The partition function of the two-matrix model as an isomonodromic tau-function

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Abstract

We consider the Itzykson-Zuber-Eynard-Mehta two-matrix model and prove that the partition function is an isomonodromic tau function in a sense that generalizes Jimbo-Miwa-Ueno's [20]. In order to achieve the generalization we need to define a notion of tau-function for isomonodromic systems where the *ad*–regularity of the leading coefficient is not a necessary requirement.

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1   Introduction

Random matrices models have been studied for years and have generated important results in many fields of both theoretical physics and mathematics. The two-matrix model

$$\mathrm{d}\mu(M_1, M_2) = \mathrm{e}^{-\mathrm{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)}\mathrm{d}M_1\mathrm{d}M_2 \tag{1.1}$$

was used to model $2D$ quantum gravity [14] and was investigated from a more mathematical point of view in [26, 17, 4, 5, 7, 12, 8]; the *partition function* of the model

$$\mathcal{Z}_N(V_1, V_2) = \int\int \mathrm{d}\mu(M_1, M_2) \tag{1.2}$$

has important properties in the large N–limit for the enumeration of discrete maps on surfaces [15] of arbitrary genus and it is also known to be a tau-function for the 2–Toda hierarchy. In the case of the Witten conjecture, proved by Kontsevich [24] with the use of matrix integrals not too dissimilar from the above one, the enumerative properties of the tau function imply some nonlinear (hierarchy of) PDEs (the KdV hierarchy for the mentioned example). On a similar level, one expects some hierarchy of PDEs for the case of the two-matrix model and possibly some Painlevé property (namely the absence of movable essential singularities). The Painlevé property is characteristic of tau-functions for isomonodromic families of ODEs that depend on parameters; hence a way of establishing such property for the partition function $\mathcal{Z}_N$ is that of identifying it with an instance of isomonodromic tau function [20, 21].

   This is precisely the purpose of this article; we capitalize on previous work that showed how to relate the matrix model to certain biorthogonal polynomials [26, 17] and how these appear in a natural fashion as the solution of certain isomonodromic family [9] .

   The paper extends to the case of the two matrix model the work contained in [9, 12, 10]; it uses, however, a different approach, closer to the recent [6].

   In [9, 12, 10, 19] the partition function of the one–matrix model (and certain shifted Töplitz determinants) were identified as isomonodromic tau functions by using *spectral residue formulæ* in terms of the spectral curve of the differential equation. Such spectral curve has interesting properties inasmuch as –in the one-matrix case– the spectral invariants can be related to the expectation values of the matrix model. Recently the spectral curve of the two matrix model [8] has been written explicitly in terms of expectation values of the two–matrix model and hence one could use their result and follow a similar path for the proof as the one followed in [12]. Whichever one of the two approaches one chooses, a main obstacle is that the definition of

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isomonodromic tau function [20, 21] relies on a genericity assumption for the ODE which fails in the case at hand, thus requiring a generalization in the definition.

According to this logic, one of the purposes of this paper is to extend the notion of tau-function introduced by Jimbo-Miwa-Ueno's [20], to the two-matrix Itzykson-Zuber model. This task is accomplished in a rather general framework in Sec. 3.

We then show that the partition function has a very precise relationship with the tau-function so introduced, allowing us to (essentially) identify it as an isomonodromic tau function (Thm. 3.4). The proof is organized as follow: in section 2 we recall a Riemann-Hilbert formulation of the problem due to [25]. In section 2.2 we transform it into a RHP with constant jumps, thus guaranteeing the existence of an ODE. As the asymptotic at infinity does not enter into the class of Jimbo-Miwa theorem for isomonodromic tau function, we generalize their result to our setting in section 3. Using Schlesinger transformations, namely the shift in the size of the matrix model from N to $N + 1$, we eventually show that the partition function equals the new defined isomonodromic tau function in section 3.2.

2 A Riemann Hilbert formulation of the two-matrix model

According to the seminal work [26, 17] and following the notations and definitions introduced in [4, 5], we consider paired sequences of monic polynomials $\{\pi_m(x), \sigma_m(y)\}_{m=0\dots\infty}$ ($m = \deg \pi_m = \deg \sigma_m$), that are biorthogonal in the sense that

$$\iint_\varkappa dx dy \pi_m(x)\sigma_n(y)e^{-V_1(x)-V_2(y)+xy} = h_m \delta_{mn}, \quad h_m \neq 0. \tag{2.1}$$

The functions $V_1(x), V_2(y)$ appearing here are referred to as *potentials*, terminology drawn from random matrix theory, in which such quantities play a fundamental role.

Henceforth, the second potential $V_2(y)$ will be chosen as a polynomial of degree $d_2 + 1$

$$V_2(y) = \sum_{J=1}^{d_2+1} \frac{v_J}{J} y^J, \quad v_{d_2+1} \neq \tag{2.2}$$

For the purposes of most of the considerations to follow, the first potential $V_1(x)$ may have very general analyticity properties as long as the manipulations make sense, but for definiteness and clarity we choose it to be polynomial as well:

$$V_1(x) = \sum_{K=1}^{d_1+1} \frac{u_K}{K} x^K, \quad u_{d_1+1} \neq 0 \tag{2.3}$$

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The symbol $\int\int_\varkappa$ stands for any linear combination of integrals of the form

$$\iint_\varkappa dxdy := \sum_j \sum_k \varkappa_{jk} \int_{\Gamma_j} dx \int_{\hat{\Gamma}_k} dy, \qquad \varkappa_{ij} \in \mathbb{C} \tag{2.4}$$

where the contours $\{\hat{\Gamma}_k\}_{k=1\ldots d_2}$ will be chosen as follows. In the y–plane, define $d_2 + 1$ "wedge sectors" $\{\hat{S}_k\}_{k=0\ldots d_2}$ such that $\hat{S}_k$ is bounded by the pairs of rays: $r_k := \{y|\arg y = \theta + \frac{2k\pi}{d_2+1}\}$ and $r_{k-1} := \{y|\arg y = \theta + \frac{2(k-1)\pi}{d_2+1}\}$, where $\theta := \arg v_{d_2+1}$. Then $\hat{\Gamma}_k$ is any smooth oriented contour within the sector $\hat{S}_k$ starting from ∞ asymptotic to the ray r_k (or any ray within the sector that is at an angle $< \frac{\pi}{2(2d_2+1)}$ to it, which is equivalent for purposes of integration), and returning to ∞ asymptotically along r_{k-1} (or at an angle $< \frac{\pi}{2(2d_2+1)}$ to it). These will be referred to as the "wedge contours". We also define a set of smooth oriented contours $\{\check{\Gamma}_k\}_{k=1,\ldots d_2}$, that have intersection matrix $\check{\Gamma}_j \cap \hat{\Gamma}_k = \delta_{jk}$ with the $\hat{\Gamma}_k$'s, such that $\check{\Gamma}_k$ starts from ∞ in sector $\hat{S}_0$, asymptotic to the ray $\check{r}_0 := \{y|\arg(y) = \theta - \frac{\pi}{d_2+1}\}$ and returns to ∞ in sector $\hat{S}_k$ asymptotically along the ray $\check{r}_k := \{y|\arg(y) = \theta + \frac{2(k-\frac{1}{2})}{d_2+1}\}$. These will be called the "anti-wedge" contours. (See Fig. 1.) The choice of these contours is determined by the requirement that all moment integrals of the form

$$\int_{\hat{\Gamma}_k} y^j e^{-V_2(y)+xy}dy, \quad \int_{\check{\Gamma}_k} y^k e^{V_2(y)-xy}dy, \quad k = 1,\ldots d_2, \quad j \in \mathbb{N} \tag{2.5}$$

be uniformly convergent in $x \in \mathbb{C}$. In the case when the other potential $V_1(x)$ is also a polynomial, of degree $d_1 + 1$, the contours $\{\Gamma_k\}_{k=1,\ldots d_1}$ in the x–plane may be defined similarly.

