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Effective inverse spectral problem for rational Lax matrices and applications

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Abstract

We reconstruct a rational Lax matrix of size $R+1$ from its spectral curve (the desingularization of the characteristic polynomial) and some additional data. Using a twisted Cauchy–like kernel (a bi-differential of bi-weight $(1-\nu,\nu)$) we provide a residue-formula for the entries of the Lax matrix in terms of bases of dual differentials of weights $\nu, 1-\nu$ respectively. All objects are described in the most explicit terms using Theta functions. Via a sequence of "elementary twists", we construct sequences of Lax matrices sharing the same spectral curve and polar structure and related by conjugations by rational matrices.

Particular choices of elementary twists lead to construction of sequences of Lax matrices related to finite–band recurrence relations (i.e. difference operators) sharing the same shape. Recurrences of this kind are satisfied by several types of orthogonal and biorthogonal polynomials. The relevance of formulæ obtained to the study of the large degree asymptotics for these polynomials is indicated.

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Contents

1 Introduction and setting **1**

2 Notation and main tools **5**

2.1 Theta functions 5

2.2 Prime form 6

3 Inverse spectral problem for rational Lax matrices **7**

3.1 Using different tensor weights 9

3.1.1 Twisting by flat line bundles 11

3.2 Residue formulæ for the Lax matrix 12

3.2.1 Left eigenvector 13

3.2.2 Structure of the Lax matrix near a branch-point of X 13

3.2.3 Change of divisor 14

3.2.4 Change of line bundle 15

3.2.5 Linear (smooth) deformations of the line bundle χ_η 15

3.3 Spectral bidifferential 16

3.4 Elementary twisting lattice and dual wave functions 17

3.4.1 Dualization 19

3.5 Expressions in terms of Theta functions 20

4 Finite band recurrence relations and commuting (pseudo) difference operators **22**

4.1 Lax and ladder matrices 25

4.2 Riemann–Hilbert problems 27

5 Commuting (pseudo)–difference operators in duality related to the two–matrix model **28**

1 Introduction and setting

The aim of this paper is to present an explicit solution of the inverse spectral problem for Lax matrices $A(x)$ of size $(R+1)\times(R+1)$ depending rationally on x. The forward problem and beautiful connections with integrable systems were explored in [3, 14, 27, 28, 1, 2, 25, 21, 22, 23, 24, 16] (to name a few); in particular it was shown in these works the important rôle of the theory of Theta functions [15] in the solution of both forward and inverse problems.

From the literature cited above we know that the forward problem (under suitable genericity assumptions) produces "spectral data" consisting of a smooth algebraic curve $\mathcal{L}$ of genus g with two meromorphic functions $X, Y : \mathcal{L} \to \mathbb{C}$, where X has degree $R+1$; $X(p) = x$ is the spectral parameter while $y = Y(p)$ is the eigenvalue of the rational matrix to be reconstructed.

The scheme of reconstruction requires that we fix two **dual tensor weights** $\nu, 1-\nu$; this means that the eigenvectors of $A(x)$ will be realized as bases in the suitable spaces of sections of ν–differentials (for the left eigenvectors) and $(1-\nu)$–differentials for the right eigenvectors. The parameter ν can be chosen integer or half-integer. This type of Baker–Akhiezer functions was considered (but in a different context) in [29] and slightly earlier in the series of papers [22, 23, 24]. In addition we need a divisor Γ (whose degree depends on ν) and an arbitrary meromorphic differential η; they will determine the local properties of the dual BA vectors by fixing the zeroes/poles and the essential singularity structure, the latter determined by $\exp 2i\pi \int \eta$.

The problem is not new and Baker–Akhiezer functions have been around for a long time (see, e.g. surveys [13, 20] or the monograph [3]); however our aim is to provide explicit **residue formulæ** for the entries of $A(x)$ and explore the intimate relation between a suitable bidifferential $\mathfrak{K}(p,q)$ of weight $(1-\nu, \nu)$ and several objects of the theory.

We then "twist" the reconstruction scheme so as to obtain recurrence relations for the BA vectors. We construct a sequence Γ_n of divisors of the same degree such that $\Gamma_{n+1} - \Gamma_n$ is an **elementary divisor** of degree zero consisting of two arbitrarily chosen points (in general). These *elementary twists* set in a general framework the original idea behind the construction of **discrete variable BA functions** suggested in the papers [21] and later utilized in [25] to develop the theory of commuting difference operators.

