2-39}$$

3 Formal Asymptotics

The cornerstone of our analysis of resonant irregular singularities in our class is the following theorem, which displays the normal formal asymptotic form of a kernel solution $\Psi' = A\Psi$ in a sectorial neighborhood of the singularity.

This theorem contains in a compact form all the relevant results of this section and the remainder of the section is devoted to its proof. The reader may want to skip after this statement to the logical development of Section 3.1.

Theorem 3.1 (Main Theorem) *Let $A(x)$ be a $M \times M$ matrix with coefficients polynomial in x of degree at most $r-1$ ($r \geq 1$), and formal series in x^{-1}*

$$A(x) = \sum_{j \leq r} A_j x^{j-1}. \tag{3-1}$$

We assume that the leading coefficient is in Jordan canonical form with the following elementary block structure

$$A_r = ({\lambda_1}^{n_1}) \cdots ({\lambda_s}^{n_s}) \ , \tag{3-2}$$

where the eigenvalues are not assumed to be distinct. For each eigenvalue we consider the subblock of A_{r-1} corresponding to the nilspace of A_r with that singular value and its Lidskii submatrix according to the definitions of the previous section[4]. Under the genericity assumption of Def. 2.1 on the Lidskii pseudovalues of these Lidskii submatrices, there exists a (uniquely determined) formal gauge Y analytically invertible at ∞ of the form

$$Y(x) = Y_0 + \sum_{j=1}^{\infty} x^{-j} Y_j \ , \quad \det(Y_0) \neq 0 \ , \tag{3-3}$$

such that the gauge transformed connection

$$D(x) := Y^{-1}AY - Y^{-1}Y' = \mathrm{diag}(D_1, \ldots, D_s) \tag{3-4}$$

is in block diagonal form according to the minimal block decomposition of A_r; each block (of size n_j) corresponding to an eigenvalue λ_j has the form

$$D_j \quad = \quad \lambda_j x^{r-1} + \frac{1}{n_j x} \sum_{J=0}^{rn_j - 1} t_{J,j} \mathcal{H}_j(x)^J - \frac{G_j}{x} \tag{3-5}$$

[4] We can regard the subleading terms of the expansion of $A(x)$ as a formal perturbation of the leading term by identifying $M(\epsilon)$ of Section 2 with $x^{-r+1}A(x)$ in the parameter $\epsilon = \frac{1}{x}$.

$$= \frac{1}{n_j x}\sum_{J=0}^{rn_j} t_{J,j}\mathcal{H}_j(x)^J - \frac{G_j}{x} \ , \quad t_{rn_j} := n_j\lambda_j \tag{3-6}$$

$$G_j := \mathrm{diag}(0,1,\ldots,n_j-1) \tag{3-7}$$

$$\mathcal{H}_j(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \ \mathrm{Mat}(n_j,n_j,\mathbb{C}) \ . \tag{3-8}$$

The system admits a formal solution in the form

$$\Psi_{form} = Y(x)\cdot \Psi^{bare}(x) \tag{3-9}$$

$$\Psi^{bare} = \mathrm{diag}\left(\exp\left(\sum_{J=1}^{rn_1}\frac{t_{J,1}}{J}\mathcal{H}_1^J\right)x^{\frac{t_{0,1}-G_1}{n_1}},\ldots,\exp\left(\sum_{J=1}^{rn_s}\frac{t_{J,s}}{J}\mathcal{H}_s^J\right)x^{\frac{t_{0,s}-G_s}{n_s}}\right) \ . \tag{3-10}$$

The proof of this theorem is accomplished in various steps contained in the remainder of this section: the hinge is the technical result contained in Thm. 3.2, which gives a refinement of the standard splitting lemma and allows to decompose the ODE (using a formal analytic gauge) into the direct sum of ODEs of the same dimension as the elementary blocks in which the Jordan form of the leading coefficient splits. In the nonresonant case each block would be of dimension one, namely the ODE would be gauged to a diagonal ODE.

Once this decomposition is achieved we find the canonical form of the formal solution for each of these blocks in Prop. 3.1 together with Prop. 3.2. The proof of Thm. 3.1 will be thus complete.

Remark 3.1 We are here dealing only with singularities at $x=\infty$ but an analogous statement can be made for (formal) series in any local parameter $(x-\gamma)$ by a linear fractional transformation of the variable x appearing in Thm. 3.1.

The first step in the proof of Thm. 3.1 is to show that we can split the connection A into block diagonal form according to the block diagonal structure of the Jordan canonical form of its leading coefficient (under the assumed genericity assumption for the subleading coefficient).

As a preliminary step we apply the standard splitting lemma inductively.

Lemma 3.1 (Standard splitting lemma in [19] or [18]) *Suppose that the eigenvalues of A_r in the series of the form (3-1) has two groups of eigenvalues $\lambda_1,\ldots,\lambda_p$ and $\lambda_{p+1},\ldots,\lambda_K$ such that $\lambda_i \neq \lambda_j$ for $i\le p$ and $j>p$. Then A_r is similar to a block diagonal matrix*

$$A_r = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1, A_2 are of dimensions $p\times p$ and $(k-p)\times(k-p)$ respectively.

Furthermore there exists a formal power series Y such that

$$B^{(1)}(x) = Y^{-1}AY - Y^{-1}Y' \ ,$$

has the block diagonal form

$$B^{(1)} = \begin{pmatrix} B_1^{(1)} & 0 \\ 0 & B_2^{(1)} \end{pmatrix}$$

where $B_1^{(1)}$, $B_2^{(1)}$ are square matrices of dimensions $m_1\times m_1$ and $m_2\times m_2$ respectively.

Repeated application of this lemma guarantees the existence of a formal gauge Y such that the transformed connection

$$B^{(1)}(x) = Y^{-1}AY - Y^{-1}Y' \ ,$$

is block diagonal and each block has only one eigenvalue in its leading term.

Re-denoting by A one of these blocks, in view of the above splitting lemma we can restrict ourselves to the case where the Jordan form of A_r has only one eigenvalue, that is

$$A_r \quad = \quad (\lambda^{n_1}) \cdots (\lambda^{n_s}) \tag{3-11}$$

and, without loss of generality, we can assume that $\lambda = 0$ by performing a scalar gauge transformation $\mathrm{e}^{-\frac{\lambda}{r+1}x^{r+1}}\mathbf{1}$. We shall now prove the following proposition that shows decomposability of $A(x)$ in block diagonal form under the genericity assumption in Def. 2.1.

Theorem 3.2 *Suppose that the leading coefficient A_r in the series (3-1) is in Jordan canonical form*

$$A_r \quad = \quad (0^{n_1}) \cdots (0^{n_s})$$

and that the Lidskii pseudovalues of A_{r-1} are generic in the sense of Def. 2.1. Then there exists a formal power series

$$Y(x) = Y_0 + \sum_{j=1}^{\infty} x^{-j} Y_j \ , \quad \det(Y_0) \neq 0 \ ,$$

such that

$$B(x) = Y^{-1}AY - Y^{-1}Y' \ ,$$

is a formal power series in block diagonal form according to the block decomposition of A_r.

[For the proof see Appendix B].

By application of Thm. 3.2 we can reduce ourselves to analyzing a single block of the final form of the connection. We accomplish this final step in the following proposition.

Proposition 3.1 Let $A(x)$ be a $n \times n$ matrix of degree $r-1$ and formal Laurent tail, such that the leading coefficient is $\lambda\mathbf{1}$ plus a nilpotent matrix of rank $n-1$. Suppose also that the Lidskii coefficient[5] of the subleading term (in the constant gauge in which A_r is in Jordan canonical form) does not vanish.

$$A(x) = x^{r-1}A_r + x^{r-2}A_{r-1} + \dots \tag{3-12}$$

Then there exists a nonsingular formal gauge $Y(x)$ of the form

$$Y(x) = C\left(\mathbf{1} + \mathcal{O}(x^{-1})\right) \ , \tag{3-13}$$

such that the connection $\partial_x - A(x)$ is formally gauged to the connection $\partial_x - D(x)$ of the form

$$D(x) = \frac{t_{rn}}{n}x^{r-1} + \frac{1}{nx}\sum_{j=0}^{rn-1} t_j \mathcal{H}^j - \frac{G}{x} \tag{3-14}$$

$$t_{rn} := n\lambda \ , \ \ G := \mathrm{diag}(0,1,2,\dots,n-1)$$

$$\mathcal{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

[5] For a leading coefficient with only one elementary Jordan block the Lidskii submatrix of the subleading term is 1×1 and consist of its lower-left corner.

Proof. First of all we assume without loss of generality that $\lambda = 0$ since we can gauge it away by a scalar gauge transformation. Next, we can conjugate the connection by a constant gauge so that the leading coefficient becomes the nilpotent Jordan block $\mathcal{N}$. Then, as shown in Prop. 2.1, we can perform a second conjugation (constant in x) which recasts the problem in the form

$$A(x) \mapsto A(x) = x^{r-1} \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \dots & \lambda_{n-2} & \lambda_{n-1} \\ 0 & 0 & \lambda_1 & \ddots & \ddots & \lambda_{n-2} \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ & \ddots & & & & \lambda_2 \\ & & & & & \lambda_1 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} + x^{r-2} \begin{bmatrix} \star & \star & \dots & & \star \\ & & & & \\ & & & & \\ \star & & \dots & & \\ \lambda_1 & \star & \dots & & \end{bmatrix} + \dots , \tag{3-15}$$

where the coefficients λ_j are defined as the coefficients of the formal expansion of the **eigenvalues** of $A(x)$ in Puiseux series of $q = x^{1/n}$

$$y_j(q) = y(\omega^j q) \tag{3-16}$$

$$y(q) = q^{(r-1)n} \sum_{j=1}^{\infty} \lambda_j q^{-j} = x^{r-1} \sum_{j=1}^{\infty} \lambda_j x^{-\frac{j}{n}} \tag{3-17}$$

In fact we should identify the matrix $x^{1-r}A(x)$ with the perturbation problem $M(\epsilon) = M(1/x)$ used in Prop. 2.1.

We use the same symbol $A(x)$ for this new connection in "canonical" form to economize on notation. At this point we can perform the shearing transformation as done in Prop. 2.1: the difference is that now the transformation is a change of gauge and not merely a conjugation. Therefore we introduce another formal connection B which will be a formal Laurent series in $q = x^{1/n}$ (by choosing in an arbitrary but fixed way the determination of the root)

$$\widetilde{B}(q) := SAS^{-1} + S'S^{-1} \tag{3-18}$$

$$S := q^G \ , \quad G := \mathrm{diag}(0, 1, 2, \dots, n-1) \ . \tag{3-19}$$

Finally we change the variable of differentiation from x to $q = x^{1/n}$

$$\tilde{\Psi}'(q) = nq^{n-1}\widetilde{B}(q)\tilde{\Psi} =: nB(q)\tilde{\Psi}. \tag{3-20}$$

where the matrix $B(q)$, as a consequence of the shearing, has the following structure and enjoys the following properties.

$$\begin{aligned} B(q) &= q^{R-1}\left[\overbrace{\lambda_1 \mathcal{C}}^{=:B_R} + \frac{1}{q}B_{R-1} + \dots\right] = \sum_{j=-\infty}^{R} B_j q^{j-1} \\ &= q^{R-1}\lambda_1\mathcal{C} + q^{R-2}\begin{bmatrix} & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_2 \\ \star & & & & \\ & \star & & & \end{bmatrix} + \dots + q^{R-n}\begin{bmatrix} & & & & \lambda_{n-1} \\ \star & & & & \\ & \star & & & \\ & & \ddots & & \\ & & & \star & \end{bmatrix} + \dots \\ \mathcal{C} &:= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{3-21}$$

$$
\begin{aligned}
&B_j = b_j \mathcal{C}^j \ , \ b_j \text{ diagonal matrices}\\
&R = rn - 1\\
&(b_{R-j})_{\ell\ell} = \lambda_j \ , j = 1, \dots n-1, \ \ell = 1, \dots, n-j\\
&\omega B(\omega q) = \Omega B(q) \Omega^{-1} \ ,\\
&\Omega := diag(1, \omega, \omega^2, \dots, \omega^{n-1}), \ \omega := \mathrm{e}^{2i\pi/n}
\end{aligned} \tag{3-22}
$$

Since the leading coefficient is nondegenerate we may apply the standard theory of asymptotic expansions as in [19, 18, 11] stating that we can find a formal solution in the form

$$
\begin{aligned}
\widetilde{\Psi} &= Z(q)\mathrm{e}^{T(q)}\\
T(q) &= \sum_{j=1}^{R} \frac{1}{j} T_j q^j + T_0 \ln(q) \qquad Z = Z_0 + \mathcal{O}(q^{-1}) \ , \quad \det Z_0 \neq 0.
\end{aligned}
$$

We can however derive a slightly improved statement, proved in Appendix C.

Lemma 3.2 *Let $B(q)$ of the form in eq. (3-21). There exists a (unique) formal solution of $\partial_q \Psi = B(q)\Psi$ satisfying the conditions*

$$\Psi(q) = Z(q)\mathrm{e}^{nT(q)} \tag{3-23}$$

$$T(q) = \sum_{j=1}^{R} \frac{t_j}{j} \Omega^j q^j + t_0 \ln(q) \ , \ t_j \ \textit{scalars} \tag{3-24}$$

$$Z(q) = \Omega^{-1} Z(\omega q) \mathcal{C} \tag{3-25}$$

$$Z(\infty) = W \ , \tag{3-26}$$

where W is the following eigenvector matrix of the cyclic permutation matrix C (see (3-21))

$$\mathcal{C} W = W \Omega^{-1} \ , \qquad W_{ij} := \omega^{-(i-1)(j-1)} \ , \qquad \omega := \mathrm{e}^{\frac{2i\pi}{n}} \ . \tag{3-27}$$

The matrix $Z(q)$ has the expansion

$$Z := \sum_{j=0}^{\infty} q^{-j} Z_j \ , \qquad Z_j = \mathcal{C}^j z_j W \ , \quad z_j = \ \textit{diagonal matrices} \ , z_0 = \mathbf{1}. \tag{3-28}$$

We can now conclude the proof of Prop. 3.1: to this end we gauge transform once more the connection $\partial_q - nB(q)$ by means of the formal gauge $Y := Z(q) W^{-1} q^G$, namely

$$\partial_q - nB(q) \mapsto \partial_q - n\tilde{D}(q) \tag{3-29}$$

$$\tilde{D}(q) := q^{-G} W \overbrace{\left(Z^{-1} B Z - \frac{1}{n} Z^{-1} Z' \right)}^{=T'(q)} W^{-1} q^G - \frac{G}{q} \ . \tag{3-30}$$

The formal series $\tilde{D}(q)$ enjoys the periodicity

$$\tilde{D}(\omega q) = \frac{1}{\omega} \tilde{D}(q) \tag{3-31}$$

Restoring the independent variable to x we have the connection $\partial_x - D(x)$ where

$$D(x) = \frac{1}{x} q \tilde{D}(q) = \frac{1}{nx} \sum_{j=0}^{R} t_j \begin{bmatrix} 0 & 0 & 0 & 0 & x \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^j - \frac{G}{nx} = \frac{1}{nx} \sum_{j=0}^{R} t_j \mathcal{H}(x)^j - \frac{G}{nx} \tag{3-32}$$

At this point we can summarize the chain of gauge transformations as

$$\begin{aligned} Y^{-1}AY &- Y^{-1}Y' = D(x) && \text{(3-33)}\\ Y(x) &= q^{-G}Z(q)W^{-1}q^{G} = \sum_{j=0}^{\infty} q^{-j}q^{-G}\mathcal{C}^j z_j q^G = && \text{(3-34)}\\ &= \sum_{j=0}^{\infty} \mathcal{H}^{-j}(x) z_j \ . && \text{(3-35)} \end{aligned}$$

Since $z_0 = \mathbf{1}$ and z_j are diagonal matrices, a direct inspection shows that the leading coefficient of $Y(x)$ is

$$Y(x) = \left(\mathbf{1} + \sum_{j=1}^{n-1}\mathcal{N}^j z_j + \mathcal{O}(x^{-1})\right) \ . \tag{3-36}$$

Therefore the formal gauge $Y(x)$ is nonsingular (note also that it contains only integer powers of x, in spite of the intermediate steps). Finally we claim that the constant term in $Y(x)$ is in the centralizer of $\mathcal{N}$[6]: indeed by inspecting the leading term of $D(x)$ we see that it is the same as the leading term of $A(x)$ (in the "canonical" form 3-15), and hence the constant term of $Y(x)$ must commute with it and so commutes also with its canonical form, which is $\mathcal{N}$. We can incorporate $\mathbf{1} + \sum \mathcal{N}^j z_j$ it in the first constant gauge, thus completing the proof. **Q.E.D.** Prop. 3.1.

Remark 3.2 For later reference we point out that the upper triangular matrix $U := \sum_{j=1}^{n-1}\mathcal{N}^j z_j$ is uniquely fixed once the connection is in the canonical form (3-15). Moreover, from the recursion relations defining y_j's it is obvious that the entries of U are rational in $\lambda_1 = t_{rn-1}$ and polynomials in the other entries of the coefficients of $A(x)$.

Before proceeding with the analysis of isomonodromic deformations we point out the almost obvious

Proposition 3.2 [Bare Isomonodromic Deformation] Introducing the matrices

$$D^\infty(x) = \frac{1}{nx}\sum_{j=0}^{rn} t_j \mathcal{H}(x)^j - \frac{G}{nx} \ , \qquad G := \mathrm{diag}(0,1,\ldots,n-1) \tag{3-37}$$

$$\mathcal{T}_J^{\infty,bare}(x) = \frac{1}{J}\mathcal{H}(x)^J \ , \ J = 1,\ldots,rn \ . \tag{3-38}$$

(where $\mathcal{H}(x)$ is the matrix appearing in eq. 3-14) we have the zero-curvature equations

$$\left[\frac{\partial}{\partial t_J} - \mathcal{T}_J^{\infty,bare}, \frac{\partial}{\partial t_K} - \mathcal{T}_K^{\infty,bare}\right] = 0 \tag{3-39}$$

$$\left[\frac{\partial}{\partial t_J} - \mathcal{T}_J^{\infty,bare}, \partial_x - D^\infty(x)\right] = 0 \ . \tag{3-40}$$

The kernel of all these connection is spanned by

$$\Psi^{bare} = \exp\left(\sum_{J=1}^{rn}\frac{t_J}{J}\mathcal{H}^J\right) x^{\frac{t_0 - G}{n}} \ . \tag{3-41}$$

[6]This does not follow from the above expression directly because the diagonal matrices z_j are not multiples of the identity.