The "partition function" is defined here to be the multiple integral

$$\mathcal{Z}_N := \frac{1}{N!} \iint_{\varkappa^N} \prod_{j=1}^N dx_j dy_j \Delta(X)\Delta(Y) \prod_{j=1}^N e^{-V_1(x_j)-V_2(y_j)+x_j y_j} \tag{2.6}$$

where $\Delta(X)$ and $\Delta(Y)$ denote the usual Vandermonde determinants and the factor $\frac{1}{N!}$ is chosen for convenience.

Such multiple integral can also be represented as the following determinant

$$\mathcal{Z}_N = \det[\mu_{ij}]_{0\le i,j\le N-1}, \quad \mu_{ij} := \int_\varkappa x^i y^j e^{-V_1(x)-V_2(y)+xy}dxdy \tag{2.7}$$

The denomination of "partition function" comes from the fact [26, 17, 9] that when $\varkappa$ coincides with $\mathbb{R} \times \mathbb{R}$ then $\mathcal{Z}_N$ coincides (up to a normalization for the volume of the unitary group) with the following matrix integral

$$\iint dM_1 dM_2 e^{-\mathrm{tr}(V_1(M_1)+V_2(M_2)-M_1 M_2)} \tag{2.8}$$

extended over the space of Hermitean matrices M_1, M_2 of size $N \times N$, namely the normalization factor for the measure $d\mu(M_1, M_2)$ introduced in 1.1.
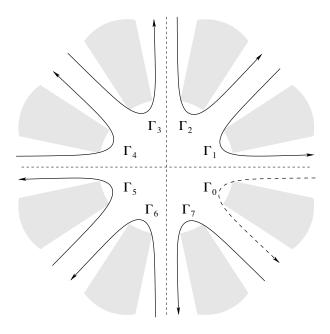
Figure 1: Wedge and anti-wedge contours for $V_2(y)$ of degree $D_2 + 1$

2.1 Riemann–Hilbert characterization for the orthogonal polynomials

A Riemann–Hilbert characterization of the biorthogonal polynomials is a crucial step towards implementing a steepest–descent analysis. In our context it is also crucial in order to tie the random matrix side to the theory of isomonodromic deformations.

We first recall the approach given by Kuijlaars and McLaughin (referred to as KM in the rest of the article) in [25], suitably extended and adapted (in a rather trivial way) to the setting and notation of the present work. We quote -paraphrasing and with a minor generalization- their theorem, without proof.

Theorem 2.1 (Kuijlaars and McLaughin asymptotic [25]). *The monic bi-orthogonal polynomial $\pi_n(x)$ is the $(1,1)$ entry of the solution $\Gamma(x)$ (if it exists) of the following Riemann-Hilbert problem for $\Gamma(x)$.*

1. The matrix $\Gamma(x)$ is piecewise analytic in $\mathbb{C} \setminus \bigsqcup \Gamma_j$;

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2. *the (non-tangential) boundary values of $\Gamma(x)$ satisfy the relations*

$$\Gamma(x)_+ = \Gamma(x)_- \begin{bmatrix} 1 & w_{j,1} & \cdots & w_{j,d_2} \\ & 1 & 0 & 0 \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad x \in \Gamma_j \qquad (2.9)$$

$$w_{j,\nu} = w_{j,\nu}(x) := e^{-V_1(x)} \sum_{k=1}^{d_2} \varkappa_{jk} \int_{\hat{\Gamma}_k} y^{\nu-1} e^{-V_2(y)+xy} dy \qquad (2.10)$$

3. *as $x \to \infty$ we have the following asymptotic expansion*

$$\Gamma(x) \sim \left(I_d + \frac{Y_{N,1}}{x} + \mathcal{O}\left(\frac{1}{x^2}\right) \right) \begin{pmatrix} x^N & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \qquad (2.11)$$

where we have defined the integers m_N, r_N as follows

$$N = m_N d_2 + r_N, \quad m_N, r_N \in \mathbb{N}, \quad 0 \le r_n \le d_2 - 1 \qquad (2.12)$$

It follows from [25] that the solution $\Gamma_N(x)$ has the following form

$$\Gamma_N(x) := \Gamma(x) := \begin{bmatrix} \pi_N(x) & \mathcal{C}_0(\pi_N) & \cdots & \mathcal{C}_{d_2-1}(\pi_N) \\ p_{N-1}(x) & \mathcal{C}_0(p_{N-1}) & \cdots & \mathcal{C}_{d_2-1}(p_{N-1}) \\ \vdots & & & \vdots \\ p_{N-d_2}(x) & \mathcal{C}_0(p_{N-d_2}) & \cdots & \mathcal{C}_{d_2-1}(p_{N-d_2}) \end{bmatrix}, \qquad (2.13)$$

$$\mathcal{C}_i(f(z)) := \frac{1}{2\pi i} \iint_\varkappa \frac{f(x)}{x-z} y^i e^{-V_1(x)-V_2(y)+xy} dy dx \qquad (2.14)$$

where the polynomials denoted above by $p_{N-1}, \ldots, p_{N-d_2}$ are some polynomials of degree not exceeding $N-1$, whose detailed properties are largely irrelevant for our discussion; we refer to [25] for these details.

By a left multiplication of this solution by a suitable constant matrix (i.e. some suitable linear combination of the last rows) we can see that the matrix

$$\widehat{\Gamma}_N := \begin{bmatrix} \pi_n & \mathcal{C}_0(\pi_n) & \cdots & \mathcal{C}_{d_2-1}(\pi_n) \\ \pi_{n-1} & \mathcal{C}_0(\pi_{n-1}) & \cdots & \mathcal{C}_{d_2-1}(\pi_{n-1}) \\ \vdots & & & \vdots \\ \pi_{n-d_2} & \mathcal{C}_0(\pi_{n-d_2}) & \cdots & \mathcal{C}_{d_2-1}(\pi_{n-d_2}) \end{bmatrix} \qquad (2.15)$$

and Γ_N are related as

$$\widehat{\Gamma}_N(x) = U_N \Gamma_N(x) \qquad (2.16)$$

where U_N is a constant matrix (depending on N and on the coefficients of the polynomials but not on x). As an immediate consequence, $\widehat{\Gamma}_N$ solves the same RHP as Γ with the exception of the normalization at infinity (2.11). The matrix U_N will not play a major role in the following proof but we mention it for completeness.

The RHP featured in Thm. 2.1 is not immediately suitable to make the connection to the theory of isomonodromic deformations as described in [20, 21]; we recall that this is the theory that describes the deformations of an ODE in the complex plane which leave the Stokes' matrices (i.e. the so–called *extended monodromy data*) invariant. The solution Γ_N (or $\widehat{\Gamma}_N$) does not solve any ODE as formulated, because the jumps on the contours are non constant. If -however- we can relate Γ_N with some other RHP with constant jumps, then its solution can be immediately shown to satisfy a polynomial ODE, which allows us to use the machinery of [20, 21]. This is the purpose of the next section.