The associated sequence of bidifferentials $\mathfrak{K}_n$ then defines a pair of sequences ρ_n, π_n of $\nu/(1-\nu)$–differentials (called **dual wave functions**) related by a particular form of **Serre duality**: such duality is realized via a **residue pairing** of the form $\operatorname*{res}_{\infty^{(+)}} \rho_n \pi_m = \delta_{mn}$, where $\infty^{(+)}$ is the divisor of positive points in the elementary twisting divisors.

If the elementary twisting divisors are chosen amongst the poles of X we obtain wave-functions solving a finite–term recurrence relation of the form

$$X\pi_n = \sum_{j=-d_-}^{d_+} \alpha_j(n) \pi_{n+j} \ . \tag{1-1}$$

These recurrence relations fall within the scope of the theory of "difference operators" extensively studied [27, 25]. Our interest has a different origin in connection with the theory of orthogonal and biorthogonal polynomials and their asymptotics for large degrees; in this perspective the wave-functions represent a (formal) asymptotic regime for the polynomials, the sequence of bidifferentials $\mathfrak{K}_n$ is intimately related to "Christoffel–Darboux-like" kernels and "Christoffel–Darboux-formulæ" that arise in those contexts.

We repeat the argument of [10] to illustrate this connection in the simplest case of the ordinary orthogonal polynomials.

Orthogonal polynomials $p_n(x)$ with respect to a weight on the real line, $w_N(x) = \mathrm{e}^{-NV(x)}\mathrm{d}x$, satisfy a three–term recurrence relation[5] [30]

$$xp_n(x) = \gamma_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_{n-1} p_{n-1}(x) \ , \tag{1-2}$$

which can be written in matrix form as $x\mathbf{p} = Q\mathbf{p}$, with $\mathbf{p}$ the semiinfinite vector of the orthogonal polynomials and Q the tri-diagonal (symmetric) matrix with entries given by the coefficients of the above recurrence relations.

In studying the large degree asymptotics one typically sends the large parameter N appearing in the measure to infinity at the same rate as the degree n of the polynomial [12]: this means that –while Q implicitly changes because of the change in the measure– we are considering the polynomials and the recurrence relations very "far down" along the diagonal. On a heuristic level one argues that the tridiagonal **semi**infinite matrix Q can be replaced by a **doubly**-infinite matrix $\mathbf{X}$ (i.e. indexed by $\mathbb{Z}$ rather than $\mathbb{N}$) of the same shape and symmetries. Consider now the associated functions $\pi_n := p_n \mathrm{e}^{-\frac{N}{2}V(x)}$: if $V(x)$ (the *potential* appearing in the measure that defines the OPs) is a polynomial of degree $d+1$ then this sequence –while still satisfying the same three-term recurrence relation– satisfies also a $2d+1$–term *differential* recurrence relation

$$\frac{1}{N}\partial_x \pi_n = c_d(n)\pi_{n+d} + \ldots + -c_d(n-d)\pi_{n-d} \tag{1-3}$$

where in matrix form the recurrence is represented by a (skew-symmetric) matrix P with d supra- and d sub-diagonals. The scaling $\frac{1}{N}$ is needed (on heuristic grounds) to assure the boundedness of the coefficients of the recurrence relation. By construction, the two matrices P, Q satisfy

$$[P, Q] = \frac{1}{N}\mathbf{1} \tag{1-4}$$

and in the $N \to \infty$ limit they commute: we thus replace them by two **commuting** doubly–infinite matrices $\mathbf{X}, \mathbf{Y}$ of the same shape and symmetries. At this point, the first problem is therefore to classify such pairs of commuting matrices and much of this has been extensively analyzed in [25]; some additional ingredients (Serre duality) can be found in [10] and are put in a general context in the present manuscript.

In applications stemming from random matrices, the so–called Christoffel-Darboux kernel has crucial importance since it generates all correlation functions [26]. The C-D kernel is nothing but the orthogonal projection operator (for the chosen measure) on the subspace of polynomials of degree $N-1$ or less and is given by

$$K_n(x, x') = \sum_{j=0}^{N-1} p_j(x) p_j(x') \ . \tag{1-5}$$

[5] In fact the situation allows a generalization to holomorphic weights on contours as explained in [10].