Proof. The above connection is nothing but the connection

$$\partial_q - \sum_{J=0}^{rn} t_J q^{J-1}\Omega^J \ ; \qquad \partial_{t_J} - \frac{1}{J} q^J \Omega^J \ , \ J = 1, \dots rn \ , \quad q = x^{\frac{1}{n}} \tag{3-42}$$

–the compatibility of which is absolutely trivial– written in a different gauge (and variable), namely by gauging with $q^{-G}W$. The kernel of these connections is trivially

$$\Phi(q) = \exp\left(\sum_{J=1}^{rn} \frac{t_J}{j} q^J \Omega^J + t_0 \ln q\right) \tag{3-43}$$

Therefore we can set (multiplying on the right by the constant invertible matrix W^{-1})

$$\Psi^{bare} = q^{-G} W \, \Phi \, W^{-1} = q^{-G} W \, \Phi \, W^{-1} q^G q^{-G} \ . \tag{3-44}$$

Since $\mathcal{H}(x) = q^{-G} W q \Omega W^{-1} q^G$, we find

$$\Psi^{bare}(x) = \exp\left(\sum_{J=1}^{rn} \frac{t_J}{J} \mathcal{H}^J\right) x^{\frac{t_0 - G}{n}} . \ \ \textbf{Q.E.D.} \tag{3-45}$$

Note that in Prop. (3.2) the sum extends to rn; namely t_{rn} is $n\lambda$, where λ is the unique eigenvalue of the leading coefficient matrix. Among the times, each t_j, $j \equiv 0 \mathrm{mod}(n)$ is "trivial" in the sense that it could be gauged away by a scalar gauge transformation. However we prefer to keep them in view of the general case, where they will not be "trivial" anymore.

In these considerations we have always placed the pole at $x = \infty$: if the pole were at a finite point $x = c$ we should modify some of the formulas above in a trivial manner which is left to the reader to verify. The analog of Prop. 3.1 localized at a finite point is (we are not considering Fuchsian singularities in this section)

Proposition 3.3 [Prop. 3.1 for finite poles] Let $A(x)$ be a $n \times n$ matrix with a pole at $x = \gamma$ of degree $r + 1 \geq 2$ and formal Taylor series, such that the leading coefficient is $\lambda\mathbf{1}$ plus a nilpotent matrix of rank $n - 1$. Suppose also that the Lidskii coefficient of the subleading term (in the constant gauge in which A_r is in Jordan canonical form) does not vanish.

$$A(x) = \frac{A_r}{(x-\gamma)^{r+1}} + \frac{A_{r-1}}{(x-\gamma)^r} + \dots \tag{3-46}$$

Then there exists a nonsingular formal gauge $Y(x)$ of the form

$$Y(x) = C\left(\mathbf{1} + \mathcal{O}((x-\gamma))\right) \ , \tag{3-47}$$

such that the connection $\partial_x - A(x)$ is formally gauged to the connection $\partial_x - D(x)$ of the form

$$D^\gamma(x) = -\sum_{J=0}^{rn} \frac{t_J}{n(x-\gamma)} \mathcal{H}^J((x-\gamma)^{-1}) + \frac{G}{n(x-\gamma)} \tag{3-48}$$

$$t_{rn} := -n\lambda \ , \ G := \mathrm{diag}(0, 1, 2, \dots, n-1) \tag{3-49}$$

$$\mathcal{H}((x-\gamma)^{-1}) = \begin{bmatrix} 0 & 0 & 0 & 0 & (x-\gamma)^{-1} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3-50}$$

Proposition 3.4 [Bare I-Defs for finite poles] Introducing the matrices

$$D^\gamma(x) = -\sum_{J=0}^{rn} \frac{t_J}{n(x-\gamma)} \mathcal{H}^J((x-\gamma)^{-1}) + \frac{G}{n(x-\gamma)} \tag{3-51}$$

$$\mathcal{T}_J^{\gamma,bare}(x) := \frac{1}{J}\mathcal{H}((x-\gamma)^{-1})^J \ , \ j = 1, \ldots, rn \tag{3-52}$$

$$\mathcal{C}^{\gamma,bare}(x) := -D^\gamma(x) \ , \tag{3-53}$$

we have the zero-curvature equations

$$\left[\frac{\partial}{\partial t_J} - \mathcal{T}_J^{\gamma,bare}, \frac{\partial}{\partial t_K} - \mathcal{T}_K^{\gamma,bare}\right] = 0 \tag{3-54}$$

$$\left[\frac{\partial}{\partial t_J} - \mathcal{T}_J^{\gamma,bare}, \partial_x - D^\gamma(x)\right] = 0 \tag{3-55}$$

$$\left[\frac{\partial}{\partial \gamma} - \mathcal{C}^{\gamma,bare}, \partial_x - D^\gamma(x)\right] = 0 \tag{3-56}$$

$$\left[\frac{\partial}{\partial \gamma} - \mathcal{C}^{\gamma,bare}, \frac{\partial}{\partial t_K} - \mathcal{T}_K^{\gamma,bare}\right] = 0 \tag{3-57}$$

The kernel of all these connection is spanned by

$$\Psi^{bare} = \exp\left(\sum_{J=1}^{rn} \frac{t_J}{J}\mathcal{H}^J((x-\gamma)^{-1})\right)(x-\gamma)^{\frac{G-t_0}{n}} \ . \tag{3-58}$$

We conclude with the remark that the number of isomonodromic times in this situation is exactly the same as in the nonresonant case, and in fact this still holds for more general Jordan block decompositions: in a sense we may consider the cases under analysis "minimally" resonant.

3.1 Isomonodromic Deformations

The general situation in which we would like to define a set of isomonodromic deformations is that of an arbitrary rational connection $A(x)$ with leading coefficients at each singularity consisting of (a conjugacy class of) an arbitrary Jordan form, under the suitable genericity assumptions on the Lidskii matrix for the subleading coefficient.

At this point of our discussion the main difficulty is rather notational than conceptual: in view of Thm. 3.1 we can approach the problem by reasoning on each block of the connection in the local formal gauge as in eq. 3-4 at each of the singularities.

As a warm-up exercise we consider the following simplified but nontrivial situation (which –for example– includes as a special case the isomonodromic-deformation formulation of the Painlevé I equation): let $\partial_x - A(x)$ be a **polynomial** connection of degree $r-1$ with leading coefficient consisting of a single Jordan block of size n. There is no loss of generality in assuming that the connection is in the canonical form (3-15): this is tantamount requiring that the leading term is diagonal in the non-resonant case and fixes conveniently the constant gauge arbitrariness. More specifically we know from the proof of Prop. 3.1 that the coefficients appearing in the canonical form (3-15) are in fact the first n times, namely we will have

$$A(x) = \frac{1}{n}x^{r-1}\begin{bmatrix} t_{rn} & t_{rn-1} & \cdots & & t_{rn-n+1} \\ & t_{rn} & \ddots & & t_{rn-n+2} \\ & & \ddots & \ddots & \\ & & & t_{rn} & t_{rn-1} \\ & & & & t_{rn} \end{bmatrix} + x^{r-1}\begin{bmatrix} & & \star \\ & & \\ \frac{t_{rn-1}}{n} & & \end{bmatrix} + \ldots \tag{3-59}$$

$$t_{rn-1} \neq 0$$

where the parameters t_{rn-j}, $j = 0, \ldots, n-1$ are the first n times in the bare form (Prop. 3.2) of the connection. This can always be achieved -if necessary- by first putting the leading term in Jordan canonical form, then determining the parameters t_{rn-j} and then applying a second appropriate constant conjugation.

Theorem 3.3 *Let $A(x)$ be a polynomial of degree $r-1$ with leading coefficient in the gauge-fixed form (3-59). There exists a unique formal solution $\Psi' = A\Psi$ of the form*

$$\Psi = \overbrace{\left(\mathbf{1} + U + \mathcal{O}(x^{-1})\right)}^{=Y(x)} \exp\left[\sum_{J=1}^{rn} \frac{t_J}{J}\mathcal{H}^J(x)\right] x^{t_0 - \frac{G}{n}} \ , \tag{3-60}$$

where U denotes a strictly upper triangular matrix uniquely determined, rational in t_{rn-1} and polynomial in the remaining coefficients of A_j's, independent of x and in the centralizer of $\mathcal{N}$.

We then define

$$\mathcal{T}_J(x) := \frac{1}{J}\left(Y\mathcal{H}^J(x)Y^{-1}\right)_+ \ , \tag{3-61}$$

where $()_+$ means the projection onto nonnegative powers of x. With these definitions, the operators

$$\partial_x - A(x),\ \partial_j - \mathcal{T}_J(x) \ ,\ J = 1, \ldots rn \tag{3-62}$$

satisfy pairwise zero-curvature conditions, which ensure the Fröbenius integrability of the Pfaffian system

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi = A\Psi \tag{3-63}$$

$$\partial_{t_J}\Psi = \mathcal{T}_J\Psi \ . \tag{3-64}$$

The matrix U appearing in the formal gauge is a constant of all the motions and hence it can be disposed of by an appropriate conjugation by $\mathbf{1}+U$.

Note that although U is constant and in the centralizer of the leading coefficient it is not true that it can be set to zero a priori because setting it to zero is a (although conserved) *algebraic constraint* on the coefficient of the connection. We should rather think of U as "Casimirs" of the flows.

Proof. The property of U being uniquely determined follows from Remark (3.2). We rewrite the Pfaffian system for Y as follows

$$\partial_x Y = A\,Y - Y\,D \tag{3-65}$$

$$\partial_{t_J} Y = \mathcal{T}_J\,Y - Y\,\mathcal{T}_J^{bare} \ , \tag{3-66}$$

where $D(x)$ and $\mathcal{T}_J^{bare}$ are as in Prop. 3.2 for

$$\Psi^{bare} := \exp\left(\sum_{J=1}^{rn} \frac{t_J}{J}\mathcal{H}^J\right) x^{\frac{t_0 - G}{n}} \tag{3-67}$$

is the solution of the Pfaffian system of Prop. 3.2. The definition of $\mathcal{T}_j$ implies that

$$\partial_{t_J} Y Y^{-1} = \mathcal{O}(x^{-1}) \ , \tag{3-68}$$

namely the upper-triangular matrix U in the constant term is actually a constant of all the flows; note that the independent entries of U are $n-1$ since the matrix commutes with $\mathcal{N}$ and depend polynomially

on the (coefficients in x of the) entries of A, rationally on t_{rn-1}.
We now want to verify the zero-curvature conditions; to this end one computes

$$
\begin{aligned}
[\partial_{t_J}, \partial_{t_K}]Y \cdot Y^{-1} &= \partial_J \mathcal{T}_K - \partial_K \mathcal{T}_J + [\mathcal{T}_K, \mathcal{T}_J] - \overbrace{\left(\partial_J \mathcal{T}_K^{bare} - \partial_K \mathcal{T}_J^{bare} + [\mathcal{T}_K^{bare}, \mathcal{T}_J^{bare}]\right)}^{\equiv 0} \quad (3\text{-}69)\\
[\partial_{t_J}, \partial_x]Y \cdot Y^{-1} &= \partial_J A - \partial_x \mathcal{T}_J + [A, \mathcal{T}_J] - \underbrace{\left(\partial_J D - \partial_x \mathcal{T}_J^{bare} + [D, \mathcal{T}_J^{bare}]\right)}_{\equiv 0} . \quad (3\text{-}70)
\end{aligned}
$$

In the above equations the LHS is $\mathcal{O}(x^{-1})$ since U is a constant of the motions, while the RHS is polynomial. Therefore both sides vanish identically. **Q.E.D.**

Note that we could conjugate the connection $A(x)$ by $\mathbf{1}+U$ since *a posteriori* this is a constant gauge of all the flows. However we could not gauge it away to begin with, since we did not know *a priori* whether it was a first integral of the isomonodromic deformations.

Remark. We would like to comment about U being a constant in the proof of theorem 3.3.

The situation is here very similar to the nonresonant case [11]: indeed the deformation equations that we have proposed here are not the most general that would preserve the leading coefficient at ∞ of the connection $A(x)$. The residual freedom in this case as well as in the nonresonant case is by gauge action of constant (in x) transformations in the centralizer of A_r.

Suppose indeed we have a general solution $\{A, \mathcal{T}\}$ to the zero curvature condition (3-63) such that $\mathcal{T}$ is a polynomial. Let their formal solution be given by

$$
\tilde{\Psi} = \left(\mathbf{1} + \tilde{U} + \mathcal{O}(x^{-1})\right) \exp\left[\sum_{J=1}^{rn} \frac{t_J}{J} \mathcal{H}^J(x)\right] x^{\frac{t_0 - G}{n}} \quad . \tag{3-71}
$$

Then $\tilde{U}$ need only be in the centralizer of A_r (i.e. of $\mathcal{N}$) but it could otherwise depend on the "times" in an arbitrary analytic way. Any such $\tilde{\Psi}$ can be obtained from multiplying a formal solution (3-60) by $C = (I + \tilde{U})(I+U)^{-1}$. This has the effect of gauge transforming the solution $\{A^0, \mathcal{T}^0\}$ by the gauge C, where $\{A^0, \mathcal{T}^0\}$ is the solution in which U is fixed. Therefore any solution to the zero curvature condition can be obtained by a gauge transformation and there is no lost of generality in assuming U is a constant.

3.1.1 Tau function

We now turn our attention to a suitable definition of **isomonodromic tau function**. To this end we employ the strategy used in [6], namely of expressing the tau function in terms of spectral invariants of the connection A. This is motivated by the fact proven in loc.cit. that the standard Miwa-Jimbo-Ueno definition can in fact be expressed in terms of spectral invariants and this formulation is more suitable to be generalized to this case.

First of all we make a few simple observations about the formal gauge Y used in eq. (3-60). Since we have

$$
\begin{aligned}
A(x) &= Y'Y^{-1} + Y\left(\frac{1}{nx}\sum_{J=0}^{rn} t_J \mathcal{H}(x)^J - \frac{G}{nx}\right)Y^{-1} = \\
&= Y\, x^{-\frac{G}{n}} W \underbrace{\left(\frac{1}{nx}\sum_{J=0}^{rn} t_J x^{\frac{J}{n}} \Omega^J - \frac{x^{\frac{G}{n}} W^{-1} G W x^{-\frac{G}{n}}}{nx}\right)}_{=:\hat{D}(x)} W^{-1} x^{\frac{G}{n}} Y^{-1} + Y'Y^{-1} \ , \quad (3\text{-}72)
\end{aligned}
$$

it follows immediately that the matrix $P(q) := Y\, x^{-\frac{G}{n}} W$ is an eigenvector matrix *mod* $q^{-rn-1+n}$ where $q = x^{\frac{1}{n}}$. It also follows that the expansion of the eigenvalues y_j of A coincides with the expansion of the eigenvalues of the bare system $\hat{D}(x)$. This leads to

$$J\,\mathcal{T}_J = \left(Y \mathcal{H}^J Y^{-1}\right)_+ = \left(x^{\frac{J}{n}} Y x^{-\frac{G}{n}} W \Omega^J W^{-1} x^{\frac{G}{n}}\right)_+ = \left(q^J P \Omega^J P^{-1}\right)_+ + \text{constant}\ , \tag{3-73}$$

where the plus in the subscript always denotes the polynomial part in x. The constant appearing in the RHS is actually present only for $J \geq rn - n + 1$, namely for the deformations along the highest n times; in any case the specific form of this constant (w.r.t. x) matrix is irrelevant for our purposes.
Note that the expression in the RHS is nothing but a combination of the spectral projectors of the matrix $A(x)$; if we denote by E_a the diagonal elementary matrix we have

$$P\Omega^J P^{-1} = \sum_{\sigma=1}^{n} \omega^{J(\sigma-1)} P E_\sigma P^{-1} = \sum_{\sigma=1}^{n} \omega^{J(a-1)} \Pi_a\ , \tag{3-74}$$

where Π_a is the rank-one projector on the eigenspace of the eigenvalue $y_a(q)$ (part of the cyclically permuted n-tuplet). By the properties of the spectrum of A we have

$$\Pi_\sigma(\omega q) = \Pi_{\sigma+1}(q). \tag{3-75}$$

This ensures that the expressions

$$q^J \sum_{\sigma=1}^{n} \omega^{J(\sigma-1)} \Pi_\sigma(q) \tag{3-76}$$

contain in fact only **integer powers of** $x = q^n$. We can write the spectral projectors by the classical formula

$$\Pi_\sigma(q) = \frac{1}{\mathrm{Tr}(\widetilde{A - y_\sigma})} \widetilde{A - y_\sigma}\ , \tag{3-77}$$

where the tilde denotes the classical adjoint (the matrix of cofactors).