2.2 A RHP with constant jumps

In [9] the biorthogonal polynomials were characterised in terms of an ODE or –which is the same– of a RHP with constant jumps. In order to connect the two formulations we will use some results contained in [13] and we start by defining some auxiliary quantities: for $1 \leq k \leq d_2$, define the d_2 sequences of functions $\{\psi_m^{(k)}(x)\}_{m\in\mathbb{N}}$ as follows:

$$\psi_m^{(k)}(x) := \frac{1}{2\pi i} \int_{\check{\Gamma}_k} ds \iint_{\varkappa} dzdw \frac{\pi_m(z)e^{-V_1(z)}}{x-z} \frac{V_2'(s) - V_2'(w)}{s-w} e^{-V_2(w)+V_2(s)+zw-xs}, \quad 1 \leq k \leq d_2,$$
(2.17)

and let

$$\psi_m^{(0)}(x) := \pi_m(x)e^{-V_1(x)}.$$
(2.18)

In terms of these define, for $N \geq d_2$, the sequence of $(d_2 + 1) \times (d_2 + 1)$ matrix valued functions $\widehat{\underset{N}{\Psi}}(x)$

$$\widehat{\underset{N}{\Psi}}(x) := \begin{bmatrix} \psi_N^{(0)}(x) & \dots & \psi_N^{(d_2)}(x) \\ \vdots & & \vdots \\ \psi_{N-d_2}^{(0)}(x) & \dots & \psi_{N-d_2}^{(d_2)}(x) \end{bmatrix}$$
(2.19)

For convenience (the notation with a hat describes only that we have removed the normalization matrix U_N) we define also

$$\underset{N}{\Psi} := U_N^{-1} \widehat{\underset{N}{\Psi}}$$
(2.20)

The relationship with the matrices Γ_N, $\widehat{\Gamma}_N$ introduced in the previous section is detailed in the following crucial theorem:

Theorem 2.2 (Factorization theorem)**.** *The following identities hold*

$$\widehat{\underset{N}{\Psi}}(x) = \widehat{\Gamma}_N(x)\Psi_0(x) \ , \quad \underset{N}{\Psi}(x) = \Gamma_N(x)\Psi_0(x)$$
$$\Psi_0(x) := V(x)W(x) \tag{2.21}$$

where

$$V := \begin{pmatrix} e^{-V_1(x)} & 0 \\ 0 & V_0, \end{pmatrix} \ , \qquad W(x) := \begin{pmatrix} 1 & 0 \\ 0 & W_0(x) \end{pmatrix} \tag{2.22}$$

and V_0, $W_0(x)$ *are the* $d_2 \times d_2$ *matrices with elements*

$$(V_0)_{jk} = \begin{bmatrix} v_2 & v_3 & \cdots & & v_{d_2+1} \\ v_3 & & & v_{d_2+1} & \\ & & \cdot\cdot\cdot & & \\ v_{d_2} & v_{d_2+1} & & & \\ v_{d_2+1} & & & & \end{bmatrix} = \tag{2.23}$$

$$= \begin{array}{cc} v_{j+k} & \text{if } j+k \le d_2+1 \\ 0 & \text{if } j+k > d_2+1, \end{array} \tag{2.24}$$

$$(W_0(x))_{jk} = \int_{\check{\Gamma}_k} y^{j-1} e^{V_2(y)-xy} dy, \quad 1 \le j,k \le d_2 \tag{2.25}$$

Proof The proof is contained in appendix A (based on [13, 3]) and is a direct verification based on matrix multiplication, noticing that the matrix V_0 is nothing but the matrix representation of $\frac{V_2'(y)-V_2'(s)}{y-s}$ as a quadratic form in the bases $1, y, y^2, \ldots, y^{d_2-1}$ and $1, s, s^2, \ldots, s^{d_2-1}$. **Q.E.D.**

The point of the matter now is that this new matrix $\underset{N}{\Psi}$ (or $\widehat{\underset{N}{\Psi}}$) solves a Riemann–Hilbert problem with *constant* jumps; this is due to the right multiplication of the factor $\Psi_0 := V(x)W(x)$. To this end we state the following theorem

Theorem 2.3. *The matrix* $\underset{N}{\Psi}$ *is the unique solution of the following RHP:*

1. *Constant Jumps:*

$$\underset{N}{\Psi}_+(x) = \underset{N}{\Psi}_-(x)\mathbf{H}^{(j)} \ , \qquad x \in \Gamma_j \tag{2.26}$$

$$\tag{2.27}$$

where

$$\mathbf{H}^{(j)} := \mathbf{I} - 2\pi i \mathbf{e}_0 \kappa^T$$

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$$\hat{\mathbf{H}}^{(j)} = (\mathbf{H}^{(j)})^{-1} = \mathbf{I} + 2\pi i \mathbf{e}_0 \kappa^T$$

$$\mathbf{e}_0 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \kappa := \begin{pmatrix} 0 \\ \varkappa_{j1} \\ \vdots \\ \varkappa_{jd_2} \end{pmatrix} \tag{2.28}$$

2. *Asymptotic behavior at infinity:*

$$\underset{N}{\Psi}(x) \sim \Gamma_N \begin{pmatrix} x^N e^{-V_1(x)} & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \Psi_0(x) \tag{2.29}$$

where

$$\Gamma_N = Id + \frac{Y_{N,1}}{x} + ... \tag{2.30}$$

and where $\Psi_0(x) := V(x)W(x)$ will be referred to as the bare *solution. Its asymptotic at infinity can be computed by steepest descent, but since it is N–independent, for the sake of brevity, we do not report on it (details are contained in [5, 13]).*

3. $\Psi'_N(x)\Psi_N^{-1} = D_N(x)$ *where $D_N(x)$ is a polynomial in x of degree d_1.*

4. $\partial_{u_K}\Psi_N(x)\Psi_N^{-1} = U_{K,N}(x)$ *is polynomial in x of degree K.*

5. $\partial_{v_J}\Psi_N(x)\Psi_N^{-1} = V_{J,N}(x)$ *is polynomial in x.*

6. $\det(\Psi_{N+1}\Psi_N^{-1}) \equiv C$ *with C a constant independent of x.*

(Sketch of) Proof. In order to prove point (1) we have to directly consider the definition of $\underset{N}{\widehat{\Psi}}$ (noticing that $\underset{N}{\Psi}$ has the same jumps since they differ only by a *left* multiplier). Using Sokhotski–Plemelji formulae and the results of Appendix B one can quickly obtain the proof.

Point (2) is a consequence of the factorization Thm. 2.2 and the asymptotic representation of Γ_N given in Thm. 2.1.

The uniqueness of the solution is proved by showing that $\det \underset{N}{\Psi}$ is constant in x; this follows from the fact that the determinant has no jumps due to the unimodularity of the jump matrices, the fact that $\det \Psi_0$ is constant and at infinity $\det \underset{N}{\Psi}$ tends to a constant. Thus by Liouville's theorem $\det \underset{N}{\Psi}$ must be constant. At this point, the argument goes, any other solution $\underset{N}{\widetilde{\Psi}}$ must be invertible as well; the ratio $\underset{N}{\Psi}\underset{N}{\widetilde{\Psi}}^{-1}$ has no jump, thus being entire, and tends to the identity at infinity. Once more, Liouville's theorem assures that then the two matrices are identically equal. The points (3,4,5,6) in the above theorem can be found in [8, 5] and the proofs will not be repeated here in detail. For example point 3 can be proven by noticing that since Ψ_N has

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constant jumps, the expression $\Psi'_N \Psi_N^{-1}$ has no jumps and is hence entire. The fact that it is in fact a polynomial of degree d_1 follows from the asymptotic behavior at infinity. In a similar fashion one can prove the other two points. **Q.E.D.**

The main upshot of points (3,4,5) is that Ψ_N is the solution of an isomonodromic deformation system: namely, we may think of the equations as deformation equations which leave the (generalized) monodromy of the ODE in (3) invariant, as defined in [20, 21]. We can thus apply the ideas of said papers to our situation, after the suitable generalizations.