Due to the Christoffel–Darboux theorem it can be expressed in terms of only two OP

$$K_N(x,x') = \gamma_N \frac{p_N(x)p_{N-1}(x') - p_{N-1}(x)p_N(x')}{x - x'} \tag{1-6}$$

and this fact is crucial in proving universality results since it allows to express the asymptotic behavior for large N in terms of a fixed (i.e. N independent) number of polynomials (in this case 2).

In the heuristic approach used in [10] (and then justified rigorously using Riemann–Hilbert techniques) the quasipolynomials π_n were replaced by meromorphic sections of a spinor bundle, namely by half-differentials on the (asymptotic) spectral curve, in this case hyperelliptic. The function x was then regarded as a meromorphic function $X(p)$ on this algebraic curve whose multiplication of the half-differentials π_n can be expressed in term of the same sequence of half-differentials, thus producing a recurrence relation. The "orthogonality" was replaced by a residue pairing between the sequence π_n of half differentials and the Serre–dual sequence $\pi_n^\star$ of half-differentials : $\operatorname{res} \pi_n \pi_n^\star = \delta_{mn}$. Similarly the kernel $K_n(x,x')$ was replaced by bidifferential of weights $(1/2, 1/2)$ that played the rôle of projection operator with respect to the residue pairing.

Such bidifferential also satisfies a "Christoffel–Darboux" theorem

$$\mathfrak{K}_N(p,p') = \gamma_N \frac{\pi_N(p)\pi^\star_{N-1}(p') - \pi_{N-1}(p')\pi^\star_N(p)}{X(p) - X(p')} \tag{1-7}$$

which is ultimately an identity for Theta functions; this is fully generalized presently in Prop. 3.2 and Prop. 4.2.

Our paper does not focus primarily on difference operators, rather we find them as a byproduct of the sequence of transformations induced on the Lax matrix by the elementary twisting; also, the eigenvectors for the Lax matrix (i.e. the Baker–Akhiezer vectors) solve certain Riemann–Hilbert problems with quasi-permutation monodromies. These were studied in [19] for their own sake, while our approach finds them as a natural byproduct of the inverse-spectral reconstruction. Riemann–Hilbert problems with quasipermutation monodromies are also related to asymptotics of (multi)orthogonal polynomials; indeed after the so-called normalization of the RH problem satisfied by the polynomials and associated functions, one is lead to an approximating asymptotic problem with quasipermutation monodromies.

The paper is organized as follows: in Section 2 we recall the basic tools from the geometry of Riemann surfaces, in particular the notion of Theta functions and prime forms, after [15].

In Section 3 we set up the inverse spectral problem for rational Lax matrices; here the problem is solved using pairs of dual Baker–Akhiezer vectors with tensor weights $\nu, 1-\nu$ where $\nu \in \frac{1}{2}\mathbb{Z}$. A residue formula for the Lax matrix in terms of spectral projectors is derived. We also derive the "generalized Toda lattice" in terms of elementary twists and express the ladder matrices and the matrices implementing the change of a line bundle in terms of suitable residue formulæ. Finally, we provide explicit expressions for the relevant twisted Cauchy kernels in terms of Theta functions and prime forms.

In Section 4 we specialize the generalized Toda lattice so as to obtain genuine finite-terms recurrence relations (difference operators); in this setting more explicit formulæ for the BA vectors are derived. The connection to Riemann–Hilbert problems with quasi–permutation monodromies is pointed out.

Finally, in Section 5 we consider the case that is potentially most relevant to the study of biorthogonal polynomials for the two–matrix model [6] and reveal a notion of duality that is well known for biorthogonal polynomials but was not known in the context of pairs of commuting difference operators.

We end this introduction pointing out that in the case that $\mathcal{L}$ has genus 0 all the formulæ can be expressed in terms of rational functions of the uniformizing parameter: this is left as exercise for the interested reader.