Note that once we have fixed the determination of the root $q = x^{\frac{1}{n}}$ and of $t_{rn-1} = (A_r)^{1/n}$, there is a unique eigenvalue which admits the (in fact convergent) Puiseux series expansion

$$y_1(q) = \frac{1}{n} \sum_{J=1}^{rn} t_J q^{J-n} + \frac{\tilde{t}_0}{nx} - \frac{1}{n} \sum_{K=1}^{\infty} K H_K q^{-n-K}\ , \tag{3-78}$$

where this formula has defined symbols for the coefficients of the negative powers of the expansion. From an algebro-geometric point of view the cyclicity of the eigenvalues near $x = \infty$ means that the spectral curve has a branch-point of order n in $\mathbb{P}^1 \times \mathbb{P}^1$ at $y = \infty$ even after desingularization; put otherwise the local parameter at near $x = \infty$ is $q = x^{\frac{1}{n}}$. This means that we have

$$t_J = -\operatorname*{res}_{x=\infty} \left(\sum_{a=0}^{n-1} x^{-J/n} \omega^{Ja} y_{a+1}(q) \right) \mathrm{d}x = -\operatorname*{res}_{\zeta_\infty} x^{-\frac{J}{n}} y \mathrm{d}x\ , \qquad 1 \leq J \leq rn \tag{3-79}$$

$$\tilde{t}_0 = -\operatorname*{res}_{x=\infty} \left(\sum_{a=1}^{n} y_a(q) \right) \mathrm{d}x = -\operatorname*{res}_{x=\infty} \mathrm{Tr}(A) \mathrm{d}x = t_0 - \frac{\mathrm{Tr}(G)}{n} = t_0 - \frac{n-1}{2} \tag{3-80}$$

$$H_J = \frac{1}{J} \operatorname*{res}_{x=\infty} \left(\sum_{a=0}^{n-1} x^{J/n} \omega^{Ja} y_a(q) \right) \mathrm{d}x = \frac{1}{J} \operatorname*{res}_{\zeta_\infty} x^{\frac{J}{n}} y \mathrm{d}x\ . \tag{3-81}$$

We remark that the first residue-formulas for t_J and H_J are taken on the x-plane (the "base-curve") and they make sense because the expressions in the brackets contain only integer powers of x due to the cyclicity of the eigenvalues: the second residue-formulas are taken on the spectral curve around the point ζ_∞ (with local parameter $x^{-1/n}$) which projects down to $x = \infty$. Since the positive orientation of a small circle around ζ_∞ is the opposite than the positive (counterclockwise) orientation in the x-plane, this explains the difference in sign. Note also that the coefficient t_0 of the bare system is not the residue of the eigenvalue but it is shifted due to the fact that the term G/x contributes to the residue (3-80). We are now ready to define the tau function for this guide-example

Proposition 3.5 The following differential is closed

$$\mathrm{d}\ln\tau = \sum_{K=1}^{rn} H_K \mathrm{d}t_K \ . \tag{3-82}$$

Before entering in the (easy) proof let us comment on the motivation and shape of the formula. The reason for the name of this differential as "isomonodromic tau function" resides in the fact that for the nonresonant case [6] a similar formula involving residues of the spectral differential $y\mathrm{d}x$ on the spectral curve of the connection is shown to coincide with the definition in [11], which is of different nature and –on the face of it– not a spectral invariant of the connection. Secondly, this sort of expressions appear throughout the literature of dispersionless integrable hierarchies, Seiberg-Witten models etc., we cannot find in the literature any statement or implication that the Miwa-Jimbo-Ueno tau function is defined by or equivalent to this ubiquitous formula.

On a more technical point we also remark that the most similar setting is that of the so-called universal Whitham hierarchy [13], where the data are an algebraic curve and a meromorphic differential plus some decorative data of local parameters near punctures.

While the main characters appear similar (we have a spectral curve, we have punctures and local parameters and a meromorphic differential $y\mathrm{d}x$) there are some significant distinctions regarding the coordinates on the phase space. In fact in the Whitham setting the periods of the differential $y\mathrm{d}x$ would be treated as coordinates on the phase space independent of the other "times". In this case the periods of the differential $y\mathrm{d}x$ are **not independent** and in fact mainly uncontrollable. What is fixed instead are the parameters of formal monodromy and the Stokes' parameters, which cannot be recovered by inspection of the spectral curve alone.

Proof of Prop. 3.5.
We compute the closure of the differential

$$-\partial_{t_J} H_K = \frac{1}{K}\mathop{\mathrm{res}}_{\zeta_\infty} x^{K/n}\partial_{t_J} y\mathrm{d}x = \frac{1}{J}\left(\sum_{a=1}^{n}\mathop{\mathrm{res}}_{q=\infty} x^{\frac{K}{n}}\omega^{K(a-1)}\partial_{t_J} y_a(x)\right)\mathrm{d}x \ . \tag{3-83}$$

In order to compute the variation of the eigenvalue we recall the classical formula describing the variation of a simple eigenvalue y of a matrix A under an infinitesimal deformation $A \mapsto A + \delta A$

$$\delta y = \frac{1}{\mathrm{Tr}(\widetilde{A-y})}\mathrm{Tr}\left(\widetilde{A-y}\ \delta A\right) \ . \tag{3-84}$$

In our case we should use the formula with $\delta A = \mathcal{T}_J' + [\mathcal{T}_J, A]$ because on the path of integration the eigenvalue is "uniformly" simple. We thus obtain

$$-\partial_{t_J} H_K = \frac{1}{K}\mathop{\mathrm{res}}_{x=\infty}\left(\sum_{b=1}^{n}\omega^{K(b-1)}x^{\frac{K}{n}}\mathrm{Tr}\left(\frac{(\widetilde{A-y_b})(x)}{\mathrm{Tr}(\widetilde{A-y_b})(x)}\left(\partial_x\mathcal{T}_J + [\mathcal{T}_J, A]\right)\right)\right)\mathrm{d}x = \tag{3-85}$$

$$= \mathop{\mathrm{res}}_{x=\infty}\sum_{b=1}^{n}\omega^{K(b-1)}\frac{1}{JK}x^{\frac{K}{n}}\mathrm{Tr}\left(\Pi_b(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{a=1}^{n}x^{\frac{J}{n}}\omega^{J(a-1)}\Pi_a(x)\right)_+\right)\mathrm{d}x = \tag{3-86}$$

$$= \frac{1}{JK}\mathop{\mathrm{res}}_{x=\infty}\mathop{\mathrm{res}}_{z=\infty}\left(\sum_{b,a=1}^{n}\omega^{K(b-1)+J(a-1)}\frac{x^{\frac{K}{n}}z^{\frac{J}{n}}}{(x-z)^2}\mathrm{Tr}\left(\Pi_b(q)\Pi_a(p)\right)\right)\mathrm{d}z\,\mathrm{d}x \ . \tag{3-87}$$

At this point the formula is almost obviously symmetric: attention is to be paid as to whether the order of residues can be interchanged. To see this, consider the kernel

$$\Omega_{ba}(x,z) := \frac{1}{(x-z)^2}\mathrm{Tr}\left(\Pi_b(x)\Pi_a(z)\right)\mathrm{d}x\mathrm{d}z \ . \tag{3-88}$$

To conclude that the order of residues can be interchanged it is sufficient to show that there is no residue on the diagonal $x = z$. To this end we note that

$$\mathrm{Tr}(\Pi_a(x)\Pi_b(z)) = \delta_{ab} + \mathcal{O}((x-z)^2). \tag{3-89}$$

Indeed the fact that $\mathrm{Tr}(\Pi_a(x)\Pi_b(x)) = \delta_{ab}$ is nothing but a statement of simplicity of eigenvalues ($x \neq \infty$): by continuity it holds also at $x = \infty$. Next, we note that if $C(x)$ is a locally differentiable matrix of eigenvectors, we have

$$\begin{aligned}\frac{\mathrm{Tr}(\Pi_a(x)\Pi_b(z))}{x-z} &= \mathrm{Tr}(\Pi_a(x)\Pi_b'(x)) + \mathcal{O}(x-z) = \\ &= \mathrm{Tr}\left(\Pi_a[C'C^{-1},\Pi_b]\right) + \mathcal{O}(x-z) = \\ &= \mathrm{Tr}(C'C^{-1}[\Pi_a,\Pi_b]) + \mathcal{O}(x-z) = \mathcal{O}(x-z) \ ,\end{aligned} \tag{3-90}$$

where we have used the cyclicity of trace. This proves that the kernel $\Omega_{ab}(x,z)$ has only a double pole without residue on the diagonal (and only for $a = b$) and this is sufficient to have independence of the order of the residues. **Q.E.D.**

Note that in fact the kernel introduced in the proof has some interesting properties because it is analytic on the whole spectral curve except on the diagonal where it has a residueless normalized double pole. This is almost the same as the fundamental bidifferential of the spectral curve, the only shortcoming being the absence of a definite normalization around an isotropic basis in the homology of the curve.

Before taking on the case of a general rational connection, we highlight the overall logic used to construct the Pfaffian system, prove Fröbenius integrability and construct the tau function:

1. we start with the **bare** zero-curvature equations in Prop. 3.2
2. we have **dressed** by the formal **nonsingular** gauge Y
3. we have defined the tau function in terms of residues of a canonical differential on the spectral curve.

The key of the proof of zero-curvature is the usual argument that the curvature of the Pfaffian system is on one hand *a priori* a polynomial expression in x whereas on the other side it is also a Laurent series, hence concluding that it must identically vanish: this argument -of course- hinges on the fact that the formal gauge Y is nonsingular. The deviation from the usual situation is not in the way we prove the compatibility, but rather in the precise form of the bare system of equations which was not known beforehand and which in the nonresonant case is in diagonal form.

4 Fuchsian resonant case

Isomonodromic deformations of resonant Fuchsian systems in full generality have been addressed in [7]. In this section, for the reader's convenience, we mainly collect those facts.

Consider a formal Fuchsian singularity (at $z = 0$)

$$F(z) = \frac{\Lambda}{z} + \sum_{j\geq 0} F_j z^j \ . \tag{4-1}$$

We assume without loss of generality that Λ is in Jordan canonical form. Since the notion of resonance for (formal) Fuchsian singularities is that two eigenvalues differ by a **nonzero integer**, we split Λ into blocks of (Jordan blocks of) eigenvalues differing only by integers (a **bouquet**). It is not difficult to show that there is a formally analytic gauge in which the connection is completely split into block diagonal form according to the decomposition into bouquets. Hence we can assume that Λ has only eigenvalues differing by integers (i.e. consists of only one bouquet of eigenvalues). By a scalar gauge transformation we can actually shift the eigenvalue to zero.

At the end of these simplifications we have a connection in which Λ is in Jordan canonical form with only integer eigenvalues which we assume in decreasing order

$$n_0 > n_1 > n_2 > \ldots > n_K \ . \tag{4-2}$$

Each eigenvalue has a certain algebraic multiplicity and a certain geometrical multiplicity (i.e. the rank of the eigenspace), which is of no particular interest to us.

We now consider a formal analytic gauge $Y(z) = \sum_{j\geq 0} Y_j z^j$ and seek the most "canonical" form under formal gauge equivalence

$$Y' = FY - YB \ , \qquad B = \frac{\Lambda}{z} + \sum_{j\geq 0} B_j z^j \ , \quad Y = \mathbf{1} + \mathcal{O}(z) \tag{4-3}$$

Our goal is to have B as simple as possible. Writing out the coefficients of the power z^{k-1} we have

$$kY_k = [\Lambda, Y_k] + F_k - B_k + \sum_{j=1}^{k-1} (F_j Y_{k-j} - Y_{k-j} B_j) \tag{4-4}$$

We rewrite this as

$$(k\,\mathrm{Id} - ad_\Lambda) Y_k = F_k - B_k + \sum_{j=1}^{k-1} (F_j Y_{k-j} - Y_{k-j} B_j) \ . \tag{4-5}$$

The linear operator $\mathcal{L}_k := (k\,\mathrm{Id} - ad_\Lambda)$ on the space of matrices $n \times n$ is invertible provided that k is not in the spectrum of ad_Λ, i.e. provided that no pair of eigenvalues of Λ differ by k. If $\mathcal{L}_k$ is invertible we can impose $B_k = 0$ since the solvability of the recurrence relation in terms of Y_k is guaranteed. If $\mathcal{L}_k$ is not invertible then B_k must be chosen so that the RHS of (4-5) is in the image of $\mathcal{L}_k$. It is not difficult to see that the image of $\mathcal{L}_k$ consists of arbitrary matrices with a zero block in the (j,ℓ) block such that $n_j - n_\ell = k$. Therefore B_k can be chosen to be zero in the complement of that block and it is then uniquely determined by (4-5) itself. By finiteness of the number of eigenvalues of Λ we can assure that $\mathcal{L}_k$ is invertible for k large enough, namely that only a finite number of B_k may need to be chosen nonzero. At the end of this procedure we always obtain a connection in the form

$$B = \frac{1}{z}\left[\begin{array}{c|c|c|c|c|c}
n_0 + \nabla & \star z^{n_0-n_1} & \star z^{n_0-n_2} & \star z^{n_0-n_3} & \ldots & \star z^{n0-n_K} \\ \hline
0 & n_1\mathbf{1} + \nabla & \star z^{n_1-n_2} & \star z^{n_1-n_3} & \ldots & \star z^{n_1-n_K} \\ \hline
0 & 0 & n_2\mathbf{1} + \nabla & \star z^{n_2-n_3} & \star z^{n_2-n_4} & \ldots \\ \hline
 & & & \ddots & & \\ \hline
 & & & & n_{K-1}\mathbf{1} + \nabla & \star z^{n_{K-1}-n_K} \\ \hline
 & & & & & n_K\mathbf{1} + \nabla
\end{array}\right] \tag{4-6}$$

where ∇ denotes a nilpotent matrix in Jordan canonical form and $\star$ denote the only possibly nonzero coefficients and **constant** in z. Each diagonal block in the above decomposition has dimension equal to the algebraic multiplicity of the corresponding integer eigenvalue of Λ. The block diagonal shearing

$$z^G := (z^{n_0}\mathbf{1}, z^{n_1}\mathbf{1}, z^{n_2}\mathbf{1}, \ldots) \tag{4-7}$$

(here each identity is of the appropriate dimension) recasts the connection to a nonresonant one with the only eigenvalue zero of the simple form

$$z^{-G}Bz^G - \frac{G}{z} = \frac{T}{z}\ ,\quad T := \left[\begin{array}{c|c|c|c|c|c} \triangledown & \star & \star & \star & \dots & \star \\ \hline 0 & \triangledown & \star & \star & \dots & \star \\ \hline 0 & 0 & \triangledown & \star & \star & \dots \\ \hline & & & \ddots & & \\ \hline & & & & \triangledown & \star \\ \hline & & & & & \triangledown \end{array}\right] \tag{4-8}$$

Note that T is constant. The $\star$'s in (4-6) and (4-8) represent the same coefficients and are defined up to action of constant gauge transformations in block diagonal form, each nonzero block of which consists of the centralizers of the diagonal blocks of F. The orbit under this centralizer group is what defines the local monodromy and hence must be preserved by the isomonodromic deformation.
The solution of last system is

$$\Phi^{bare} = z^T. \tag{4-9}$$

In other words a (formal) solution of the original system is

$$\Psi = Y(z)z^G z^T\ , \tag{4-10}$$

where G is the diagonal matrix of integers used in the shearing (inducing the grading) and in general does not commute with T. Since T is upper (semi)triangular it can be put in Jordan canonical form T_{can} by an upper triangular matrix P;

$$\Psi = Yz^G P z^{T_{can}} P^{-1} \tag{4-11}$$

Since $z^G P z^{-G}$ is analytically invertible, it can be reabsorbed in the definition of Y, so that without loss of generality we can always assume the formal solution (4-10) to have T in Jordan canonical form and hence we will denote T by J in the sequel. At this point the residual arbitrariness of this (formal) solution is by multiplication on the right by a z-independent matrix S in the centralizer of J and such that $z^G S z^{-G}$ is analytic; this implies that S must be at the same time in the centralizer of J and in block upper-triangular form according to the minimal decomposition of G into blocks which are multiple of the identity matrix.

To put it in a different way, S must be in the intersection of the positive root spaces of Ad_{z^G} with the centralizer of J.

Summarizing this discussion and restoring the generality of all the steps we have obtained the

Proposition 4.1 [Formal solution for resonant Fuchsian singularities] Let $A = \frac{A_0}{z} + \mathcal{O}(1)$ be the matrix of a Fuchsian singularity at $z = 0$. Then there exists a (formal) solution Ψ of the form

$$\Psi = Y(z)z^G z^J \tag{4-12}$$

where

1. $Y(z)$ is analytically (formally) invertible at $z = 0$
2. J is in Jordan canonical form with distinct eigenvalues with real part in the interval $[0, 1)$
3. G is an integer valued diagonal matrix such that –within each block corresponding to the same eigenvalue of T– the integers form a weakly decreasing sequence which distinguishes the eigenvalues of the same bouquet.
4. The spectrum of A_0 coincides with the spectrum of $G + J$ and moreover $A_0 = Y_0(G + J){Y_0}^{-1}$

Such solution is unique up to ordering of the eigenvalues of J and by multiplication on the right by a z-independent matrix lying in the intersection of the centralizer of J and the nonnegative root subspace of Ad_{z^G}[7]. Moreover the gauge $Y(z)$ is actually a convergent series if $A(z)$ is convergent in a punctured disk around the regular singularity [19].

4.1 Isomonodromic deformation of resonant Fuchsian singularities

In this section we rephrase part of the content of [7]. Suppose we have an isomonodromic family of resonant Fuchsian connections with poles at points γ_j, $j = 1, \dots$ and let $\Phi(z;\boldsymbol{\gamma})$ be the fundamental solution of the family. At each pole γ_j and by virtue of the previous discussion culminated in Prop. 4.1 there is a z-independent nonsingular matrix C_j for which

$$\Phi = Y_j(z)(z-\gamma_j)^{G_j}(z-\gamma_j)^{J_j}C_j \ . \tag{4-13}$$

The matrix C_j is defined modulo the group described in Prop. 4.1. The monodromy is then

$$M_j = C_j^{-1}\mathrm{e}^{2i\pi J_j}C_j \ . \tag{4-14}$$

We see that under a monodromy preserving deformation the matrices C_j can vary arbitrarily by left multiplication of a matrix in the centralizer of J_j (which is the same as the centralizer of $\mathrm{e}^{2i\pi J_j}$ since the eigenvalues of J_j by construction do not differ by integers).