In the next section we shall define a proper notion of isomonodromic tau function: it should be pointed out that the definition of [20, 21] cannot be applied as such because –as showed in [5]– the ODE that the matrix $\underset{N}{\Psi}$ (or $\underset{N}{\widehat{\Psi}}$) solves, has a highly degenerate leading coefficient at the singularity at infinity.

In the list, the crucial ingredients are the differential equations (in x or relatively to the parameters u_K and v_J). First, the fact that $D_N(x)$ is a polynomial comes from explicit computation (See [8] for example). The result concerning the determinant of $R_N(x)$ can also be found in [8] where one has: $\det(\Psi_{N+1}\Psi_N^{-1}) = \det(a_N(x)) = Cste$. The properties concerning the differential equations relatively to parameters can be found in [8] too. Under all these assumptions, we will show that the proof of Jimbo-Miwa-Ueno can be adapted and that we can define a suitable τ-function in the same way Jimbo-Miwa-Ueno did it.

3   Definition of the τ-function

In this section, we will place ourselves in a more general context than the one described above; we will show that under few assumptions one can define a good notion of tau-function.

More generally we will denote with t_a the isomonodromic parameters or "isomonodromic times" (in our case they are the u_K's and the v_J's) and a subscript a or b is understood as a derivation relatively to t_a or t_b. For a function f of the isomonodromic times we will denote by the usual symbol its differential

$$\mathrm{d}f = \sum_a \partial_{t_a} f \mathrm{d}t_a = \sum_a f_a \mathrm{d}t_a \tag{3.1}$$

Our setup falls in the following framework that it is useful to ascertain from the specifics of the case at hands. Suppose we are given a matrix whose behavior at infinity is:

$$\Psi(x) \sim Y(x)\,\Xi(x)\ ,\ \ Y(x) := \left(\mathbf{1} + \frac{Y_1}{x} + \frac{Y_2}{x^2} + \dots\right)x^S \tag{3.2}$$

where $\Xi(x) = \Xi(x; \mathbf{t})$ is some explicit expression (the "bare" isomonodromic solution) and S is a matrix independent of the isomonodromic times. Note that this asymptotic falls exactly in

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the one we get for $\Psi_N(x)$). This implies that if we define the one–form-valued matrix $\mathcal{H}(x; \mathbf{t})$ by

$$\mathcal{H}(x; \mathbf{t}) = \mathrm{d}\Xi(x; \mathbf{t})\, \Xi(x; \mathbf{t})^{-1} \tag{3.3}$$

then $\mathcal{H}(x) = \sum \mathcal{H}_a \mathrm{d}t_a$ (we suppress explicit mention of the $\mathbf{t}$ dependence henceforth) is some solution of the zero-curvature equations:

$$\partial_a \mathcal{H}_b - \partial_b \mathcal{H}_a = [\mathcal{H}_a, \mathcal{H}_b] \tag{3.4}$$

This result is a direct computation using $\mathcal{H}_a = (\partial_a \Xi)\Xi^{-1}$ and $\partial_b(\Xi^{-1}) = -\Xi^{-1}(\partial_b \Xi)\Xi^{-1}$. We will **assume** (which is the case in our setting) that all $\mathcal{H}_a$ are **polynomials** in x. We will also use that the dressed deformations Ω_a given by $\Psi_a = \Omega_a \Psi$ are polynomials, which is the case in our setup thanks to properties (4, 5) of Thm. 2.3. Moreover, according to the asymptotic they are given by:

$$\Omega_a = (Y\mathcal{H}_a Y^{-1})_{pol}. \tag{3.5}$$

In this very general (and generic) setting we can formulate the definition of a "tau function" as follows

Definition 3.1. *The tau-differential is the one-form defined by*

$$\omega := \sum \omega_a \mathrm{d}t^a := \sum_a \mathrm{res}\,\mathrm{tr}\left(Y^{-1}Y'\mathcal{H}_a\right) \mathrm{d}t^a \tag{3.6}$$

The main point of the matter is that -without any further detail- we can now prove that the tau-differential is in fact closed and hence locally defines a function.

Theorem 3.1. *The tau-differential is a closed differential and locally defines a τ–function as*

$$\mathrm{d}\log\tau = \omega \tag{3.7}$$

Proof. We need to prove the closure of the differential. We first recall the main relations between the bare and dressed deformations

$$\partial_a Y = \Omega_a Y - Y\mathcal{H}_a \;\; ; \qquad Y\mathcal{H}_a Y^{-1} = \Omega_a - \mathcal{R}_a \; ; \qquad \mathcal{R}_a := \partial_a Y Y^{-1} \tag{3.8}$$

We note that -by construction- $\Omega_a = (Y\mathcal{H}_a Y^{-1})_{pol}$ is a polynomial while $\mathcal{R}_a = \mathcal{O}(x^{-1})$ *irrespectively of the form of S.* We compute the cross derivatives directly

$$
\begin{aligned}
\partial_a \omega_b &= \mathrm{res}\mathrm{Tr}\Bigg(-Y^{-1}\left(\Omega_a Y - Y\mathcal{H}_a\right)Y^{-1}Y'\mathcal{H}_b + Y^{-1}\left(\Omega_a Y - Y\mathcal{H}_a\right)' \mathcal{H}_b + Y^{-1}Y'\partial_a \mathcal{H}_b\Bigg) \\
&= \mathrm{res}\mathrm{Tr}\Bigg(\mathcal{H}_a Y^{-1}Y'\mathcal{H}_b + Y^{-1}\Omega_a' Y\mathcal{H}_b - Y^{-1}Y'\mathcal{H}_a\mathcal{H}_b - \mathcal{H}_a'\mathcal{H}_b + Y^{-1}Y'\partial_a H_b\Bigg)
\end{aligned}
$$

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$$= \operatorname{resTr}\left(Y^{-1}Y'\left([\mathcal{H}_b,\mathcal{H}_a] + \partial_a\mathcal{H}_b\right) + Y^{-1}\Omega_a'Y\mathcal{H}_b - \overbrace{\mathcal{H}_a'\mathcal{H}_b}^{\text{polynomial}}\right)$$

$$= \operatorname{resTr}\left(Y^{-1}Y'\left([\mathcal{H}_b,\mathcal{H}_a] + \partial_a\mathcal{H}_b\right) - \Omega_a'\mathcal{R}_b\right) \tag{3.9}$$

where, in the last step, we have used that $Y\mathcal{H}_bY^{-1} = \Omega_b - \mathcal{R}_b$ and that the contribution coming from Ω_b vanishes since it is a polynomial. Rewriting the same with $a \leftrightarrow b$ and subtracting we obtain

$$\partial_a\omega_b - \partial_b\omega_a = \operatorname{resTr}\left(2Y^{-1}Y'[\mathcal{H}_b,\mathcal{H}_a] - \Omega_a'\mathcal{R}_b + \Omega_b'\mathcal{R}_a + Y^{-1}Y'\left(\partial_a\mathcal{H}_b - \partial_b\mathcal{H}_a\right)\right)$$

$$= \operatorname{resTr}\left(Y^{-1}Y'[\mathcal{H}_b,\mathcal{H}_a] - \Omega_a'\mathcal{R}_b + \Omega_b'\mathcal{R}_a + Y^{-1}Y'\left(\overbrace{\partial_a\mathcal{H}_b - \partial_b\mathcal{H}_a + [\mathcal{H}_b,\mathcal{H}_a]}^{=0 \text{ by the ZCC } 3.4}\right)\right)$$

$$= \operatorname{resTr}\left(Y^{-1}Y'[\mathcal{H}_b,\mathcal{H}_a] - \Omega_a'\mathcal{R}_b + \Omega_b'\mathcal{R}_a\right) \tag{3.10}$$