2 Notation and main tools

2.1 Theta functions

For a given smooth genus-g curve $\mathcal{L}$ with a fixed choice of symplectic homology basis of a and b-cycles, we denote by ω_j the normalized basis of holomorphic differentials

$$\oint_{a_j} \omega_\ell = \delta_{j\ell} \ , \qquad \oint_{b_j} \omega_\ell = \tau_{j\ell} = \tau_{\ell j} \ . \tag{2-1}$$

We will denote by Θ the theta function

$$\Theta(\mathbf{z}) := \sum_{\vec{n}\in\mathbb{Z}^g} \mathrm{e}^{i\pi\vec{n}\cdot\tau\vec{n} - 2i\pi\mathbf{z}\cdot\vec{n}} \tag{2-2}$$

For brevity we will often omit any symbolic reference to the Abel map: namely if $p \in \mathcal{L}$ is a point and it appears as argument of a Theta-function, it will be understood that the Abel map (with a certain basepoint) was applied.

We denote by $\mathcal{K}$ the vector of Riemann constants (also depending on the choice of the basepoint)

$$\mathcal{K}_j = -\sum_{\ell=1}^{g}\left[\oint_{a_\ell}\omega_\ell(p)\int_{p_0}^{p}\omega_j(q) - \delta_{j\ell}\frac{\tau_{jj}}{2}\right] \tag{2-3}$$

where in this expression the cycles a_j are realized as loops with basepoint p_0 and the inner integration is done along a path lying in the canonical dissection of the surface along the chosen representatives of the basis in the homology of the curve.

The crucial property of $\mathcal{K}$ is that for a nonspecial divisor Γ of degree g, $\Gamma = \sum_{j=1}^{g} \gamma_j$, the "function"

$$f(p) = \Theta(p - \Gamma - \mathcal{K}) \tag{2-4}$$

has zeroes precisely and only at $p = \gamma_j$, $j = 1 \dots g$.

We will also have to use Theta functions with (complex) characteristics; for any two complex vectors $\vec{\epsilon}, \vec{\delta}$ the theta function with these (half) characteristics is defined via

$$\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z}) := \exp\left(2i\pi\left(\frac{\epsilon\cdot\tau\cdot\epsilon}{8} + \frac{1}{2}\epsilon\cdot\mathbf{z} + \frac{1}{4}\epsilon\cdot\delta\right)\right)\Theta\left(\mathbf{z} + \frac{\vec{\delta}}{2} + \tau\frac{\vec{\epsilon}}{2}\right) \tag{2-5}$$

Here the (half) characteristics of a point $\zeta \in \mathbb{C}^g$ are defined by

$$2\zeta = \delta + \tau\epsilon \tag{2-6}$$

where the factor of 2 is purely conventional so that half integer characteristics have integer (half)-characteristics. This modified Theta function has the following periodicity property : for $\lambda, \mu \in \mathbb{Z}^g$

$$\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z} + \lambda + \tau\mu) = \exp\left[i\pi(\vec{\epsilon}\cdot\lambda - \vec{\delta}\cdot\mu) - i\pi\mu\cdot\tau\cdot\mu - 2i\pi\mathbf{z}\cdot\mu\right]\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z}) \tag{2-7}$$

Note also the symmetry

$$\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z}) = \Theta\begin{bmatrix}-\vec{\epsilon}\\ -\vec{\delta}\end{bmatrix}(-\mathbf{z}) \tag{2-8}$$

2.2 Prime form

The prime form $E(\zeta, \zeta')$ is defined as follows [15]

Definition 2.1 *The prime form $E(\zeta, \zeta')$ is the $(-1/2, -1/2)$ bi-differential on $\mathcal{L} \times \mathcal{L}$*

$$E(\zeta, \zeta') = \frac{\Theta_\Delta(\mathfrak{u}(\zeta) - \mathfrak{u}(\zeta'))}{h_\Delta(\zeta) h_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}(\zeta')} \tag{2-9}$$

$$h_\Delta(\zeta)^2 := \sum_{k=1}^{g} \partial_{\mathfrak{u}_k} \ln \Theta_\Delta\Big|_{\mathfrak{u}=0} \omega_k(\zeta) \ , \tag{2-10}$$

where ω_k are the normalized Abelian holomorphic differentials, $\mathfrak{u}$ is the corresponding Abel map and $\Delta = \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is a half–integer odd characteristic (the prime form does not depend on which one).