Under a continuous monodromy preserving deformation we have near each pole γ_j

$$\dot\Phi\Phi^{-1} = \dot\gamma_j\, Y_j\frac{G_j+J_j}{z-\gamma_j}{Y_j}^{-1} + Y_j(z-\gamma_j)^{G_j}\dot C_j{C_j}^{-1}(z-\gamma_j)^{-G_j}{Y_j}^{-1} + \dot Y_j Y_j^{-1} \ . \tag{4-15}$$

The last term is analytic at γ_j. The first term has a simple pole and is the standard term in Schlesinger deformations. The second term may have poles of higher order: indeed $\dot C_j C_j$ needs only to belong to the centralizer Lie algebra of J_j but not necessarily to the centralizer of G_j nor to its nonnegative root subspace. A case in which the second term is certainly absent is when the centralizer of J_j is the Abelian algebra of diagonal matrices, which corresponds to the case of nonresonant Fuchsian singularities; another case is when J_j has only one Jordan block, for in that case the centralizer is upper triangular and hence certainly in the nonnegative root space of Ad_{z^G}. In case J_j contains more than one irreducible block with the same eigenvalue then the centralizer is not upper triangular (see Figure 1 for representations of such a centralizer) and hence conjugation by $(z-\gamma_j)^{G_j}$ may have poles of higher degree (at most the index of resonance, i.e., the maximum integer difference between two eigenvalues).

This situation is the most general as explained in [7]. We can regard the dependence of C_j on the deformations as "pure gauge" in the sense that it is arbitrarily defined and can always be disposed of by a **rational** gauge equivalence which *does not move the position of the poles*, or -which is the same- by solving a certain Riemann–Hilbert problem.

Suppose indeed that $\Phi, \tilde\Phi$, are two isomonodromic families of a resonant Fuchsian connection with the same monodromy representation $\{M_j\}$ **and** with same residual spectrum at the Fuchsian singularities

$$Sp(\operatorname*{res}_{z=\gamma_j} \Phi'\,\Phi^{-1}) = Sp(\operatorname*{res}_{z=\gamma_j} \tilde\Phi'\,\tilde\Phi^{-1}) \ , \ \forall j \ . \tag{4-16}$$

This implies that the matrices J_j (the Jordan form of the monodromies) and G_j are the same for the two families. This also implies that

$$G(z) := \tilde\Phi(z)\Phi^{-1}(z) \tag{4-17}$$

[7] The operator Ad_{z^G} splits $GL(n,\mathbb{C})$ into eigenspaces with eigenvalues $z^{G_{ii}-G_{jj}}$; the direct sum of the subspaces with nonnegative exponent for this eigenvalue is our definition of nonnegative root subspace. For example if the entries of G are strictly decreasing then the nonnegative root subspaces are the upper semitriangular matrices.

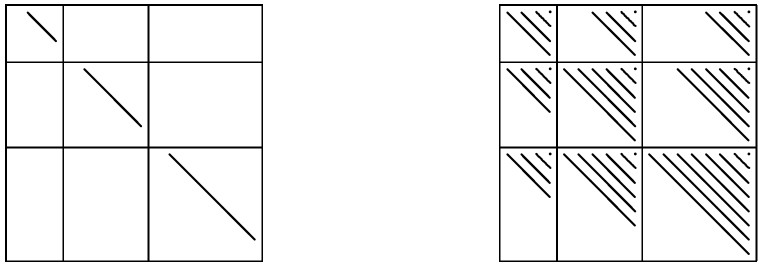

Figure 1: A nilpotent matrix in Jordan canonical form and the shape of its centralizer: the oblique segments represent entries with the same numerical value [1].

is a matrix function defined on the punctured plane and single-valued. One sees from the asymptotic representation of $\tilde{\Phi}$ and Φ that

$$G(z) = \tilde{Y}_j (z-\gamma_j)^{G_j} \tilde{C}_j C_j^{-1} (z-\gamma_j)^{-G_j} Y_j + \mathcal{O}(z-\gamma_j) \tag{4-18}$$

where we have used that $H_j := \tilde{C}_j C_j^{-1}$ is in the centralizer group of J_j because $C_j^{-1} \mathrm{e}^{2i\pi J_j} C_j = M_j = \tilde{M}_j = \tilde{C}_j^{-1} \mathrm{e}^{2i\pi J_j} \tilde{C}_j$. This shows that $G(z)$ has at worst poles at the γ_j's and hence it is rational.

We can think of $G(z)$ as the solution of the following RH problem

$$G_+(z) = G_-(z)\, Y_j\, (z-\gamma_j)^{G_j} H_j (z-\gamma_j)^{-G_j} Y_j^{-1}\ , \quad z \in \{|z-\gamma_j| = \epsilon\} \tag{4-19}$$

where $G_-(z)$ is analytically invertible in each disk and equal to $\widetilde{Y}_j(z) {Y_j}^{-1}(z)$ (recall that both $\widetilde{Y}_j$ and Y_j are actual convergent series around γ_j [19]). Since the jump matrices are analytic in each punctured small disk, the "exterior" part of the solution of this RH problem can be analytically extended to a rational function $G(z)$ in the punctured plane $\mathbb{C}P^1 \setminus \cup\{\gamma_j\}$.

Vice-versa suppose that given an isomonodromic family Φ we want to pass to another isomonodromic family $\tilde{\Phi}$ with the same monodromy matrices M_j and same residual spectrum (4-16). To this end we should find a solution to the Riemann–Hilbert problem (4-19) with preassigned arbitrary matrices H_j depending on the deformation parameters (but independent of z) and in the centralizer group of J_j. The "exterior" part G_+ of the solution of the RH problem defines by analytical continuation a rational function $G = G_+$ which transforms the family Φ in a family $\tilde{\Phi}$ with $\tilde{C}_j = H_j C_j$. The solvability of this RH problem can be assured by the argument which will be used later in the more general setting of Thm. 5.3 (see Remark 5.1).

This discussion means that for resonant Schlesinger systems the Schlesinger equations are still consistent and any other isomonodromic family is obtained from a solution of Schlesinger equations by a rational gauge equivalence constructed from a solution of (4-19).

5 General rational resonant connections

We now address the full-fledged general case. As mentioned earlier the conceptual difficulty is to *par* with the one involved in the Isomonodromic deformation system studied in Thm. 3.3 but we need to set up a good deal of notation.

Let $\mathcal{D}$ be an effective divisor on $\mathbb{C}P^1$

$$\mathcal{D} = r_\infty \infty + \sum_{\gamma \in \mathbb{C}} (r_\gamma + 1)\gamma\ , \quad r_\infty \geq 1,\ r_\gamma \geq 0. \tag{5-1}$$

It is understood that only a finite number of points γ appear in the above sum. We now consider an arbitrary rational connection of the form

$$
\begin{aligned}
A &:= \sum_{\gamma \in supp(\mathcal{D})} A_\gamma(x) \\
A_\gamma(x) &:= \sum_{J=0}^{r_\gamma} \frac{A_{\gamma,J}}{(x-\gamma)^{J+1}} \\
A_\infty(x) &:= \sum_{J=1}^{r_\infty} x^{J-1} A_{\infty,J}
\end{aligned}
\tag{5-2}
$$

where the leading terms of the singularity at each irregular singularity ($r_\gamma > 0$) may have an arbitrary Jordan canonical form provided that the Lidskii pseudo-eigenvalues of the second leading term subordinated to the Jordan form of the leading term do not vanish and are distinct. No restriction is assumed on the residues at the Fuchsian singularities. Note also that we are assuming that there is at least one irregular singularity and we have placed it at ∞.

5.1 Generalized monodromy data for resonant singularities

We now proceed along the lines of [18] to define the generalized monodromy data for our system of linear ODE

$$
\frac{d\Psi}{dx} = A\Psi \tag{5-3}
$$

Let s_γ be the number of blocks in the Jordan form of A_{γ,r_γ} and n_i^γ be the dimensions of the blocks for a non-Fuchsian singularity ($r_\gamma \geq 1$). Denote by R_γ be the diagonal matrix

$$
R_\gamma = \operatorname{diag}\left(\frac{-t_{0,1}^\gamma}{n_1^\gamma}, \frac{-t_{0,1}^\gamma+1}{n_1^\gamma}, \dots, \frac{-t_{0,1}^\gamma+n_1^\gamma-1}{n_1^\gamma}, \dots, \frac{-t_{0,s}^\gamma}{n_s^\gamma}, \frac{-t_{0,s}^\gamma+1}{n_s^\gamma}, \dots, \frac{-t_{0,s}^\gamma+n_s^\gamma-1}{n_s^\gamma}\right)
$$

where $t_{0,k}^c$ are scalars.

We now rephrase Thm. 3.1 and the results in Section 4 to construct formal solutions of (5-3) near each pole.

Proposition 5.1 *Let A be given by (5-2) such that at each pole $\gamma \in \mathcal{D}$, A_{γ,r_γ} has the Jordan normal form*

$$
[A_{\gamma,r_\gamma}] = (\lambda_{\gamma,1}^{n_1^\gamma}) \cdots (\lambda_{\gamma,s}^{n_s^\gamma})
$$

and such that the genericity condition in Def. 2.1 is satisfied at each pole if $r_\gamma \neq 0$. Then for each pole such that $r_\gamma \neq 0$, there exists a unique formal series Y^γ

$$
Y^\gamma(x-\gamma) \;=\; Y_0^\gamma + \sum_{j=1}^{\infty} (x-\gamma)^j Y_j^\gamma \ , \quad \det(Y_0^\gamma) \neq 0 \tag{5-4}
$$

such that

$$
\Psi^\gamma = Y^\gamma \exp(Q_\gamma((x-\gamma)^{-1}))(x-\gamma)^{R_\gamma} \tag{5-5}
$$

is a formal solution to the ODE (5-3), where $Q_\gamma(x-\gamma)$ is the matrix

$$Q_\gamma = \begin{pmatrix} \sum_{j=1}^{rn_1^\gamma} t_{j,1}^\gamma \frac{\mathcal{H}_{\gamma,1}^j}{j} & 0 & \dots & 0 \\ 0 & \sum_{j=1}^{r\,n_2^\gamma} t_{j,2}^\gamma \frac{\mathcal{H}_{\gamma,2}^j}{j} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sum_{j=1}^{rn_s^\gamma} t_{j,s}^\gamma \frac{\mathcal{H}_{\gamma,s}^j}{j} \end{pmatrix} \tag{5-6}$$

where $\mathcal{H}_{\gamma,i}$ are $n_i^\gamma \times n_i^\gamma$ matrices defined as in (3-14) and $t_{j,k}^\gamma$ are scalars (for $\gamma=\infty$ we must replace $(x-\gamma)^{-1}$ by x in the above expressions).

When $r_\gamma = 0$, there exist a solution of the form

$$\Psi^\gamma = Y^\gamma (x-\gamma)^{G_\gamma} (x-\gamma)^{J^\gamma} \ . \tag{5-7}$$

where G_γ is a matrix determined by the integral differences between the eigenvalues of A_{γ,r_γ} as in section 4, Y^γ is a convergent series as in (5-4) and J^γ is a constant Jordan matrix.

The uniqueness of Y^γ can be seen by first making a gauge transformation of A such that A_{γ,r_γ} is in Jordan canonical form.

We can now use this formal solution to define the monodromic data of the ODE (5-3). We have the following

Theorem 5.1 *There exists a finite number of asymptotic sectors $\mathcal{S}_i^\gamma$ with vertex at $x=\gamma$, such that $\{\mathcal{S}_i^\gamma\}$ is a covering at $x=\gamma$. Moreover, there exist fundamental solutions $\Psi_i^{(\gamma)}$ of the linear ODE*

$$\frac{d\Psi}{dx} = A\Psi$$

asymptotic to the formal series solution of the form (5-5) or (5-7) near $x=\gamma$.

The proof of this result can be found in [18] or [19]. Such solutions Ψ_i^γ are analytic outside of the divisor $\mathcal{D}$ and on the universal cover of $\mathbb{C}P^1 \setminus \mathcal{D}$.

In particular, we can now cover the punctured Riemann sphere $\mathbb{CP}^1/\mathcal{D}$ with sectors $\mathcal{S}_j^\gamma$, $\gamma \in \mathcal{D}$ and define the stokes data by using the asymptotic form (5-5) and the sectors $\mathcal{S}_j^c$.

We have the following

Definition 5.1 *Let $\{\mathcal{S}_i^\gamma\}$ be a covering of the punctured Riemann sphere $\mathbb{CP}^1/\mathcal{D}$ that satisfies the conditions in Thm. 5.1 and let Ψ_i^γ be the fundamental solution that is asymptotic to the formal solution of the form (5-5) in $\mathcal{S}_i^\gamma$. Let*

$$C_{kl}^{\alpha\beta} = (\Psi_k^\alpha)^{-1}\,\Psi_l^\beta \ .$$

We shall call $\left\{\mathcal{S}_i^\gamma, C_{kl}^{\alpha\beta}\right\}$ a **Stokes phenomenon** *of the ODE (5-3). This, together with the set of variables $t_{0,j}^c$ at each pole $x=\gamma$, $r_\gamma \neq 0$, and the J^γ for $r_\gamma=0$, is called the* **monodromic data** *of the ODE (5-3).*

Here the matrices $C_{kl}^{\alpha\beta}$ contain the usual definition of Stokes' matrices and connection and monodromy on the same basis (and redundantly).

5.2 Isomonodromic deformations

In view of Thm. 3.3, at each singularity $\gamma \in \mathcal{D}$ we can find a formal nonsingular gauge Y_γ that gauges $A(x)$ to the localized version of the bare form $D_{(\gamma)}(x)$ advocated Prop. 3.3. The bare connection/deformation at $x = \gamma$ is now the direct sum of as many bare systems of the form given in Prop. 3.4 as the number of blocks in which the system has been decomposed. All these bare deformation/differential equations are then dressed by the same dressing matrix Y^γ.

In order to be more explicit consider an irregular singularity $x = \gamma$. Suppose the Jordan form of the leading coefficient at $x = \gamma$ is as in Prop. 5.1. In symbolic notation we denote by $\mathbf{t}_\gamma$ the times "attached" to the pole at $x = \gamma$. The number of them is

$$\#\{\mathbf{t}_\gamma\} = rn \ , \tag{5-8}$$

namely the same as if the system were nonresonant.

The following theorem allows us to consider the direct sum of many "bare" Pfaffian systems of the form in Lemmas 3.2, 3.4 and obtain a solution of another Pfaffian systems after a "dressing" using a formal gauge equivalence.

Theorem 5.2 *Suppose A is a matrix-valued function given by (5-2) and Y^γ is a formal series holomorphic and invertible at γ*

$$\begin{aligned} Y^\gamma(x-\gamma) &= Y_0^\gamma + \sum_{j=1}^{\infty} (x-\gamma)^j Y_j^\gamma \ , \quad \det(Y_0) \neq 0 \\ Y^\infty(x) &= Y_0^\infty + \sum_{j=1}^{\infty} x^{-j} Y_j^\infty \end{aligned}$$

such that

$$B^\gamma(x) = (Y^\gamma)^{-1} A Y^\gamma - (Y^\gamma)^{-1} (Y^\gamma)' \ ,$$

is a formal power series in $(x-\gamma)$ in block diagonal form according to the block decomposition of A_{γ,r_γ}, and similar expression for ∞.

Let $\mathcal{T}^\gamma(x)$ ($\mathcal{T}^\infty(x)$) be matrix-valued 1-forms polynomial in $(x-\gamma)^{-1}$ (x, respectively) such that

$$[\partial_x - B^\gamma(x), \mathrm{d} - \mathcal{T}^\gamma(x)] = \mathcal{O}(x-\gamma) \tag{5-9}$$

$$[\partial_x - B^\infty(x), \mathrm{d} - \mathcal{T}^\infty(x)] = \mathcal{O}(x^{-1}) \ . \tag{5-10}$$

Then the Pffaffian system

$$[\partial_x - A(x)]\, \Psi = 0 \tag{5-11}$$

$$[\mathrm{d} - \mathcal{M}(x)]\, \Psi = 0 \tag{5-12}$$

is Fröbenius compatible, where $\mathcal{M}(x)$ is the matrix-valued rational 1-form such that the singular part of $\mathcal{M}$ near each pole is given by

$$\mathcal{M}(x) = \Big(Y^\gamma \mathcal{T}^\gamma(x-\gamma)\, (Y^\gamma)^{-1} \Big)_{pp} + \mathcal{O}(1), \quad x \to \gamma \neq \infty \tag{5-13}$$

where X_{pp} denotes the principal part of X at $x = \gamma$. Near $x = \infty$, $\mathcal{M}$ is given by

$$\mathcal{M}(x) = \Big(Y^\infty \mathcal{T}^\infty(x)\, (Y^\infty)^{-1} + \mathrm{d}Y^\infty\, (Y^\infty)^{-1} \Big)_+ + \mathcal{O}(x^{-1}), \quad x \to \infty \tag{5-14}$$

where X_+ is the polynomial part of X and Y_0^∞ is some analytic function in the deformation parameters.

Proof. Let $\mathcal{M}(x)=\sum\mathcal{M}_{\nu}(x)dt_{\nu}$: here the label ν is a generic label used for the collection of all times at all singularities. According to this definition each $\mathcal{M}_{\nu}$ is singular only at one of the poles $\gamma\in(\mathcal{D})$. Let ν be the pole of $\mathcal{M}_{\nu}(x)$. A simple calculation shows that

$$[\partial_{\nu},\partial_{x}]\,Y^{\nu}\left((\hat{Y})^{\nu}\right)^{-1}=\partial_{\nu}A-\partial_{x}\mathcal{M}_{\nu}(x)+[A,\mathcal{M}_{\nu}(x)]-(\partial_{\nu}B^{\nu}-\partial_{x}T_{\nu}^{\nu}+[B^{\nu},\mathcal{T}_{\nu}^{\nu}(x)])$$

where $M_{\nu}(x)=\left(Y^{\nu}\mathcal{T}_{\nu}\left(Y^{\nu}\right)^{-1}\right)_{pp}$.