Note that, up to this point, we only used the zero curvature equations for the connection $\nabla = \sum(\partial_a - \mathcal{H}_a)dt^a$ and the fact that $\mathcal{H}_a$ are polynomials in x. We thus need to prove that the last quantity in (3.10) vanishes: this follows from the following computation, which uses once more the fact that $\mathcal{H}_a$ and Ω_a are all polynomials. Indeed, we have $\operatorname{resTr}(\mathcal{H}_a'\mathcal{H}_b) = 0$ and hence (using (3.8))

$$0 = \operatorname{res tr}(\mathcal{H}_a'\mathcal{H}_b) = \operatorname{resTr}\left(\left(Y\mathcal{H}_aY^{-1}\right)'Y\mathcal{H}_bY^{-1}\right) - \operatorname{resTr}\left(Y'\mathcal{H}_a\mathcal{H}_bY^{-1}\right) + \operatorname{resTr}\left(\mathcal{H}_aY^{-1}Y'\mathcal{H}_b\right)$$

$$= \operatorname{resTr}\left(\left(\Omega_a - \mathcal{R}_a\right)'\left(\Omega_b - \mathcal{R}_b\right)\right) + \operatorname{resTr}\left(Y^{-1}Y'\left[\mathcal{H}_b,\mathcal{H}_a\right]\right)$$

$$= \operatorname{resTr}\left(\overbrace{\Omega_a'\Omega_b}^{\text{poly}} - \mathcal{R}_a'\Omega_b - \Omega_a'\mathcal{R}_b + \overbrace{\mathcal{R}_a'\mathcal{R}_b}^{=\mathcal{O}(x^{-2})} + Y^{-1}Y'[\mathcal{H}_b,\mathcal{H}_a]\right)$$

$$= \operatorname{resTr}\left(-\mathcal{R}_a'\Omega_b - \Omega_a'\mathcal{R}_b + Y^{-1}Y'[\mathcal{H}_b,\mathcal{H}_a]\right) = 0 \tag{3.11}$$

Using integration by parts (and cyclicity of the trace) on the first term here above, we obtain precisely the last quantity in (3.10). The Theorem is proved. **Q.E.D.**

3.1 Application to our problem

We now apply the general definition above to our setting, with the identifications $\Psi = \Psi_N$, $Y = \Gamma_N$ (as a formal power series at ∞) and $\Xi = \Psi_0$. We will write Y_N instead of Γ_N in the expressions below to emphasize that we consider its asymptotic expansion at ∞. In order to apply the generalized Def. 3.1 to our setting we identify

- Γ_N of Thm 2.3 with Y of eq. (3.2)

- the times t_a used in section 3 with the coefficients v_j and u_K of the two potentials (2.3, 2.2).

- The "bare isomonodromic solution" $\Xi(x; \mathbf{t})$ of Section 3 with the bare solution $\Psi_0(x; \vec{u}, \vec{v})$ appearing in Thm. 2.3.

This reduces the definition of the tau function to the one below

Definition 3.2. *The τ-function is defined by the following PDE*

$$d(\log \tau_N) = \operatorname*{Res}_{x \to \infty} \operatorname{Tr} \left(Y_N^{-1} Y_N' d(\Psi_0) \Psi_0^{-1} \right) \tag{3.12}$$

where Y_N is the formal asymptotic expansion of Γ_N at infinity

$$Y_N = \widetilde{Y}_N \begin{pmatrix} x^N & 0 & 0 \\ 0 & x^{-m_N-1} Id_{r_N} & 0 \\ 0 & 0 & x^{-m_N} Id_{d_2-r_N} \end{pmatrix} \tag{3.13}$$

Remark 3.1. *The matrix S of the previous section (eq. (3.2)) in our case becomes:*

$$S = S_N = \begin{pmatrix} N & 0 & 0 \\ 0 & (-m_N-1) Id_{r_N} & 0 \\ 0 & 0 & -m_N Id_{d_2-r_N} \end{pmatrix} \tag{3.14}$$

as follows by inspection of eq. (3.13).

The partial derivatives of $\ln \tau_N$ with respect to the times u_K, v_J split into two sets which have different form:

$$\partial_{u_K} \log \tau_N = - \operatorname*{Res}_{x \to \infty} \operatorname{Tr} \left(Y_N^{-1} Y_N' \frac{x^K}{K} \mathbf{E_{11}} \right) \tag{3.15}$$

$$\partial_{v_J} \log \tau_N \;\; = \;\; \operatorname*{Res}_{x \to \infty} \operatorname{Tr} \left(Y_N^{-1} Y_N' \partial_{v_J} (\Psi_0) \Psi_0^{-1} \right) \tag{3.16}$$

These two equations are the specializations of Def. 3.1 using that $\Xi(x; \mathbf{t}) = \Psi_0(x; \vec{u}, \vec{v})$ and thus $\mathcal{H}(x) = \mathrm{d}\Xi\Xi^{-1}$ takes on two forms depending on which of the times we are differentiating with. The block form of the derivatives $\partial_{u_K} \Psi_0 \Psi_0^{-1}$ or $\partial_{v_J} \Psi_0 \Psi_0^{-1}$ follows simply by the block-form of Ψ_0 (in Thm. 2.2), noticing that the dependence on the u_K's is only in the $(1,1)$ entry $((\Psi_0)_{11} = \mathrm{e}^{-V_1(x)})$, while the dependence on the v_J's is only in the $d_2 \times d_2$ lower-right block.

One can notice that the situation we are looking at is a generalization of what happen in the one-matrix case. In the 1-matrix model, the matrix S is zero and therefore Y_N are (formal) Laurent series. The matrix Ψ_0 matrix is absent in that case since there is only one potential and thus one recovers the usual definition of isomonodromic tau function (see [12]). Note also that in

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the derivation with respect to v_J we have obtained the second equality using the block diagonal structure of Ψ_0 (first row/column does not play a role). It is remarkable that the two systems are completely decoupled, i.e. that in the first one the matrix Ψ_0 (containing all the dependence in V_2) disappears and that in the second one the matrix A_0 (containing the potential V_1) also disappears. Now that we have defined a proper generalization of isomonodromic tau functions corresponding to our setting, we still need to make the link with the partition function.

3.2 Discrete Schlesinger transformation: Tau-function quotient

In this section we investigate the relationship between the tau-function of Def. 3.2 and the partition function $\mathcal{Z}_N$ of the matrix model.

We anticipate that the two object turn out to be the same (up to a nonzero factor that will be explicitly computed, Thm. 3.4): the proof relies on two steps, the first of which we prepare in this section. These are

- proving that they satisfy the same recurrence relation in N

- identifying the initial conditions for the recurrence relation.