By using similar argument as in the proof of Thm. 3.3, we see that

$$\begin{aligned}[]
[\partial_{x}-A(x),\partial_{\nu}-\mathcal{M}_{\nu}(x)] &= \mathcal{O}(x-\nu), \quad x\to\nu\\
[\partial_{x}-A(x),\partial_{\nu}-\mathcal{M}_{\nu}(x)] &= \mathcal{O}(x^{-1}), \quad x\to\infty
\end{aligned}$$

Therefore, by Liouville's theorem, we have

$$[\partial_{x}-A(x),\partial_{\nu}-\mathcal{M}_{\nu}(x)]=0$$

Similarly, we have

$$\begin{aligned}[]
[\partial_{\nu}-\mathcal{M}_{\nu}(x),\partial_{\mu}-\mathcal{M}_{\mu}(x)] &= \mathcal{O}(x-\nu), \quad x\to\nu=\mu\\
[\partial_{\nu}-\mathcal{M}_{\nu}(x),\partial_{\mu}-\mathcal{M}_{\mu}(x)] &= \mathcal{O}(x^{-1}), \quad x\to\nu=\mu=\infty
\end{aligned}$$

when the pole of $\mathcal{M}_{\nu}(x)$ and $\mathcal{M}_{\mu}(x)$ are the same point.

Now consider the case when the poles of $\mathcal{M}_{\nu}(x)$ and $\mathcal{M}_{\mu}(x)$ are not the same. Then near $x=\nu$, we have

$$\begin{aligned}
\partial_{\nu}Y^{\nu} &= \mathcal{M}_{\nu}(x)Y^{\nu}+Y^{\nu}\mathcal{T}_{\nu}^{\nu}\\
\partial_{\mu}Y^{\nu} &= \mathcal{M}_{\mu}(x)Y^{\nu}
\end{aligned}$$

We also have similar equations with ν replaced by μ. From this we have

$$[\partial_{\nu},\partial_{\mu}]\,Y^{\nu}\left(Y^{\nu}\right)^{-1}=\partial_{\nu}\mathcal{M}_{\nu}(x)-\partial_{\mu}\mathcal{M}_{\nu}(x)+[\mathcal{M}_{\mu}(x),\mathcal{M}_{\nu}(x)]$$

near $x=\nu$ and similar equation with μ replacing ν. By using similar argument as before, we see that

$$\partial_{\nu}\mathcal{M}_{\nu}(x)-\partial_{\mu}\mathcal{M}_{\nu}(x)+[\mathcal{M}_{\mu}(x),\mathcal{M}_{\nu}(x)]=0$$

this concludes the proof of the theorem. **Q.E.D.**

We can now consider isomonodromic deformations of the ODE

$$\frac{d\Psi}{dx}=A\Psi$$

Let $\{\mathcal{S}_{k}^{\alpha},\Psi_{k}^{\alpha}\}$ be a covering and fundamental solutions as in Def. 5.1 and let $\left\{C_{kl}^{\alpha\beta},J^{\gamma},t_{0,k}^{\gamma}\right\}$ be the monodromic data associated to it. The isomonodromic problem is the following. Given $\{\mathcal{S}_{k}^{\alpha},\Psi_{k}^{\alpha}\}$, how should one deform the solutions Ψ_{k}^{α} such that the monodromic data $\left\{C_{kl}^{\alpha\beta},J^{\gamma},t_{0,k}^{\gamma}\right\}$ is fixed?

The isomonodromic problem is only defined when a covering and solutions $\{\mathcal{S}_{k}^{\alpha},\Psi_{k}^{\alpha}\}$ is chosen and should be thought of as deformations of the fundamental solutions. For if the monodromic data of $\{\mathcal{S}_{k}^{\alpha},\Psi_{k}^{\alpha}\}$ is fixed under deformations, then by multiplying Ψ_{k}^{α} to the right by matrices C_{k}^{α} that depend on the deformation parameters t, one obtains a pair $\{\mathcal{S}_{k}^{\alpha},\Psi_{k}^{\alpha}C_{k}^{\alpha}\}$ in which the monodromic data is no longer fixed.

We can now prove the following theorem, which classifies the monodromy preserving deformations when the genericity condition in Def. 2.1 is satisfied.

In the following we denote the matrix-differential one form of the bare deformations as

$$\mathcal{T}^{\gamma,bare}(x) = \mathrm{diag}\left(\mathcal{C}_1^\gamma \mathrm{d}\gamma + \sum_{J=1}^{rn_1^\gamma} T_1^{\gamma,bare}\mathrm{d}t_{J,1}^\gamma, \ldots, \mathcal{C}_{s_\gamma}^\gamma \mathrm{d}\gamma + \sum_{J=1}^{rn_{s_\gamma}^\gamma} T_{s_\gamma}^{\gamma,bare}\mathrm{d}t_{J,s_\gamma}^\gamma\right) \tag{5-15}$$

where the notation is as in Props. 3.2, 3.4.

Theorem 5.3 *Let A be a rational matrix-valued function given by (5-2), let $\mathcal{D}_0$ be the divisor of poles such that $r_\gamma = 0$ for $\gamma \in \mathcal{D}_0$ (the Fuchsian singularities) and D_1 be the divisor of higher order poles (the irregular ones). Suppose the genericity condition in Def. 2.1 is satisfied at each pole $\gamma \in \mathcal{D}_1$ and that $\infty \in \mathcal{D}_1$.*

Let $\{\mathcal{S}_k^\alpha, \Psi_k^\alpha\}$ be a covering and fundamental solutions that satisfies the conditions in Def. 5.1. Then the monodromic data of the ODE (5-3) defined by $\{\mathcal{S}_k^\alpha, \Psi_k^\alpha\}$ are preserved if and only if Ψ_k^α satisfies the differential equations

$$\mathrm{d}\Psi_k^\alpha = \mathcal{T}\Psi_k^\alpha \tag{5-16}$$

where $\mathcal{T}$ is a matrix-valued one-form as follows

$$\begin{aligned}\mathcal{T} \;=\;& \left(Y^\infty \mathcal{T}^{\infty,bare}(Y^\infty)^{-1} + \mathrm{d}Y_0^\infty\,(Y_0^\infty)^{-1}\right)_+ + \sum_{\gamma\in\mathcal{D}_1}\left(Y^\gamma \mathcal{T}^{\gamma,bare}(Y^\gamma)^{-1}\right)_{pp} + \\ +\;& \sum_{\gamma\in\mathcal{D}_0} A_{\gamma,0}(x-\gamma)^{-1}\mathrm{d}\gamma + \left(Y^\gamma(x-\gamma)^{G_\gamma}\mathrm{d}H^\gamma(H^\gamma)^{-1}(x-\gamma)^{-G_\gamma}\,(Y^\gamma)^{-1}\right)_{pp}\end{aligned}$$

where $T^{\gamma,bare}$ is defined in (5-15) and Y^γ, G_γ, and Y^∞ are given in Prop. 5.1, while H^γ is a matrix in the centralizer of J_γ (constant in x) which may depend analytically on the deformation parameters. X_{pp} denotes the principal part of X around the corresponding pole and X_+ denotes the polynomial part of X. The exterior derivative d in the above expressions denotes derivatives with respect to the t in proposition 5.1 and the position of the poles γ, but with $dt_{0,k}^\gamma = 0$.

The dependence of Y_0^∞ and H^γ on the parameters can be arbitrary provided that H^γ remains in the centralizer of J_γ and Y_0^∞ is invertible.

Proof. The proof is essentially the same in [11]. By using similar argument as in the proof of Thm. 5.2, we see that in the case of $r_\gamma = 0$, we have

$$[\partial_x, \partial_\gamma]\,Y^\gamma\,(Y^\gamma)^{-1} + Y^\gamma\,[\partial_x, \partial_\gamma]\left((x-\gamma)^{G_\gamma}H^\gamma\right)\left(Y^\gamma(x-\gamma)^{G_\gamma}H^\gamma\right)^{-1} = \partial_x\mathcal{T}_\gamma(x) - \partial_\gamma A(x) + [A(x), \mathcal{T}_\gamma(x)]$$

near $x = \gamma$ where $\mathcal{T}_\gamma$ is the coefficient of $\mathrm{d}\gamma$ in $\mathcal{T}$. The left hand side is a positive series in $x-\gamma$ since the second term is a well-defined function of x and the deformation parameters, and Y^γ is a positive series in $x-\gamma$. By considering the singular behavior of the right hand side near each pole and then apply Liouville's theorem, one sees that the right hand side must be zero. Similarly, one can show the commutativity between the parameters t, γ when one of the parameters involved is the position of a simple pole.

For the cases that do not involve derivatives of the position of a simple pole, we can apply Prop. 3.4 and Thm. 5.2 and see that the Pffafian system

$$\left\{\begin{array}{l}[\partial_x - A(x)]\,\Psi = 0\\ [\mathrm{d} - \mathcal{T}(x)]\,\Psi = 0\end{array}\right. \tag{5-17}$$

is integrable.

Let Ψ_k^α and Ψ_l^β be solutions of (5-17) as in Def. 5.1, where α, $\beta \in \mathcal{D}$ and may not be the same point. Since Ψ_k^α and Ψ_l^β solve the same equations (5-17), we have

$$\mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1} = \mathrm{d}\Psi_l^\beta \left(\Psi_k^\beta\right)^{-1} = \mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1} + \Psi_k^\alpha \mathrm{d}C_{kl}^{\alpha\beta} \left(C_{kl}^{\alpha\beta}\right)^{-1} \left(\Psi_k^\alpha\right)^{-1}$$

hence $\mathrm{d}C_{kl}^{\alpha\beta} \left(C_{kl}^{\alpha\beta}\right)^{-1} = 0$ and the monodromic data is preserved.

Conversely, let Ψ_k^α be deformed in such a way that the monodromic data defined by $\{\mathcal{S}_k^\alpha, \Psi_k^\alpha\}$ is preserved. We see that

$$\mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1} = \mathrm{d}\Psi_l^\beta \left(\Psi_k^\beta\right)^{-1}$$

for α, $\beta \in \mathcal{D}$, where α and β may not be the same point. Therefore $\mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1}$ is a globally defined meromorphic 1-form

$$\mathcal{M}(x) := \mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1} \ . \tag{5-18}$$

Its asymptotic behavior near each pole are given by (note that $\mathrm{d}Q_\gamma = \mathcal{T}^{\gamma,bare}$)

$$\begin{aligned}
\mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1} &= \mathrm{d}\Psi_l^\beta \left(\Psi_l^\beta\right)^{-1} \sim \mathrm{d}Y^\gamma \left(Y^\gamma\right)^{-1} + Y^\gamma \mathrm{d}Q_\gamma \left(Y^\gamma\right)^{-1} \\
&= \left(Y^\gamma \mathrm{d}Q_\gamma \left(Y^\gamma\right)^{-1}\right)_{pp} + \mathcal{O}(1) \qquad x \to \gamma, \quad r_\gamma > 0 \qquad (5\text{-}19) \\
\mathrm{d}\Psi_k^\alpha \left(\Psi_k^\alpha\right)^{-1} &= A_{\gamma,0}(x-\gamma)^{-1}\mathrm{d}\gamma + \left(Y^\gamma (x-\gamma)^{G_\gamma} \mathrm{d}H^\gamma (H^\gamma)^{-1} (x-\gamma)^{-G_\gamma} \left(Y^\gamma\right)^{-1}\right)_{pp} + \mathcal{O}(1) \\
&\quad x \to \gamma, \quad r_\gamma = 0 \qquad (5\text{-}20)
\end{aligned}$$

for all α, β and $\gamma \in \mathcal{D}$, where $X \sim W$ means that X is asymptotic to W near the corresponding point. One can also write down a similar equation for $x \to \infty$.

The equations (5-19) determines the 1-form $\mathcal{M}$ up to the addition of a constant in x.

To determine this constant, we can look at the behavior of $\mathcal{M}$ near $x = \infty$. This is given by

$$\left(\mathrm{d}Y^\infty \left(Y^\infty\right)^{-1} + Y^\infty \mathrm{d}Q_\infty \left(Y^\infty\right)^{-1}\right)_+ \quad x \to \infty$$

We see that $\mathcal{M}$ has the form of $\mathcal{T}$ in the theorem. **Q.E.D.**

Remark 5.1 The essence of Thm. 5.3 is that the dependence of Y_+^∞ and H^γ on the deformations can be regarded as gauge arbitrariness, and fixing it yields a consistent Pfaffian system.

Suppose that we have an initial value problem for the Pfaffian system (5-17) $A^{(0)}(x)$ and that we consider two evolutions, one in which H^γ are constants in the parameters and one in which they are preassigned arbitrary analytic functions with values in the prescribed centralizers. Let us denote by Ψ and $\widetilde{\Psi}$ the two kernel solutions of the Pfaffian system. Define then the function $G(z, \mathbf{t})$ (here $\mathbf{t}$ denotes collectively all the isomonodromic deformation parameters) as follows

$$G(x, \mathbf{t}) = \widetilde{\Psi}_k^\alpha(x) (\Psi_k^\alpha)^{-1}(x) \ . \tag{5-21}$$

Since both Stokes' phenomena are the same, this implies that $G(z)$ is a single-valued analytic invertible function on the punctured domain $\mathbb{CP}^1 \setminus \mathcal{D}$. Since the essential singularities of Ψ and $\widetilde{\Psi}$ have the same asymptotic expansion in the same sectors of the irregular singularities, this implies that $G(x)$ has no singularity there. The only possible singularities are poles at the Fuchsian singularities. The same considerations used in Section (4) show that in general $G(x)$ has poles of order equal to the resonance index at the given Fuchsian singularity (in case of nonresonant singularity $G(x)$ must be analytic there). This shows that the RH problem for G discussed in

Sect. 4 admits a solution for arbitrary choice of the group elements H^γ in the centralizer of the local monodromy matrices. In other words the dependence of the H^γ's on the parameters $\mathbf{t}$ is "pure gauge" and can be completely gauged away by means of a **rational** gauge equivalence $G(x, \mathbf{t})$ which does not alter the singularity structure nor the position of the poles of $A(x)$.

In order to fix part of this arbitrariness we can assume that the coefficient A_{∞,r_∞} is in the gauge-fixed form (3-59) within each block: this fixes an overall gauge Y^∞ uniquely (up to the centralizer of A_{∞,r_∞}) for our connection. At this point the procedure of Thm. 3.1 for the singularity at ∞ produces a unique formal gauge Y_∞ in which the leading coefficient is necessarily in the centralizer of A_{∞,r_∞}; the entries of this leading coefficient will be constants of the motions exactly as in the previous guide-example, and hence could be set to zero in the sense that this would be a consistent reduction of the problem (but of no practical advantage).

Corollary 5.1 (Normalized isomonodromic deformations) *Let $A(x)$ be as in Thm. 5.3. Then the one form obtained from eq. (5-16) by setting $\mathrm{d}Y^\infty \equiv 0 \equiv \mathrm{d}H^\gamma$ defines an integrable Pfaffian system.*

6 Isomonodromic Tau function

The authors of [11] gave a definition of tau function in terms of a suitable closed differential on the space of times. The definition was expressed in terms of (formal) residues of the formal local gauge (which we denote Y^γ) around each singularity of the connection. It is shown in [6] that that definition is equivalent to one which manifests the spectral nature of the tau function. In fact the definition in [6] is much more convenient and easily generalizable to the present context

We first need to fix the gauge arbitrariness of our deformation system by requiring that the deformations occur according to the integrable Pfaffian system specified in Corollary 5.1.

Let us set up the notation: we consider the **spectral curve** (characteristic polynomial) of the connection, namely the set of $\mathbb{C}^2$

$$E(x, y) := \det(y - A(x)) = \sum_{j=0}^{n} y^j P_{n-j}[x; A] = 0 \ . \tag{6-1}$$

We think of this curve Σ in $\mathbb{P}^1 \times \mathbb{P}^1$ and we regard the two functions $x, y : \Sigma \to \mathbb{C}P^1$ as meromorphic functions on the curve, or projections. The x projection $x : \Sigma \to \mathbb{C}P^1$ has ramification points at all the $\gamma \in \mathcal{D}_1$ (irregular singularities) and at ∞, together with other ramification point whose position is not known a priori and in general depends on the "times".