We start by investigating the relationship between τ_N and τ_{N+1}; this analysis is essentially identical to the theory developed in [21] and used in [6], but we report it here for the convenience of the reader.

From the fact that the Ψ_N has constant jumps, we deduce that $\Psi_{N+1}\Psi_N^{-1}$ is an entire function. Moreover asymptotically it looks like:

$$
\begin{aligned}
\Psi_{N+1}\Psi_N^{-1} &= \tilde{Y}_{N+1}
\begin{pmatrix}
x^{N+1}e^{-V_1(x)} & 0 & 0 \\
0 & x^{-m_{N+1}-1}Id_{r_{N+1}} & 0 \\
0 & 0 & x^{-m_{N+1}}Id_{d_2-r_{N+1}}
\end{pmatrix}
\Psi_0(x) \\
&\quad \Psi_0(x)^{-1}
\begin{pmatrix}
x^{-N}e^{V_1(x)} & 0 & 0 \\
0 & x^{m_N+1}Id_{r_N} & 0 \\
0 & 0 & x^{m_N}Id_{d_2-r_N}
\end{pmatrix}
\tilde{Y}_N
\end{aligned}
\tag{3.17}
$$

$$
\Psi_{N+1}\Psi_N^{-1} = \tilde{Y}_{N+1}
\begin{pmatrix}
x & 0 & 0 & 0 \\
0 & Id_{r_N-1} & 0 & 0 \\
0 & 0 & x^{-1} & 0 \\
0 & 0 & 0 & Id_{d_2-1-r_N}
\end{pmatrix}
\tilde{Y}_N
\tag{3.18}
$$

Thus, remembering that $\tilde{Y}_N$ is a series x^{-1}, Liouville's theorem states that $\Psi_{N+1}\Psi_N^{-1}$ is a polynomial of degree one, and hence, for some constant matrices R_N^0, R_N^1 we must have

$$
\Psi_{N+1}\Psi_N^{-1} = R_N(x) = R_N^0 + xR_N^1
\tag{3.19}
$$

From the fact that $\det(R_N)$ does not depend on x (last property Thm. 2.2), we know that $R_N^{-1}(x)$ is a polynomial of degree at most one as well (this is easy if one consider the expression of the inverse of a matrix using the co-matrix).

Comparing the asymptotics of Ψ_{N+1} and $R_N(x)\Psi_N$ term-by-term in the expansion in inverse powers of x and after some elementary algebra one obtains ([20] Appendix A):

$$R_N(x) = E_{\alpha_0} x + R_{N,0} \qquad \text{and} \qquad R_N^{-1}(x) = E_1 x + R_{N,0}^{-1} \tag{3.20}$$

Here we have introduced the notation $\alpha_0 = r_N + 1$ which corresponds to the index of the column where the coefficient x^{-1} is to be found in the asymptotic of $\Psi_{N+1}\Psi_N^{-1}$. The notation E_j denotes the matrix with 1 in position j, j and 0 elsewhere. These notations are the standard notation used originally by Jimbo-Miwa in a Schlesinger transformation. The matrix $(R_{N,0})_{\alpha,\beta}$ is given by:

	$\beta = \alpha_0$	$\beta = 1$	$\beta \neq \alpha_0, 1$
$\alpha = \alpha_0$	$\frac{-(Y_{N,2})_{\alpha_0,1} + \sum_{\gamma \neq \alpha_0}(Y_{N,1})_{\alpha_0,\gamma}(Y_{N,1})_{\gamma,1}}{(Y_{N,1})_{\alpha_0,1}}$	$-(Y_{N,1})_{\alpha_0,1}$	$-(Y_{N,1})_{\alpha_0,\beta}$
$\alpha = 1$	$\frac{1}{(Y_{N,1})_{\alpha_0,1}}$	0	0
$\alpha \neq \alpha_0, 1$	$-\frac{(Y_{N,1})_{\alpha,1}}{(Y_{N,1})_{\alpha_0,1}}$	0	$\delta_{\alpha,\beta}$

$$\tag{3.21}$$

and $(R_{N,0}^{-1})_{\alpha,\beta}$ is given by:

	$\beta = \alpha_0$	$\beta = 1$	$\beta \neq \alpha_0, 1$
$\alpha = \alpha_0$	0	$(Y_{N,1})_{\alpha_0,1}$	0
$\alpha = 1$	$-\frac{1}{(Y_{N,1})_{\alpha_0,1}}$	$-\frac{(Y_{N,2})_{\alpha_0,1}}{(Y_{N,1})_{\alpha_0,1}} + (Y_{N,1})_{1,1}$	$-\frac{(Y_{N,1})_{\alpha_0,\beta}}{(Y_{N,1})_{\alpha_0,1}}$
$\alpha \neq \alpha_0, 1$	0	$(Y_{N,1})_{\alpha,1}$	$\delta_{\alpha,\beta}$

$$\tag{3.22}$$

While the formulae above might seem complicated, we will use the two important observations (that can be obtained easily with the definitions without the explicit formulae given above):

$$E_{\alpha_0} R_{N,0}^{-1} + R_{N,0} E_1 = R_{N,0}^{-1} E_{\alpha_0} + E_1 R_{N,0} = 0 \tag{3.23}$$

$$R_N^{-1}(x) R_N'(x) = R_{N,0}^{-1} E_{\alpha_0} \text{ does not depend on } x.$$

The recurrence relation satisfied by the sequence $\{\tau_N\}$ is derived in the next theorem.

Theorem 3.2. *Up to multiplication by functions that do not depend on the isomonodromic parameters (i.e. independent of the potentials V_1, V_2) the following identity holds*

$$\frac{\tau_{N+1}}{\tau_N} = (Y_1)_{1,\alpha_0} \tag{3.24}$$

Proof The proof follows [21] but we report it here for convenience of the reader. Consider the following identity

$$\Psi_{N+1} = Y_{N+1}\Psi_0 = R_N Y_N \Psi_0 \tag{3.25}$$

This implies that

$$Y_{N+1} = R_N Y_N \tag{3.26}$$

Taking the derivative with respect to x gives:

$$Y_{N+1}^{-1} Y_{N+1}' = Y_N^{-1} R_N^{-1} R_N' Y_N + Y_N^{-1} Y_N' \tag{3.27}$$

Therefore we have:

$$
\begin{aligned}
d\log\tau_{N+1} - d\log\tau_N &= \operatorname*{Res}_{x\to\infty} \operatorname{Tr}((Y_N^{-1} R_N^{-1} R_N' Y_N + Y_N^{-1} Y_N' - Y_N^{-1} Y_N)d(\Psi_0)\Psi_0^{-1}) \\
&= \operatorname*{Res}_{x\to\infty} \operatorname{Tr}(Y_N^{-1} R_N^{-1} R_N' Y_N d(\Psi_0)\Psi_0^{-1})
\end{aligned}
\tag{3.28}
$$

We now need to "transfer" the exterior derivative from Ψ_0 to Y_N. This can be done using that $\Psi_N = Y_N \Psi_0$, so that

$$d\Psi_N = d(Y_N)\Psi_0 + Y_N d(\Psi_0)$$

Equivalently:

$$Y_N d\Psi_0 \Psi_0^{-1} Y_N^{-1} = d(\Psi_N)\Psi_N^{-1} - dY_N Y_N^{-1} \tag{3.29}$$