If we perform an appropriate desingularization of the curve above each pole $x = \gamma$ of the divisor $\mathcal{D}_1$ (including ∞) we obtain a branching structure in which the n sheets of the projection $x : \Sigma \to \mathbb{C}P^1$ are glued precisely according to the dimensions of the Jordan cells of the leading coefficient of the connection and cyclically permuted by a small loop of x around γ. For example, suppose the Jordan form of $A_{(\gamma),r}$ is (we set $r_\gamma = r$ for brevity since the discussion is local anyway)

$$[A_{(\gamma),r}] = ({\lambda_1}^5)^2 {\lambda_2}^3 \lambda_3 (\lambda_4)^2 \tag{6-2}$$

This means that there are 2 Jordan blocks of size $n_1 = 5 = n_2$ and eigenvalue λ_1, one Jordan block of size 3 and eigenvalue λ_2 etc. (this would correspond to $n = 5 \times 2 + 3 + 1 + 1 \times 2 = 16$). Under our assumptions for the Lidskii matrix of the subleading term, these eigenvalues split into *distinct* cyclic $k-$tuplets where k is the size of the block and the number of $k-$tuplets equals the number of blocks of that size. In the example we will have two groups of five-plets $y_{1,j}, y_{2,j},\ j = 1 \ldots n_1 = n_2$ that have the common asymptotics

$$y_{a,j} \simeq \lambda_1 (x - \gamma)^{-1-r}(1 + o(1)). \tag{6-3}$$

Both groups admit a Puiseux series expansion in powers of $\xi := (x-\gamma)^{\frac{1}{n_1}}$ and each group forms a cyclic n_1-tuplet

$$y_{a,j} = \frac{\lambda_1}{\xi^{n_a(r+1)}}\left(1+\sum_{k=1}^{\infty} c_{a,k}\xi^k\right) \tag{6-4}$$

$$y_{a,j}(\mathrm{e}^{2i\pi/n_a}\xi) = y_{a,j+1}(\xi). \tag{6-5}$$

The two groups are distinct in the sense that the first coefficients of the expansion in the brackets differ from each other $c_{a,1} \neq c_{b,1}$, $a \neq b$. The first $r \cdot n_a = n_a\, r_c$ coefficients are actually the isomonodromic times of our problem, or more precisely (carrying on with this example)

$$y_{a,1} = \frac{1}{n_a \xi^{n_a(r_c+1)}}\left(-t_{a,n_a r} - \sum_{K=1}^{n_1 r-1} t_{a,n_a r-K}\xi^K + \dots\right) = -\sum_{J=1}^{n_a r}\frac{t_{a,J}}{n_a\xi^{n_a+J}} - \frac{\widetilde{t}_{a,0}}{n_a\xi^{n_a}} + \sum_{K=1}^{\infty}\frac{K\, H_{a,K}}{n_a}\xi^K\ . \tag{6-6}$$

This means that we can "extract" the isomonodromic times (as well as other parameters of formal monodromy) with the residue

$$\begin{aligned}
t_{a,J} \quad &= \quad -\mathop{\mathrm{res}}_{x=\gamma}\left(\sum_{\sigma=0}^{n_a-1}(\omega_a)^{J\sigma} y_{a,\sigma+1}\, n_a \xi^{\frac{J}{n_a}}\right)\mathrm{d}x = -\mathop{\mathrm{res}}_{\zeta^1_\gamma} y(x-\gamma)^{\frac{J}{n_a}}\mathrm{d}x\ ,\\
&\qquad J = 1,\dots,n_a r\ ,\quad \omega_a := \mathrm{e}^{2i\pi/n_a}\\
\widetilde{t}_{a,0} \quad &= \quad t_0 - \frac{n_a-1}{2} = -\mathop{\mathrm{res}}_{x=\gamma}\left(\sum_{\sigma=0}^{n_a-1} y_{a,\sigma+1}\right)\mathrm{d}x = -\mathop{\mathrm{res}}_{\zeta^a_\gamma} y\mathrm{d}x
\end{aligned} \tag{6-7}$$

Note that the fractional power is in fact a well-defined local coordinate on the (desingularization of the) spectral curve. The notation is that ζ^a_γ is the point on the spectral curve that projects to $\gamma = x(\zeta^a_\gamma)$ and a distinguishes the different (in this case $a = 1,2$) points projecting to the same γ (and corresponds to different Jordan cells of the same dimension, in this example $n_a = 5$, $a = 1,2$).

We now define the isomonodromic tau function by the following equations

$$\frac{\partial}{\partial t_{a,K}}\ln\tau : \quad = \quad H_{a,K} = \frac{1}{K}\mathop{\mathrm{res}}_{x=\gamma}\left(\sum_{\sigma=0}^{n_a-1} y_{a,\sigma+1}(\omega_a)^{J\sigma}(x-\gamma)^{K/n}\right)\mathrm{d}x \tag{6-8}$$

$$\frac{\partial}{\partial\gamma}\ln\tau : \quad = \quad \frac{1}{2}\mathop{\mathrm{res}}_{x=\gamma}\mathrm{Tr}A^2(x)\mathrm{d}x \tag{6-9}$$

Quite clearly the above definition should be repeated for all times and all singularities of our connection, where for the Fuchsian singularities the only pertinent equation is (6-9): then we should *prove* that the definition is well-posed, namely that the differential defined by these equation is closed.

In fact -as it turns out- the proof of this fact is no different from [6] and the previous simple case of polynomial connection with single Jordan leading block, needing just some modifications in the notation.

It is based mainly on the following

Lemma 6.1 Let $A(x)$ be a rational matrix and let $\widetilde{M}$ denote the classical adjoint of a matrix M. Let $y(x)$ be the (multivalued) eigenvalue of $A(x)$. Define the following expression

$$\Omega := \frac{\mathrm{Tr}\left(\widetilde{(y-A)}(x)\,\widetilde{(y'-A)}(x')\right)}{\mathrm{Tr}\left(\widetilde{(y-A)}(x)\right)\,\mathrm{Tr}\left(\widetilde{(y'-A)}(x')\right)}\frac{\mathrm{d}x\mathrm{d}x'}{(x-x')^2}\ , \tag{6-10}$$

where $y(x)$ and $y'(x)$ are two determination of the eigenvalues above the point x. Then Ω is a well defined bidifferential on the spectral curve, with a double pole without residue on the diagonal away from the singularities of A and from the branch-points of $y(x)$.

Note that the lemma states that -in spite of the $(x-x')^2$ denominator, there are no singularities unless $y(x') \to y'(x)$ as $x' \to x$ (i.e. unless y and y' belong to the same sheet of the x-projection).

Proof. This is essentially the same argument used in the proof of Prop. (3.5). The only -a priori- singularities of Ω are at the poles of A and possibly at the branch-points of y -which are of no interest to us- **and** on the points above $x = x'$. We must prove that if $y(x)$ and $y(x')$ are on different sheets then there is no pole and if they belong to the same sheet then there is a double pole without residue. Let x and x' be in a common neighborhood which does not include any branch-points; then we can distinguish the r sheets of the spectral curve $y_1, \ldots, y_r$. The expressions

$$\Pi_a(x) := \frac{\widetilde{A - y_a}(x)}{\mathrm{Tr}(\widetilde{A - y_a})(x)} , \tag{6-11}$$

are then the rank-one spectral projectors on the (one-dimensional) eigenspace and they are well defined in said neighborhood. As we already remarked we have

$$\mathrm{Tr}(\Pi_a(x)\Pi_b(x')) = \delta_{ab} + \mathcal{O}((x-x')^2) \tag{6-12}$$

which proves the assertion. **Q.E.D.**

Remark 6.1 In fact one may prove a stronger assertion (which is not important for our purposes) that Ω is a well defined differential on the spectral curve everywhere except the diagonal $(x, y) = (x', y')$.

The bidifferential Ω appears naturally when computing the closure of the differential of the tau function and the absence of residue on the diagonal is the key feature that allows proving such closure.

We now remark that –in the same way as in Section 3.1.1– the formal gauge $Y^\gamma(x)(x-\gamma)^{G_\gamma} W^\gamma$ near any irregular singularity coincides with the eigenvector matrix up to high order. More precisely, near $x = \gamma$ ($r_\gamma > 0$) we have

$$A(x) = (Y^\gamma)^{-1} D^\gamma(x) Y^\gamma - (Y^\gamma)^{-1} \frac{\mathrm{d}}{\mathrm{d}x} Y^\gamma \tag{6-13}$$

where $D^\gamma(x)$ is the block diagonal matrix (we momentarily suppress the explicit reference to γ from the labels for notational convenience)

$$\begin{aligned} D(x) &= \begin{pmatrix} -\displaystyle\sum_{J=0}^{rn_1} t_{J,1} \frac{\mathcal{H}_1^J}{n_1(x-\gamma)} & & \\ & \ddots & \\ & & -\displaystyle\sum_{J=0}^{rn_s^\gamma} t_{J,s} \frac{\mathcal{H}_s^J}{n_s(x-\gamma)} \end{pmatrix} + \begin{pmatrix} \dfrac{G_1}{x-\gamma} & & \\ & \ddots & \\ & & \dfrac{G_s}{x-\gamma} \end{pmatrix} = \qquad (6\text{-}14) \\ &= \mathrm{diag}(D_1(x), \ldots, D_s(x)) \qquad (6\text{-}15) \\ & \quad G_j := \mathrm{diag}\left(0, \frac{1}{n_j}, \ldots, \frac{n_j - 1}{n_j}\right) \qquad (6\text{-}16) \end{aligned}$$

Define now the block-diagonal matrices

$$G := \mathrm{diag}(G_1, \ldots, G_s) \tag{6-17}$$

$$W := \mathrm{diag}(W_1, \ldots, W_2) , \quad W_j := [\omega_j^{-(\ell-1)(k-1)}]_{\ell,k} , \quad \omega_j := \mathrm{e}^{\frac{2i\pi}{n_j}} . \tag{6-18}$$

Then

$$
\begin{aligned}
A &= Y x^{-G} W \mathrm{diag}(\hat{D}_1(x), \dots, \hat{D}_s(x)) W^{-1} x^G Y^{-1} + Y' Y^{-1} && \text{(6-19)}\\
\hat{D}_j(x) : &= -\frac{1}{n_j(x-\gamma)} \sum_{K=0}^{r n_j} t_K (x-\gamma)^{\frac{K}{n_j}} {\Omega_j}^K - \frac{(x-\gamma)^{G_j} {W_j}^{-1} G_j W_j (x-\gamma)^{-G_j}}{n_j(x-\gamma)} && \text{(6-20)}\\
\Omega_j : &= \mathrm{diag}(1, \omega_j, {\omega_j}^2, \dots, \omega_j^{n_j-1}) && \text{(6-21)}
\end{aligned}
$$

As in Section 3.1.1 this implies that $Z := Y(x-\gamma)^{-G} W$ coincides with an eigenvector matrix P up to order $(x-\gamma)^{1-r+\epsilon}$ where $\epsilon = \min \left\{ \frac{1}{n_j} \right\}$ and in turn this implies that the deformation matrices are

$$
\begin{aligned}
J \mathcal{T}_{J,j} &= \left(Y \mathrm{diag}(0, \dots, {\mathcal{H}_j}^J, \dots, 0) Y^{-1} \right)_{pp} = \left((x-\gamma)^{-\frac{J}{n_j}} Z \mathrm{diag}(0, \dots, W_j {\Omega_j}^J {W_j}^{-1}, \dots, 0) Z^{-1} \right)_{pp} = \\
&= \left(P \mathrm{diag}(0, \dots, W_j {\Omega_j}^J {W_j}^{-1}, \dots, 0) P^{-1} \right)_{pp} && \text{(6-22)}\\
j &= 1, \dots, s, \quad J = 1, \dots, r\, n_j && \text{(6-23)}\\
\mathcal{C} &= \left(Y\, D\, Y^{-1} \right)_{pp} = A_\gamma(x) && \text{(6-24)}
\end{aligned}
$$

For the deformation of parameters at ∞ the principal part is to be replaced by the polynomial part and then the identity (6-22) is valid only up to the x-independent constant. Let us denote by $y_{j,\sigma}(x)$ the eigenvalues which are asymptotic to

$$
y_{j,\sigma}(x) = -\frac{t_{r n_j, j}}{n_j (x-\gamma)^{r+1}} - {\omega_j}^\sigma \frac{t_{r n_j - 1, j}}{n_j (x-\gamma)^{r+1-\frac{1}{n_j}}} + \dots \ ; \quad \sigma = 0, \dots, n_j - 1 \qquad \text{(6-25)}
$$

Note that these n_j eigenvalues are cyclically permuted by a small loop around the singularity γ on the base curve. Moreover suppose now that there are two such cyclic multiplets with the same order n_j and same leading term $t_{r n_j}$ (which means that the leading term at the singularity of $A(x)$ has two Jordan blocks of the same size and the same eigenvalue); then our genericity assumption (2.1) precisely implies that the next-to-leading coefficients displayed in (6-25) will distinguish the two cyclic groups. We denote accordingly the corresponding spectral projectors

$$
\Pi_{j,\sigma}(x) := \frac{(\widetilde{A - y_{j,\sigma}})(x)}{\mathrm{Tr}(\widetilde{A - y_{j,\sigma}})(x)} \ . \qquad \text{(6-26)}
$$

Then the previous deformation matrices can be written as (restoring the label γ)

$$
\begin{aligned}
\mathcal{T}^\gamma_{J,j} &= \frac{1}{J} \left(\sum_{\sigma=0}^{n_j^\gamma - 1} (x-\gamma)^{-\frac{J}{n_j^\gamma}} {\omega_{\gamma,j}}^{J\sigma} \Pi_{j,\sigma} \right)_{pp} , \quad \omega_{\gamma,j} := \mathrm{e}^{2i\pi/n_j^\gamma} && \text{(6-27)}\\
\mathcal{C}_\gamma &= A_\gamma(x) = (A(x))_{pp} \ . && \text{(6-28)}
\end{aligned}
$$

Note that the expressions here above which may in principle contain fractional powers of the local parameter, in fact do not because of the cyclicity properties.

Equation (6-25) (and an analogue for $x = \infty$) could be rephrased in more geometrical terms by saying that the desingularized spectral curve Σ above each point $x = \gamma$ has s(=number of Jordan cells) distinct points ζ^j_γ which are all branchpoins of order n^γ_j for the map $x : \Sigma \to \mathbb{C}P^1$, for which a local parameter is $q_{\gamma,j} := (x-\gamma)^{1/n_j^\gamma}$ (or $q_{\infty,j} := x^{-1/n_j^\infty}$). With these notations in place we can finally prove

Theorem 6.1 (Tau function for minimally resonant irregular/Fuchsian isomonodromic deformations) *The following differential is closed and hence is the differential of a locally defined function*

$$\mathrm{d}\ln\tau := \sum_{j=0}^{s_\infty}\sum_{J=1}^{rn_j^\infty}\frac{1}{J}\mathrm{d}t^\infty_{J,j}\mathop{\mathrm{res}}_{\zeta^j_\infty} x^{J/n_j^\infty}y\mathrm{d}x+\sum_{\gamma\in\mathcal{D}}\sum_{j=0}^{s_\gamma}\sum_{J=1}^{rn_j^\gamma}\mathrm{d}t^\gamma_{J,j}\frac{1}{J}\mathop{\mathrm{res}}_{\zeta^j_\gamma}\frac{y\mathrm{d}x}{(x-\gamma)^{-J/n_j^\gamma}}+\frac{1}{2}\sum_{\gamma\in\mathcal{D}}\mathop{\mathrm{res}}_{x=\gamma}\mathrm{Tr}(A^2)\mathrm{d}\gamma \quad (6\text{-}29)$$

Here the residues of the differential $y\mathrm{d}x$ *are taken at points of the spectral curve (i.e. choosing the appropriate eigenvalue* $y_{\gamma,\sigma}$*) and are taken with respect to the local parameters by going around the point* $x=\gamma$ *of the base-curve* n_k^γ *times.*

Moreover the isomonodromic times and the parameters of formal monodromy for the irregular singularities are obtained by the residues

$$t^\infty_{J,j} = -\mathop{\mathrm{res}}_{\zeta^j_\infty} x^{-J/n_j^\infty}y\mathrm{d}x\ ,\quad j=1,\dots s_\infty,\ J=1,\dots,n_j^\infty \quad (6\text{-}30)$$

$$t^\gamma_{J,j} = -\mathop{\mathrm{res}}_{\zeta^j_\gamma}(x-\gamma)^{J/n_j^\gamma}y\mathrm{d}x\ ,\quad j=1,\dots,s_\gamma,\ J=1,\dots,n_j^\gamma,\ \infty\neq\gamma\in\mathcal{D} \quad (6\text{-}31)$$

$$t^\gamma_{0,j} = -\mathop{\mathrm{res}}_{\zeta^j_\gamma}y\mathrm{d}x+\frac{n_j^\gamma-1}{2}\ , \quad (6\text{-}32)$$

Proof. The residue-formulæ for the isomonodromic times follow from the expansion of the eigenvalues in the respective local parameter. The parameter of formal monodromy $t_{0,\gamma}$ have a correction which was explained in (3-80) and follows simply from

$$-\mathop{\mathrm{res}}_{\gamma}\mathrm{Tr}D_j^\gamma(x)\mathrm{d}x = -\mathop{\mathrm{res}}_{\zeta^j_\gamma}y\mathrm{d}x = t^\gamma_{0,j}-\frac{n_j^\gamma-1}{2}\ , \quad (6\text{-}33)$$

where $D_{j,\gamma}$ was introduced in (6-15). The proof uses the same arguments adopted earlier but it is only more involved due to the presence of multiple singularities and times. First of all we note that the residues can be pushed down on the base-curve

$$\frac{1}{J}\mathop{\mathrm{res}}_{\zeta^j_\infty}(x-\gamma)^{-J/n_j^\gamma}y\mathrm{d}x = \frac{1}{J}\mathop{\mathrm{res}}_{x=\gamma}\left(\sum_{\ell=0}^{n_j^\gamma}(x-\gamma)^{-J/n_j^\gamma}(\omega_{\gamma,k})^{J\sigma}y_{\gamma,k,\sigma}\right)\mathrm{d}x \quad (6\text{-}34)$$

because the sum in the bracket is a *bona fide* (local) function of x without fractional powers, due to the periodicity of the cyclic n_j^γ-plet of eigenvalues. Here the symbol $y_{\gamma,k,\sigma}$ denotes the σ-member of the k-th multiplet (corresponding to the k-th block in the Jordan-cell decomposition) of eigenvalues near $x=\gamma$. We compute the closure of the differential

$$JK\partial_{t^\gamma_{J,j}}\partial_{t^\mu_{K,k}}\ln\tau = J\mathop{\mathrm{res}}_{x=\mu}\left(\sum_{\sigma=0}^{n_k^\mu-1}(\omega_{\mu,k})^{K\sigma}(x-\mu)^{-\frac{K}{n_k^\mu}}\partial_{t^\gamma_{J,j}}y_{\mu,k,\sigma}\right)\mathrm{d}x =$$

$$= J\mathop{\mathrm{res}}_{x=\gamma}\left(\sum_{\sigma=0}^{n_k^\mu-1}(\omega_{\mu,k})^{K\sigma}(x-\mu)^{-\frac{K}{n_k^\mu}}\mathrm{Tr}\left(\frac{\widetilde{A-y_{\mu,k,\sigma}}}{\mathrm{Tr}(\widetilde{A-y_{\mu,k,\sigma}})}\frac{\partial A}{\partial t^\gamma_{J,j}}\right)\right)\mathrm{d}x =$$

$$= J\mathop{\mathrm{res}}_{x=\gamma}\left(\sum_{\sigma=0}^{n_k^\mu-1}{\omega_{\mu,k}}^{K\sigma}(x-\mu)^{-\frac{K}{n_k^\mu}}\mathrm{Tr}\left(\Pi_{\mu,k,\sigma}(x)\left(\frac{\mathrm{d}\mathcal{T}^\gamma_{J,j}}{\mathrm{d}x}-[A,\mathcal{T}^\gamma_{J,j}]\right)\right)\right)\mathrm{d}x =$$

$$
\begin{aligned}
&= \operatorname*{res}_{x=\mu} \sum_{\sigma=0}^{n_k-1} {\omega_{\mu,k}}^{K\sigma}(x-\mu)^{-\frac{K}{n_k^\mu}} \mathrm{Tr}\left(\Pi_{\mu,k,\sigma}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{\rho=0}^{n_j^\gamma-1}(x-\gamma)^{-\frac{J}{n_j^\gamma}}{\omega_j}^{J\rho}\Pi_{\gamma,j,\rho}\right)_{pp}\right)\mathrm{d}q = \\
&= \operatorname*{res}_{x=\mu}\operatorname*{res}_{z=\gamma}\left(\sum_{\sigma=0}^{n_k^\mu-1}\sum_{\rho=0}^{n_j^\gamma-1}{\omega_{j,\mu}}^{J\rho}{\omega_{k,\mu}}^{K\sigma}(z-\gamma)^{-\frac{J}{n_j^\gamma}}(x-\mu)^{-\frac{K}{n_k^\mu}}\frac{\mathrm{Tr}\left(\Pi_{\mu,k,\sigma}(x)\Pi_{\gamma,j,\rho}(z)\right)}{(z-x)^2}\right)\mathrm{d}z\mathrm{d}x
\end{aligned}
$$