Inserting these identities in the tau quotient we obtain the relation

$$d\log\tau_{N+1} - d\log\tau_N = \operatorname*{Res}_{x\to\infty} \operatorname{Tr}\left(R_N^{-1}R_N' d(\Psi_N)\Psi_N^{-1} - R_N^{-1}R_N' dY_N Y_N^{-1}\right) \tag{3.30}$$

The first term is residueless at ∞ since $d\Psi_N \Psi_N^{-1}$ is polynomial in x and $R_N^{-1}R_N'$ does not depend on x. Therefore we are left only with:

$$d\log\tau_{N+1} - d\log\tau_N = -\operatorname*{Res}_{x\to\infty} \operatorname{Tr}(R_N^{-1}R_N' dY_N Y_N^{-1}) \tag{3.31}$$

A direct matrix computation using the explicit form of R_N yields

$$d\log\tau_{N+1} - d\log\tau_N = d\log((Y_{N,1})_{1,\alpha_0}) \tag{3.32}$$

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and hence

$$\frac{\tau_{N+1}}{\tau_N} = (Y_1)_{1,\alpha_0} \tag{3.33}$$

The last equality is to be understood up to a multiplicative constant not depending on the parameters u_K and v_J in τ. **Q.E.D.**

In order to complete the first step we need to express the entry $(Y_1)_{1,\alpha_0}$ in terms of the ratio of two consecutive partition functions. This is accomplished in the following section.

Theorem 3.3. *For the matrix Γ_N the asymptotic expansion at infinity (2.11) is such that*

$$(Y_{N,1})_{1,\alpha_0} = (v_{d_2+1})^{S_N} h_N = (v_{d_2+1})^{S_N} \frac{\mathcal{Z}_{N+1}}{\mathcal{Z}_N} \tag{3.34}$$

where S_N and $\alpha_0 \in \{0, 1, \ldots, d_2 - 1\}$ are defined by the following relation

$$N = d_2 S_N + \alpha_0 - 1 \tag{3.35}$$

Proof In order to compute $(Y_{N,1})_{1,\alpha_0}$ it is sufficient to compute the leading term of the expansion at ∞ appearing in the first row of the matrix Γ_N. Recalling the expression (2.13), we start by the following direct computation using integration by parts

$$\iint_\kappa dzdw\, \pi_N(z) z^i w^{k-1} e^{-V_1(z)-V_2(z)+zw} = \iint_\kappa dzdw\, \pi_N(z) e^{-V_1(z)} w^{k-1} e^{-V_2(w)} \frac{d^i}{dw^i}(e^{zw})$$

$$= (-1)^i \iint_\kappa dzdw\, \pi_N(z) e^{-V_1(z)+zw} \frac{d^i}{dw^i}\left(w^{k-1} e^{-V_2(w)}\right)$$

$$= \iint_\kappa dzdw\, \pi_N(z) q_{d_2 i+k-1}(w) e^{-V_1(z)-V_2(z)+zw} \tag{3.36}$$

where $q_{d_2 i+k-1}(w)$ is a polynomial of the indicated degree whose leading coefficient is $v_{d_2+1}^i$. The last RHS is 0 if $d_2 i + k - 1 < N$ because of orthogonality. If $d_2 i + k - 1 = N$ the integral gives $v_{d_2+1}^i h_N$ by the normality conditions concerning our biorthogonal set. This computation allows us to expand the Cauchy transform of $(\Gamma_N)_{1,\alpha_0}$ near ∞ as follows:

$$\begin{aligned}
\mathcal{C}(p_N w_{\alpha_0}(x)) &= & \frac{1}{2\pi i} \iint_\kappa dzdw \frac{\pi_N(z)}{z-w} w^{\alpha_0-1} e^{-V_1(z)-V_2(z)+zw} \\
&= & -\sum_{i=0}^{S_N-1} \frac{1}{2\pi i} \iint_\kappa dzdw\, \pi_N(z) \frac{z^i}{x^{i+1}} w^{\alpha_0-1} e^{-V_1(z)-V_2(z)+zw} \\
& + & \frac{1}{2\pi i} \frac{1}{x^{S_N+1}} \iint_\kappa dzdw \frac{\pi_N(z)}{x-z} z^{S_N} w^{\alpha_0-1} e^{-V_1(z)-V_2(z)+zw} + \mathcal{O}(x^{-S_N-2}) \\
& & (3.37)
\end{aligned}$$

By orthogonality the first sum vanishes term-by-term and the leading coefficient of the second term is $v_{d_2+1}^{S_N} h_N$. **Q.E.D.**

We are finally in the position of formulating and proving the main theorem of the paper.

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Theorem 3.4. *The isomonodromic τ-function and the partition function are related by:*

$$\forall N \in \mathbb{N} : \mathcal{Z}_N = (v_{d_2+1})^{\sum_{j=0}^{N-1} S_j} \tau_N \tag{3.38}$$

where we recall that S_j is given by the decomposition of $j + 1$ in the Euclidian division by d_2:
$S_j = E\left[\frac{j+1}{d_2}\right]$. *A short computation of the power in v_{d_2+1} gives:*

$$\forall N \in \mathbb{N} : \mathcal{Z}_N = (v_{d_2+1})^{d_2 \frac{\alpha_N(\alpha_N-1)}{2} + \alpha_N(N-\alpha_N d_2)} \tau_N \tag{3.39}$$

where $\alpha_N = E\left[\frac{N}{d_2}\right]$

Proof. Recall that the τ-function is only defined up to a multiplicative constant not depending on N nor on the coefficients u_k and v_j. With this in mind and combining Thm. 3.2 and Thm. 3.3 we find that

$$\frac{\tau_{N+1}}{\tau_N} \overset{\text{Thm. } 3.2}{=} (Y_1)_{1,\alpha_0} \overset{\text{Thm. } 3.3}{=} (v_{d_2+1})^{S_N} \frac{\mathcal{Z}_{N+1}}{\mathcal{Z}_N} \tag{3.40}$$

where $N = d_2 S_N + \alpha_0 - 1$. Hence for every n_0:

$$\tau_N \mathcal{Z}_{n_0} = \mathcal{Z}_N \tau_{n_0} (v_{d_2+1})^{\sum_{j=n_0}^{N-1} S_j} \tag{3.41}$$

One would like to take $n_0 = 0$ because it enables explicit computations. As we will prove now there is a way of extending naturally all the reasoning down to 0. Indeed, the RHP for Γ_N (Thm. 2.1) is perfectly well–defined for $N = 0$ and has solution

$$\Gamma_0 = \begin{pmatrix} 1 & \mathcal{C}_0(1) & \mathcal{C}_1(1) & \dots & \mathcal{C}_{d_2-1}(1) \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \ddots & 1 \end{pmatrix}. \tag{3.42}$$

Consequently we can take

$$\tau_N \mathcal{Z}_0 = (v_{d_2+1})^{\sum_{j=0}^{N-1} S_j} \mathcal{Z}_N \tau_0 \tag{3.43}$$

Also note that $\mathcal{Z}_0 \equiv 1$ (by definition).