This formula is symmetric because we can exchange the order of the residues; indeed if $\mu \neq \gamma$ the order of residues is certainly irrelevant, whereas if $\mu = \gamma$ we can exchange the order because the residue of the bidifferential vanishes due to Lemma 6.1. In these computations we have assumed that both γ, μ are finite poles, but the proof goes through similarly with minor modification only in the notation and local parameter if one or both coincide with the pole at ∞ and if one or both the deformations are translations of the position of the (finite) pole. **Q.E.D.**

As an immediate corollary of the proof we have

Corollary 6.1 (Hessian of the Tau function) *The second derivatives of the Tau function are expressed in terms of the spectral curve according to the formulæ(for the points at ∞ the formula needs obvious modifications).*

$$
\begin{aligned}
\frac{\partial^2 \ln\tau}{\partial t^\gamma_{J,j}\partial t^\mu_{K,k}} &= \operatorname*{res}_{x=\mu}\operatorname*{res}_{z=\gamma}\left(\sum_{\sigma=0}^{n_k^\mu-1}\sum_{\rho=0}^{n_j^\gamma-1}{\omega_{j,\gamma}}^{J\rho}{\omega_{k,\mu}}^{K\sigma}(z-\gamma)^{-\frac{J}{n_j^\gamma}}(x-\mu)^{-\frac{K}{n_k^\mu}}\frac{\mathrm{Tr}\left(\Pi_{\mu,k,\sigma}(x)\Pi_{\gamma,j,\rho}(z)\right)}{(z-x)^2}\right)\mathrm{d}z\mathrm{d}x \\
\frac{\partial^2 \ln\tau}{\partial t^\gamma_{J,j}\partial \mu} &= \operatorname*{res}_{x=\mu}\operatorname*{res}_{z=\gamma}\left(\sum_{\rho=0}^{n_j^\gamma-1}{\omega_{j,\gamma}}^{J\rho}(z-\gamma)^{-\frac{J}{n_j^\gamma}}\frac{\mathrm{Tr}\left(A(x)\Pi_{\gamma,j,\rho}(z)\right)}{(z-x)^2}\right)\mathrm{d}z\mathrm{d}x \\
\frac{\partial^2 \ln\tau}{\partial\gamma\partial\mu} &= \operatorname*{res}_{x=\mu}\operatorname*{res}_{z=\gamma}\left(\frac{\mathrm{Tr}\left(A(x)A(z)\right)}{(z-x)^2}\right)\mathrm{d}z\mathrm{d}x
\end{aligned}
\tag{6-35}
$$

A An example: Airy-like equations

In this appendix we provide an example of isomonodromic deformation of an irregular singularity which is of almost trivial nature but illustrates the machinery developed earlier. We consider an arbitrary polynomial $V(x)$ of degree $n+1$ and the following scalar ODE (note that the simplest nontrivial case is Airy's equation)

$$V'(\partial_x)f = xf(x) \tag{A-1}$$

$$V(z) = \sum_{j=1}^{n+1}\frac{v_j}{j}z^j \tag{A-2}$$

The matrix first order ODE associated to this equation is

$$\Psi' := \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \frac{x-v_1}{v_{n+1}} & \frac{-v_2}{v_{n+1}} & \cdots & \frac{-v_n}{v_{n+1}} \end{bmatrix}\Psi \tag{A-3}$$

The matrix $A(x)$ is in fact the companion matrix of the polynomial $V'(y)$ and thus the spectral curve is simply the rational curve (genus zero)

$$V'(y) = x \ . \tag{A-4}$$

The only singularity is at $x = \infty$ and it is irregular; the leading coefficient is nilpotent with canonical form $0^2(0)^{d-2}$. In a certain sense this is a *non-example* of our setting because the Lidskii submatrix of the subleading term does not satisfy our genericity assumption. However it is sufficient to perform a single shearing gauge transformation to recast the problem to a non-resonant one and this feature is the only essential ingredient to carry out the analysis as in Sect. 3.1. Following the same steps as in Prop. 3.1 (with the only difference that now we have to perform the inverse shearing), we obtain that there exists a formal solution of the form ($q = x^{\frac{1}{n}}$, $t_{n+1} := n(v_{n+1})^{-1/n}$)

$$\begin{aligned}
\Psi(x) \quad &= \quad (t_{n+1}/n)^{-G} Y(x) \mathrm{e}^{Q(x)} x^{\frac{t_0+G}{n}} = (t_{n+1}/n)^{-G} q^G Z(q) \mathrm{e}^{T(q)} W^{-1} \ , \qquad G := \mathrm{diag}(0,1,\ldots,n-1) \\
& Y(x) := \sum_{j=0}^{\infty} \begin{bmatrix} & & x^{-1} \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}^j z_j \qquad Q(x) := \sum_{j=1}^{n+1} \frac{t_j}{j} \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ x & & & \end{bmatrix}^j =: \sum_{j=1}^{n+1} \frac{t_j}{j} \mathcal{H}^j(x) \\
& Z(q) := \sum_{j=0}^{\infty} q^{-j} \mathcal{C}^{-j} z_j W \ , \quad T(q) := \sum_{j=1}^{n+1} \frac{t_j}{j} q^j \Omega^{-j} + t_0 \ln(q) \\
& \mathcal{C} := \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{bmatrix} \ , \qquad W := [\omega^{-(i-1)(j-1)}]_{i,j} \ , \quad \omega := \mathrm{e}^{\frac{2i\pi}{n}} \\
& z_0 := \mathbf{1} \ , \quad z_j = \text{ diagonal matrices.}
\end{aligned} \tag{A-5}$$

Note that the asymptotic representation is unique and is an explicit function of the parameters t_j because the matrix A is completely determined by them and the diagonal matrices z_j are constructed by a determined recursive procedure along the lines of Prop. 3.1. Of the two asymptotic presentations the one in fractional powers is the usual one in the context of Airy-like equations, whereas the one in terms of integer powers is the one we have used in the present paper. The coefficients t_j are the expansion of the eigenvalue in fractional powers

$$\begin{aligned}
y(x) \quad &= \quad \frac{1}{n} \sum_{j=1}^{n+1} t_j x^{\frac{j}{n}-1} \\
t_j \quad &+ \quad \delta_{j0} \frac{n-1}{2} = -\operatorname*{res}_{x=\infty} x^{-j/n} y \mathrm{d}x \\
&= \quad -\operatorname*{res}_{y=\infty} (V'(y))^{-\frac{j}{n}} y V''(y) \mathrm{d}y = \begin{cases} \dfrac{n}{n-j} \operatorname*{res}_{y=\infty} (V'(y))^{1-\frac{j}{n}} \mathrm{d}y & , j < n \\ t_n = -\dfrac{v_n}{v_{n+1}} \\ t_{n+1} = n(v_{n+1})^{-1/n} \end{cases}
\end{aligned} \tag{A-6}$$

note that $t_0 = \frac{1-n}{2}$ because the curve is of genus zero, indeed

$$t_0 + \frac{n-1}{2} = -\operatorname*{res}_{x=\infty} \mathrm{Tr} D_{bare}(x) \mathrm{d}x = -\operatorname*{res}_{x=\infty} A(x) \mathrm{d}x = 0 \ . \tag{A-7}$$

where D_{bare} is the bare form of the differential equation (see (A-9)); the residues of the traces of D_{bare} and A coincide because they are connected by a formally analytically invertible gauge giving a $\mathcal{O}(x^{-2})$ extra term which does not contribute to the residue.

One can write an explicit integral representation of the solutions in terms of Fourier–Laplace integrals

$$f_k(x) := \int_{\Gamma_k} \mathrm{e}^{xy-V(y)} \ \Rightarrow \ \Psi_{j,k}(x) = \int_{\Gamma_k} y^j \mathrm{e}^{xy-V(y)} = \partial_x^j f_k(x) \tag{A-8}$$

where the contours Γ_k can be chosen in n "homologically" independent ways [4]: in the case $n = 2$ one recognizes the standard integral representation of Airy's functions. The formal asymptotic representation in fractional powers as well as the Stokes matrices can be obtained by the steepest descent method, whereas the integer powers asymptotics cannot. The bare Pfaffian system is given by

$$\left(\partial_x - \sum_{j=1}^{n+1} \frac{t_j}{nx}\mathcal{H}^j - \frac{G}{nx}\right)\Psi_{bare} = 0\ , \qquad \left(\partial_{t_j} - \frac{1}{j}\mathcal{H}^j(x)\right)\Psi_{bare} = 0 \tag{A-9}$$

$$\Psi_{bare} := \mathrm{e}^{Q(x)} x^{\frac{G}{n}} \tag{A-10}$$

A.1 Isomonodromic deformations

In this case there is no monodromy in the usual sense, only Stokes' matrices, and these are the preserved data under the deformation. The parameters of deformations are the t_j or -which is the same after a change of coordinates- the coefficients v_j of the "potential" (this terminology comes from the application to random matrices). However our connection $A(x)$ is not gauge–fixed, but it could be done so by conjugating it by the constant coefficient of the series $Y(x)$ (which is an explicit function of the t_j's).

The deformation equations are easy to describe because we have an explicit solution. Indeed it is immediate from the integral representation (A-8) that

$$\partial_{v_j}\Psi = -\frac{1}{j}\partial_x^j \Psi \tag{A-11}$$

and hence the matrices

$$\mathcal{T}_j := -\frac{1}{j}\partial_x^j \Psi\, \Psi^{-1} \tag{A-12}$$

trivially satisfy the zero curvature conditions; note that they are polynomials of degree at most 1 except for T_{n+1} which is of degree 2. The deformation equations in terms of the parameters t_j can be obtained by using the Jacobian of the change of coordinate from v_j to t_j. Since these deformations are for a connection non gauge-fixed the deformation equations are in the more general form of Thm. 5.3.

A.2 Tau function

Since the spectral curve is rational, the tau function actually coincides with the tau function of the Whitham hierarchy

$$\mathrm{d}\ln\tau = \sum_{j=1}^{n+1} H_j \mathrm{d}t_j \tag{A-13}$$

$$H_j := -\frac{1}{j}\mathop{\mathrm{res}}_{x=\infty} x^{j/n} y\mathrm{d}x = \frac{n}{j(n+j)}\mathop{\mathrm{res}}_{y=\infty} (V'(y))^{1+\frac{j}{n}}\mathrm{d}y \tag{A-14}$$

It is an exercise left to the reader to check that an integral is

$$\ln\tau = \frac{1}{2}\sum_{j=1}^{n-1} t_j H_j \frac{n+1-j}{n+1} \tag{A-15}$$

For the case of (translated/dilated/gauged) Airy's equation ($n = 2$)

$$v_3 f'' + v_2 f' + v_1 f = xf \tag{A-16}$$

the tau function is

$$\ln \tau_{Ai} = -\frac{2{t_1}^3}{3t_3} = \frac{(4v_1v_3 - v_2^2)^3}{768{v_3}^4} \tag{A-17}$$

In all cases $n > 2$ it is always a rational expression in the parameters t_j or v_j. Note that this equation could be transformed to the standard Airy equation by a scalar gauge transformation, a translation and a dilation, which would eliminate all the parameters. The first really nontrivial case is the "hyper"-Airy equation

$$v_4 f''' + v_3 f'' + v_2 f' + v_1 f = xf \ , \tag{A-18}$$

for which

$$\ln \tau_{hAi} = -\frac{3t_2(12{t_1}^2 t_4 + {t_2}^3)}{8{t_4}^2} \ . \tag{A-19}$$

B Proof of Thm. 3.2

In order to prove Thm. 3.2 we need to prove the following Prop. B.1, which in turn relies on two simple Lemmas (B.1 and B.2) of linear algebra.

Proposition B.1 *If the Lidskii pseudovalues of A_{r-1} are generic, then we can find a gauge H linear in x^{-1} such that the first two leading terms of*

$$\tilde{A}(x) = H^{-1}AH - H^{-1}H' \ , \tag{B-1}$$

have the following form

$$\tilde{A}_r = (0^{n_1}) \cdots (0^{n_s}) = A_r \tag{B-2}$$

$$\tilde{A}_{r-1} = \begin{pmatrix} \tilde{A}_{r-1,1} & 0 & \dots & 0 \\ 0 & \tilde{A}_{r-1,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \tilde{A}_{r-1,s} \end{pmatrix} \tag{B-3}$$

where $\tilde{A}_{r-1,i}$ are square matrices of dimensions $n_i \times n_i$ such that the only non-zero entries of $\tilde{A}_{r-1,i}$ are the $\left(\tilde{A}_{r-1,i}\right)_{n_i,1}$, that is, the bottom left hand corner.

Proof. Let $H = H_0 + H_1 x^{-1}$. We first partition H_0 and H_1 into $s \times s$ blocks according to the decomposition of A_r. We will denote these blocks by $H^{(0)}_{jk}$ and $H^{(1)}_{jk}$ respectively. The block $H^{(i)}_{jk}$ is a rectangular matrix of dimension $n_j \times n_k$. We shall similarly partition A_{r-1} into $s \times s$ blocks and call these A_{jk} for simplicity (i.e. we suppress the index $r - 1$).

The coefficients of x^r and x^{r-1} in (B-1) can now be written in the form

$$\begin{aligned} A_r H_0 - H_0 A_r &= 0 \\ A_r H_1 - H_1 A_r &= A_{r-1} H_0 - H_0 \tilde{A}_{r-1} \end{aligned}$$

We can write these equations in block form, in which the jk^{th} block of both sides are given by

$$\begin{aligned}\mathcal{N}_j H^{(0)}_{jk} - H^{(0)}_{jk}\mathcal{N}_k &= 0 \\ \mathcal{N}_j H^{(1)}_{jk} - H^{(1)}_{jk}\mathcal{N}_k &= \sum_{l=1}^{s} A_{jl} H^{(0)}_{lk} - H^{(0)}_{jk}\tilde{A}_{r-1,k}\end{aligned} \tag{B-4}$$

where $\mathcal{N}_j$ is the $n_j \times n_j$ dimensional shift matrix

$$\mathcal{N}_j := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We want to find H_0 and H_1 such that (B-4) is satisfied with $\tilde{A}_{r-1}$ given by (B-2). Let us denote the following linear operator from the space of $n_j \times n_k$ rectangular matrix to the same space by $\mathcal{L}_{jk}$

$$\mathcal{L}_{jk} : \mathbf{Mat}(n_j, n_k) \to \mathbf{Mat}(n_j, n_k) \tag{B-5}$$

$$X \mapsto \mathcal{N}_j X - X\mathcal{N}_k \tag{B-6}$$

The first equation in (B-4) means that $H^{(0)}_{jk}$ is in the kernel of $\mathcal{L}_{jk}$, while right hand side of the second equation in (B-4) has to be in the image of $\mathcal{L}_{jk}$.

The rest of the proof consist in the (tedious) check that the right hand side of eq. (B-4) is guaranteed to lie in the image of the left-hand side (i.e. of $\mathcal{L}_{jk}$) under the genericity assumption in Def. 2.1.

We first need to analyze the structure of the kernel and the image of $\mathcal{L}_{jk}$; this analysis is contained in the following two lemmas, whose proof is left to the reader and follows from a straightforward calculation.

Lemma B.1 *The kernel of $\mathcal{L}_{jk}$ is of dimension $min(n_j, n_k)$ and it is spanned by $n_j \times n_k$ matrices J of the form*

$$J = \begin{pmatrix} 0 & 0 & \dots & J_1 & \dots & J_{n_j-1} & J_{n_j} \\ 0 & 0 & \dots & 0 & J_1 & \dots & J_{n_j-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & J_1 \end{pmatrix} \quad n_j < n_k \tag{B-7}$$

$$J = \begin{pmatrix} J_1 & \dots & J_{n_k-1} & J_{n_k} \\ 0 & J_1 & \dots & J_{n_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & J_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix} \quad n_k < n_j \tag{B-8}$$

where J_i are arbitrary constants. That is, the kernel is spanned by $n_j \times n_k$ matrices with top right hand corner of the form

$$\tilde{J} = \begin{pmatrix} J_1 & \dots & J_{m-1} & J_m \\ 0 & J_1 & \dots & J_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & J_1 \end{pmatrix}$$

and all other entries zero, where $m = min(n_j, n_k)$

The next lemma (left as exercise) characterizes the image of the operator $\mathcal{L}_{jk}$.

Lemma B.2 *The image of $\mathcal{L}_{jk}$ is of codimension $min(n_j, n_k)$ and it is given by $n_j \times n_k$ matrices Q such that*

$$\sum_{i=1}^{p} Q_{n_j-p+i,i} = 0, \quad p = 1, \dots, m \tag{B-9}$$

where $m = min(n_j, n_k)$.