We can compute τ_0 directly from Def. 3.2 because of the particularly simple and explicit expression of $\Psi_0 = \Gamma_0 \Psi_0$.

$$\mathrm{d}\ln\tau_0 = \mathrm{res}\mathrm{Tr}\left(Y_0^{-1} Y_0' \mathrm{d}\Psi_0 \Psi^{-1}\right) \tag{3.44}$$

We claim that this expression is identically zero (and hence we can define $\tau_0 \equiv 1$); indeed,

$$Y_0^{-1}Y_0' = \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.45}$$

and

$$\mathrm{d}\Psi_0(x)\Psi_0^{-1}(x) = \begin{pmatrix} \star & 0 & \dots & 0 \\ 0 & \star & \dots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \dots & \star \end{pmatrix} \tag{3.46}$$

so that the trace of the product is always zero (even before taking the residue). **Q.E.D.**

The presence of the power in v_{d_2+1} is due to a bad normalisation of the partition function itself ($\mathcal{Z}_N$) and can be easily canceled out by taking $v_{d_2+1} = 1$ from the start (it is just a normalization of the weight function). This finding is consistent with the work of Bergere and Eynard [2] since all their results concerning the partition function and its derivatives with respect to parameters have special cases for u_{d_1+1} and v_{d_2+1}. It also signals the fact that the RHP is badly defined when $v_{d_2+1} = 0$ because the contour integrals involved diverge and the whole setup breaks down. Indeed if $v_{d_2+1} = 0$ this simply means that V_2 is a polynomial of lower degree and thus the RHP that we should set up should be of smaller size from the outset.

4   Outlook

In this article, we have restricted ourselves to contours going from infinity to infinity. This allows us to use integration by parts without picking up any boundary term. A natural extension of this work could be to see what happens when contours end in the complex plane, and especially study what happens when the end points moves (models with hard edges). This generalization is important in the computation of the gap probabilities of the Dyson model [27], which correspond to a random matrix model with Gaussian potentials but with the integration restricted to intervals of the real axis.

Acknowledgments

A Factorization of Ψ_N

Starting from the definition of the last d_2 columns of $\widehat{\underset{N}{\Psi}}$ (2.19) we observe that

$$
\begin{aligned}
\psi_m^{(k)}(x) \quad &:= \frac{1}{2i\pi} \int_{\check{\Gamma}_k} ds \iint_{\varkappa} \frac{\pi_m(z)}{x-z} \frac{V_2'(s) - V_2'(w)}{s-w} \mathrm{e}^{-V_1(z)-V_2(w)+V_2(s)+zw-xs} dwdz \quad \text{(A.1)}\\
&= \sum_{p,q} v_{q+p} \frac{1}{2i\pi} \iint_{\varkappa} \frac{\pi_m(z)}{x-z} w^{p-1} \mathrm{e}^{-V_1(z)-V_2(w)+zw} \int_{\check{\Gamma}_k} ds\, s^{q-1} \mathrm{e}^{V_2(s)-xs}\\
&= \sum_{p,q} (\widehat{\Gamma}_N)_{m,p} (V_0)_{p,q} (W_0)_{q,k} = (\widehat{\Gamma}_N\, V_0 W_0)_{m,k} \quad \text{(A.2)}
\end{aligned}
$$

This proves Thm. 2.2.

B Bilinear concomitant as intersection number

We recall very briefly the result of [3] stating that

$$
\frac{V_2'(\partial_x) - V_2'(-\partial_z)}{\partial_x + \partial_z} w(x) f(z) \bigg|_{z=x} = \int_\Gamma \int_{\check{\Gamma}} \frac{V_2'(\eta) - V_2'(s)}{\eta - s} \mathrm{e}^{x(\eta-s)-V_2(\eta)+V_2(s)} = 2i\pi\Gamma\#\check{\Gamma} = \text{constant}.
$$
(B.1)

The last identity is obtained by integration by parts and shows that the bilinear concomitant is just the intersection number of the (homology classes) of the contours $\Gamma, \check{\Gamma}$. More precisely we get that:

$$
\begin{aligned}
&\frac{d}{dx} \int_\Gamma \int_{\check{\Gamma}} dsd\eta \, \frac{V_2'(\eta) - V_2'(s)}{\eta - s} \mathrm{e}^{x(\eta-s)-V_2(\eta)+V_2(s)}\\
=\;&\int_\Gamma \int_{\check{\Gamma}} dsd\eta \, (V_2'(\eta) - V_2'(s)) \mathrm{e}^{x(\eta-s)-V_2(\eta)+V_2(s)}\\
=\;&\int_\Gamma \int_{\check{\Gamma}} dsd\eta \, \frac{\partial}{\partial\eta}(-\mathrm{e}^{-V_2(\eta)}) \mathrm{e}^{x\eta} \mathrm{e}^{-xs+V_2(s)} - \int_\Gamma \int_{\check{\Gamma}} d\eta ds \, \frac{\partial}{\partial s}(\mathrm{e}^{V_2(s)}) \mathrm{e}^{-xs} \mathrm{e}^{x\eta-V_2(\eta)}\\
=\;&x \int_\Gamma \int_{\check{\Gamma}} dsd\eta \, \mathrm{e}^{x\eta-xs-V_2(\eta)+V_2(s)} - x \int_\Gamma \int_{\check{\Gamma}} dsd\eta \mathrm{e}^{x\eta-xs-V_2(\eta)+V_2(s)}\\
=\;&0
\end{aligned}
$$
(B.2)

The matrix expression shows that the pairing is indeed a duality since the determinant is nonzero. The undressing matrix Ψ_0 (that was originally introduced in Thm. 2.3) is thus

$$
\Psi_0 = \left[\begin{array}{c|ccccc} 1 & & & & & \\ \hline & v_2 & v_3 & \cdots & & v_{d_2+1} \\ & v_3 & & & v_{d_2+1} & \\ & & & \cdot\,^{\cdot\,^{\cdot}} & & \\ & v_{d_2} & v_{d_2+1} & & & \\ & v_{d_2+1} & & & & \end{array}\right] \left[\begin{array}{c|cccc} 1 & & & & \\ \hline & f_1 & f_2 & \cdots & f_{d_2} \\ & f_1' & f_2' & \cdots & f_{d_2}' \\ & \vdots & & & \vdots \\ & f_1^{(d_2-1)} & \cdots & & f_{d_2}^{(d_2-1)} \end{array}\right] \tag{B.3}
$$

where the Wronskian subblock in the second term is constructed by choosing d_2 homologically independent contour classes for the integrations $\check{\Gamma}$;

$$
f_k(x) := \int_{\check{\Gamma}_k} e^{-xs+V_2(s)} ds , \quad k = 1, \ldots, d_2 . \tag{B.4}
$$

The dressing matrix Ψ_0 exhibits a Stokes' phenomenon (of Airy's type) which is the inevitable drawback of removing the x-dependence from the jump matrix. We can now compute the jumps and see that it does not depend on x. For the k-th column we have:

$$
\psi_m^{(k)}(x) := \frac{1}{2\pi i} \int_{\check{\Gamma}_k} ds \iint_{\varkappa} dz dw \frac{\pi_m(z) e^{-V_1(z)}}{x-z} \frac{V_2'(s) - V_2'(w)}{s-w} e^{-V_2(w)+V_2(s)+zw-xs}, \quad 1 \le k \le d_2 \tag{B.5}
$$

gives:

$$
\begin{aligned}
\psi_m^{(k)}(x)_+ &= \psi_m^{(k)}(x)_- + \psi_m^{(0)}(x) \int\int ds dw \frac{V_2'(s) - V_2'(w)}{s-w} e^{-V_2(w)+V_2(s)+x(w-s)} \tag{B.6} \\
&= \psi_m^{(k)}(x)_- + \psi_m^{(0)}(x) \sum_{j=1}^{d_2} \varkappa_{\ell j}(\Gamma_j^{(y)} \# \check{\Gamma}_k) , \tag{B.7}
\end{aligned}
$$

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