This means that the "dual diagonals" to the ones appearing in the characterization of the kernel of $\mathcal{L}_{jk}$ are "traceless".
We now look at the term $V^{jk} = \sum_{l=1}^{s} A_{jl} H_{lk}^{(0)}$ in (B-4). To simplify the notation, we shall denote the entry in the bottom left corner of A_{jl} by $a(j,l)$ and we write $H_{lk}^{(0)}$ in the form (B-7) denoting the elements on the diagonals by h_i^{jk} and $h_i^{jk} = 0$ if $i > min(n_j, n_k)$ or $i \leq 0$. (i.e., the J_i in (B-7)). Let $m_{jk} = min(n_j, n_k)$ and consider the terms

$$\sum_{i=1}^{p} V_{n_j-p+i,i}^{jk}, \quad p = 1, \dots, m_{jk}$$

in V^{jk}, as these terms determine whether the right hand side of (B-4) is in the image of $\mathcal{L}_{jk}$.

Let $c_{jk} = min(n_j - n_k, 0)$, a simple calculation then shows that

$$\sum_{i=1}^{p} V_{n_j-p+i,i}^{jk} = \sum_{l=1}^{s} a(j,l) h_{c_{jk}+p}^{lk} + R_p^k, \quad p = 1, \dots, m_{jk} \tag{B-10}$$

where R_p^k is a term that involves only h_q^{lk} for $l = 1, \dots, s$ and $q < p + c_{jk}$ and $h_q^{lk} = 0$ if $q \leq 0$. Recall that l_1 is the number of blocks with the same size n_1, l_2 is the number of blocks with the same size n_{l_2}, etc. Let $\tilde{H}^k$ be the column vector with entries

$$\left(\tilde{H}^k\right)_{\sum_{j=1}^{i}(s-w_{j-1})+q} = h_i^{qk}, \quad i = 1, \dots, n_k, \quad q = 1, \dots, w_{i-1}$$

where $w_i = \sum_{q=1}^{p_i} l_q$ and p_i is the biggest number such that $n_{w_i} > i$.

More explicitly, $\tilde{H}^k$ is the column vector

$$\tilde{H}^k = \begin{pmatrix} h_1^{1,k} \\ h_1^{2,k} \\ \vdots \\ h_1^{s,k} \\ \hline h_i^{1,k} \\ \vdots \\ h_i^{w_{i-1},k} \\ \hline \vdots \\ \hline h_{n_k}^{1,k} \\ \vdots \\ h_{n_k}^{w_{n_k-1},k} \end{pmatrix} \tag{B-11}$$

Now let $\tilde{V}^k$ be the following column vector

$$\left(\tilde{V}^k\right)_{\sum_{j=1}^{i}(s-w_{j-1})+q} = \sum_{m=1}^{i} V^{qk}_{n_q-i+m,m}, \quad i=1,\ldots,n_k, \quad q=1,\ldots,w_i$$

That is, $\tilde{V}^k$ is obtained from the vector $\tilde{H}^k$ by replacing h_i^{qk} by the corresponding 'sums of diagonal' in V^{jk}.

$$\tilde{V}^k = \left(\begin{array}{c} V^{1k}_{n_1,1} \\ V^{2k}_{n_2,1} \\ \vdots \\ V^{sk}_{n_s,1} \\ \hline \vdots \\ \hline \sum_{q=1}^{i} V^{1k}_{n_1-i+q,q} \\ \vdots \\ \sum_{q=1}^{i} V^{w_{i-1},k}_{n_{w_i}-i+q,q} \\ \hline \vdots \\ \hline \sum_{q=1}^{n_k} V^{1k}_{n_1-n_k+q,q} \\ \vdots \\ \sum_{q=1}^{n_k} V^{w_{n_{k-1}},k}_{q,q} \end{array}\right) \tag{B-12}$$

If we denote the bottom left hand diagonal by the 'first diagonal', the next one by the 'second diagonal' and so forth, and call the sum of their elements the 'trace of the diagonal', then the i^{th} block in $\tilde{V}^k$ consists of the traces of the i^{th} diagonals.

We can now write the equations (B-10) for $j=1,\ldots,s$ in matrix form

$$U^k\tilde{H}^k = \tilde{V}^k \tag{B-13}$$

From (B-10), we see that the matrix U^k is of the form

$$U^k = \begin{pmatrix} U_1^k & 0 & \ldots & 0 \\ \star & U_2^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \ldots & \star & U^k_{n_k} \end{pmatrix}$$

where $\star$ denotes some non-zero entries. The matrices U_i^k are given by

$$U_i^k = \left[\begin{array}{ccc|c} a(1,1) & \ldots & a(1,w_{n_k-1}) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a(w_{n_k-1},1) & \ldots & a(w_{n_k-1},w_{n_k-1}) & 0 \\ \hline 0 & \ldots & 0 & 0 \end{array}\right]$$

that is, U_i^k is the w_{i-1}^2 matrix that has the principal block submatrix of the Lidskii matrix of size $w^2_{n_k-1}$ on its top left hand corner and zero elsewhere.

We now consider the term $H^{(0)}_{jk}\tilde{A}_{r-1,k}$ in (B-4). Let λ_i be the bottom left hand entry of $\tilde{A}_{r-1,i}$, that is, its only non-zero entry. We can now write this term in matrix multiplication form as in (B-13)

$$\tilde{U}^k\tilde{H}^k = \tilde{V}^k \tag{B-14}$$

where $\tilde{U}^k$ is the following matrix

$$\tilde{U}^k = \begin{pmatrix} \tilde{U}_1^k & 0 & \dots & 0 \\ \star & \tilde{U}_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \dots & \star & \tilde{U}_{n_k}^k \end{pmatrix}$$

where $\tilde{U}_i^k$ is the diagonal matrix given by

$$\begin{aligned} \left(\tilde{U}_i^k\right)_{jj} &= 0, j \leq w_{n_k+1} \\ \left(\tilde{U}_i^k\right)_{jj} &= \lambda_k, j > w_{n_k+1} \end{aligned}$$

The right hand side of (B-4) can then be written in the matrix multiplication form

$$\left(U^k - \tilde{U}^k\right)\tilde{H}^k = \tilde{V}^k, \quad k = 1, \dots, s$$

By Lemma B.2, the condition that the right hand side of (B-4) is in the image of the operator $\mathcal{L}_{jk}$ means that the determinant of $U^k - \tilde{U}^k$ has to be zero. This determinant is given by

$$\det\left(U^k - \tilde{U}^k\right) = \prod_{i=1}^{n_k} \det\left(U_i^k - \tilde{U}_i^k\right)$$

The first non-zero entry of $\tilde{U}_i^k$ is on the $w_{n_k-1}+1$ row while the first zero entry of U_i^k is on the w_{n_k} row, therefore, $U_i^k - \tilde{U}_i^k$ is given by

$$U_i^k - \tilde{U}_i^k = \left[\begin{array}{cc|ccc|c} a(11) & \dots & a(1, w_{n_k}+1) & \dots & a(1, w_{n_k-1}) & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \hline a(w_{n_k}+1, 1) & \dots & a(w_{n_k}+1, w_{n_k}+1) - \lambda_k & \dots & a(w_{n_k}+1, w_{n_k-1}) & 0 \\ \dots & \ddots & \dots & \ddots & \dots & \vdots \\ a(w_{n_k-1}, 1) & \dots & a(w_{n_k-1}, w_{n_k}+1) & \dots & a(w_{n_k-1}, w_{n_k-1}) - \lambda_k & 0 \\ \hline 0 & \dots & 0 & \dots & 0 & -\lambda_k I \end{array}\right]$$

where I is the identity matrix of dimension $(w_{i-1} - w_{n_k-1})^2$.

We see that the determinants $\det\left(U_i^k - \tilde{U}_i^k\right)$ are just products of λ_k and the pseudo-characteristic polynomials of the Lidskii submatrices. Therefore the genericity assumption in Def. 2.1 would imply that the determinant of $U^k - \tilde{U}^k$ is zero and that the right hand side of (B-4) lies in the image of $\mathcal{L}_{jk}$. **Q.E.D.**

Proof of Thm. 3.2.
The proof is a corollary of Prop. B.1. We can assume that the genericity condition holds and that A_r and A_{r-1} are given by (B-2) in Prop. B.1.

Let Y be a power series such that

$$Y(x) = Y_0 + \sum_{j=1}^{\infty} x^{-j} Y_j \ , \quad \det(Y_0) \neq 0 \ ,$$

and

$$B(x) = Y^{-1}AY - Y^{-1}Y' \ , \tag{B-15}$$

for some Laurent series B in x. The coefficient of x^{r-i} in (B-15) then gives

$$A_r Y_i - Y_i A_r = A_{r-1}Y_{i-1} - Y_{i-1}A_{r-1} + \sum_{l=2}^{i} A_{r-l}Y_{i-l} - Y_{i-l}B_{r-l} - (r-i)Y_{i-r-1}$$

where we have set for convenience $Y_k \equiv 0$ if $k < 0$.

Assuming that B is in block diagonal form, we will show that (B-15) is solvable. We first write these equations into block form, as in the proof of proposition B.1, in which the (j,k) blocks of both sides are given by

$$\begin{aligned}\mathcal{N}_j Y_{jk}^{(i)} - Y_{jk}^{(i)}\mathcal{N}_k &= A_{r-1,j}Y_{jk}^{(i-1)} - Y_{jk}^{(i-1)}A_{r-1,k} + \left(\sum_{l=2}^{i} A_{r-l}Y_{i-l} - (r-i)Y_{i-r-1}\right)_{jk} \\ &- \sum_{l=2}^{i} Y_{jk}^{(i-l)}B_{kk}^{(r-l)} \end{aligned} \tag{B-16}$$

where X_{jk} denotes the jk^{th} block of X. Provided that the right hand side is in the image of $\mathcal{L}_{jk}$, (B-16) determines $Y_{jk}^{(i)}$ up to the addition of an element in the kernel of $\mathcal{L}_{jk}$. We now are going to show that this arbitrariness in the solution of the i^{th} equation can be used to guarantee the solvability of the $(i+1)^{th}$ equation, namely in such a way that the right hand side of (B-16) lies in the image of $\mathcal{L}_{jk}$ in the $(i+1)^{th}$ equation.

Let J be a matrix in the kernel of $\mathcal{L}_{jk}$ as in (B-7), and $A_{r-1,j}$ be the $n_j \times n_j$ matrix

$$A_{r-1,j} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_j & \dots & 0 & 0 \end{pmatrix}$$

where $\lambda_j \neq 0$. When the genericity condition is satisfied, we also have $\lambda_j \neq \lambda_k$ if $n_j = n_k$, $j \neq k$.

Recall that the bottom left hand diagonals are called the “first diagonal”, the next one 'second diagonal' and so forth, and the sums of their elements are called the “traces of the diagonal”. A simple calculation shows that the trace of the l^{th} diagonal of $A_{r-1,j}Y_{jk}^{(i-1)} - Y_{jk}^{(i-1)}A_{r-1,k}$ is

$$\lambda_j J_{c(k,j)+l} - \lambda_k J_{c(j,k)+l}$$

where $c(l,m) = min(n_l - n_m, 0)$ and $J_k = 0$ if $k \leq 0$. Note that if the blocks have the same size then $c(l,m) = 0$, but then $\lambda_j \neq \lambda_k$ due to our genericity assumption: if they have different sizes then either $c(l,m)$ or $c(m,l)$ is strictly negative, and hence the system for the J_k's is triangular with determinant a power of λ_j (or λ_k, depending on which one is the “long side”). In any case we see that the traces of the first m_{jk} diagonals can be chosen to be any value by a suitable choice of J, therefore one can always choose $Y_{jk}^{(i-1)}$ such that the right hand side of (B-16) lies in the image of $\mathcal{L}_{jk}$. Hence (B-15) is solvable with a block-diagonal B. **Q.E.D.**

C Proof of Lemma 3.2

This lemma is the counterpart of Prop. 2.1 in the setting of formal gauge equivalence.

We start by remarking that the "periodicity" properties of Z and T' are compatible with the periodicity properties of B

$$nB = nZT'Z^{-1} + Z'Z^{-1} \tag{C-1}$$

$$\omega T'(\omega q) = \mathcal{C}T'(q)\mathcal{C}^{-1} \; . \tag{C-2}$$

The differential equation is conveniently rewritten as

$$\frac{1}{n} Z' = BZ - ZT' \ . \tag{C-3}$$

Let us expand Z in (formal) Laurent series

$$Z(q) = \sum_{j=0}^{\infty} q^{-j} Z_j \ , \quad Z_0 := W \ . \tag{C-4}$$

The periodicity of Z implies the following structure for the coefficients of the power series expansion

$$Z_j = \mathcal{C}^j z_j W \ , \quad z_j = \text{ diagonal matrices.} \tag{C-5}$$

We first claim that any such series can be factorized as follows

$$\begin{aligned}
\sum_{j=0}^{\infty} q^{-j} \mathcal{C}^j z_j W &= \sum_{\ell=0}^{\infty} q^{-\ell} \mathcal{C}^\ell p_\ell W \times \sum_{k=0}^{\infty} q^{-k} \gamma_k \Omega^{-k} = & \text{(C-6)} \\
&= \sum_{\ell=0}^{\infty} q^{-\ell} \mathcal{C}^\ell p_\ell W \times \exp\left(\sum_{k=0}^{\infty} q^{-k} \delta_k \Omega^{-k} \right) & \text{(C-7)} \\
\mathrm{Tr}(p_\ell) &= 0 \ , \quad \gamma_0 = 1 \ , \quad p_0 = \mathbf{1} \ , \quad \gamma_j = \text{ scalars} \ . & \text{(C-8)}
\end{aligned}$$

Indeed, by comparing the two sides term-wise in the expansion one finds the recursion relations

$$\begin{aligned}
n\gamma_j &= \mathrm{Tr}(z_j) \ , & \text{(C-9)} \\
p_j &= z_j - \gamma_j \mathbf{1} - \sum_{k=1}^{j-1} p_k \gamma_{j-k} \ . & \text{(C-10)}
\end{aligned}$$

Once the coefficients γ_j are known the coefficients δ_k in the exponential of last expression can be similarly defined in a recursive and unique fashion.
If we plug this factorized expression in the equation and compare the terms of the same power we obtain for the coefficient of q^{K-1}

$$\sum_{j=0}^{K} \left(B_{R-j} \mathcal{C}^{K-j} p_{K-j} W - t_{R-j} \mathcal{C}^{K-j} p_{K-j} W \Omega^{-j-1} \right) = 0 \ , \quad K = 0, \ldots R. \tag{C-11}$$

We recognize that this recurrence relation (for $K \leq R$) is exactly the same appearing in the proof of Prop. 2.1, which implies that $t_R = \lambda_1$ in eq. (3-17). Since $B_R = t_R \mathcal{C}$ and $\mathcal{C}^{-j} W = W \Omega^j$ we obtain an equation for p_K

$$p_K - \mathcal{C}^{-1} p_K \mathcal{C} = \frac{1}{t_R} \sum_{j=1}^{K} \left(t_{R-j} \mathcal{C}^{-j-1} p_{K-j} \mathcal{C}^{j+1} - \mathcal{C}^{-K-1} B_{R-j} \mathcal{C}^{K-j} p_{K-j} \right) \ . \tag{C-12}$$

This equation fixes the diagonal matrix p_K up to addition of a multiple of the identity (i.e. modulo the trace), but since p_K is traceless, this equation suffices in fixing it unambiguously. Moreover this equation also determines t_{R-K} from the fact that the LHS is by default a traceless matrix. Hence

$$t_{R-K} = \frac{1}{n} \mathrm{Tr} \left(\mathcal{C}^{-K-1} B_{R-K} + \sum_{j=1}^{K-1} \left(\mathcal{C}^{-K-1} B_{R-j} \mathcal{C}^{K-j} p_{K-j} \right) \right) \ . \tag{C-13}$$

This also shows that $t_{R-K} = \lambda_{K+1}$ (as per eq. (3-17)) since the above is the same recurrence relation that defines λ_K, $K \leq R$.

For the coefficients of q^{-K-1}, $K \geq 1$ we have instead the following relations (we set $t_{-j} \equiv 0$ for convenience in writing the following formula)

$$\sum_{j=-K}^{R} \left(t_j \mathcal{C}^{K+j} p_{K+j} \mathcal{C}^{-j} W - B_j \mathcal{C}^{K+j} p_{K+j} W \right) = \tag{C-14}$$

$$= \frac{K}{n} \mathcal{C}^K p_K W + \frac{1}{n} \sum_{\ell=0}^{K-1} (\ell - K) \delta_{\ell-K} \mathcal{C}^\ell p_\ell \mathcal{C}^{K-\ell} W \tag{C-15}$$

Multiplying both sides on the left by $\mathcal{C}^{-K}$ and on the right by W^{-1} we obtain

$$t_R p_{K+R} - t_R \mathcal{C}^{-1} p_{K+R} \mathcal{C} = -\frac{K}{n} p_K + \sum_{j=-K}^{R-1} \left(t_j \mathcal{C}^j p_{K+j} \mathcal{C}^{-j} - \mathcal{C}^{-K} B_j \mathcal{C}^{K+j} p_{K+j} \right) + \tag{C-16}$$

$$-\frac{1}{n} \sum_{\ell=0}^{K-1} (\ell - K) \delta_{K-\ell} \mathcal{C}^\ell p_\ell \mathcal{C}^{K-\ell} W \tag{C-17}$$

This equation determines p_{K+R} (modulo trace as discussed above). Moreover tracing both sides gives an equation for δ_K

$$K \delta_K = -\mathrm{Tr} \left(\sum_{j=-K}^{R-1} \mathcal{C}^{-K} B_j \mathcal{C}^{K+j} p_{K+j} \right) \ , \ K \geq 1 \ . \tag{C-18}$$

This determines the series Z uniquely. **Q.E.D.**

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