The prime form $E(\zeta, \zeta')$ is antisymmetric in its arguments and it is a section of an appropriate line bundle, i.e. it is multiplicatively multivalued on $\mathcal{L} \times \mathcal{L}$:

$$E(\zeta + a_j, \zeta') = E(\zeta, \zeta') \ , \qquad E(\zeta + b_j, \zeta') = E(\zeta, \zeta') \exp\left(-\frac{\tau_{jj}}{2} - \int_{\zeta}^{\zeta'} \omega_j\right) \tag{2-11}$$

In our notation for the (half)-characteristics, the vectors α, β appearing in the definition of the prime form are actually integer valued. We also note for future reference that the half order differential h_Δ is also multivalued according to

$$h_\Delta(p + a_j) = \mathrm{e}^{i\pi\alpha_j} h_\Delta(p) \tag{2-12}$$

$$h_\Delta(p + b_j) = \mathrm{e}^{-i\pi\beta_j} h_\Delta(p). \tag{2-13}$$

3 Inverse spectral problem for rational Lax matrices

The goal of this section is to explore the inverse spectral problem, namely how to reconstruct a matrix rationally dependent on x from the knowledge of its spectral curve and some additional data. Since the construction is quite symmetric we can also deal with the dual situation without any extra effort, thus treating the spectral parameter and the eigenvalues on the same footing.

We work with the following data

1. A smooth curve $\mathcal{L}$ of genus g.
2. Two meromorphic functions X, Y with polar divisors $\mathfrak{X}, \mathfrak{Y}$ of degrees $R + 1$ and $S + 1$.
3. A (generic) divisors Γ of degree $g + R$ (not necessarily positive).

The main tool is the following adaptation of the Cauchy kernel [15, 22, 23, 24]

Proposition 3.1 *For a generic choice of divisor Γ there exists a unique kernel $K(p, \xi)$ which is a function w.r.t. the point p and a differential w.r.t. the point ξ with the divisor properties*

$$(K(p, \xi))_p \geq -\Gamma + \mathfrak{X} - \xi \tag{3-1}$$

$$(K(p, \xi))_\xi \geq \Gamma - \mathfrak{X} - p \tag{3-2}$$

such that $\operatorname*{res}_{\xi=p} K(p, \xi) = 1$. The subscripts above indicate in which variable the divisor properties are considered.

The proof follows easily from the Riemann–Roch theorem; we will write explicitly the expression of this kernel (in a generalized setting of which the current one is a particular case) in terms of Theta functions later on.

Remark 3.1 *[Linear equivalence] In fact we could be slightly more general in the formulation of the above proposition, since what matters there is only the equivalence class (modulo principal divisors) of $\Gamma - \mathfrak{X}$. In particular we could use in (3-1, 3-2) two different divisors $\mathcal{D}, -\widetilde{\mathcal{D}}$ (both of degree $-g-1$) as long as $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are equivalent.*

In that case, however, $\operatorname*{res}_{\xi=p} K(p, \xi) = f(p)$ would be a meromorphic function with divisor $(f) = \widetilde{\mathcal{D}} - \mathcal{D}$; there is only one such function (generically) up to scalar multiplication. Hence the normalization would have to be fixed in some other ad hoc *way.*

Example 3.1 *If the divisor $\mathfrak{X}$ consist of $R+1$ distinct points $x_0,\dots x_R$ and Γ is a positive divisor, then the expression for K is*

$$K(p,\xi) = C\det\left[\begin{array}{c|cccc} \rho_{p\infty_0}(\xi) & \rho_{p\infty_0}(\gamma_1) & \cdots & \rho_{p\infty_0}(\gamma_{g+R}) \\ \vdots & & & \\ \rho_{p\infty_R}(\xi) & \rho_{p\infty_R}(\gamma_1) & \cdots & \rho_{p\infty_R}(\gamma_{g+R}) \\ \hline \omega_1(\xi) & \omega_1(\gamma_1) & \cdots & \omega_1(\gamma_{g+R}) \\ \vdots & & & \\ \omega_g(\xi) & \omega_g(\gamma_1) & \cdots & \omega_g(\gamma_{g+R}) \end{array}\right] \tag{3-3}$$

where $\rho_{pq}(\xi)$ stands for the (unique) normalized Abelian differential of the third kind with first–order poles at p,q and residues ± 1; the constant C depends on the divisors $\mathfrak{X},\Gamma$ and is chosen so that the residue at $\xi = p$ is 1.

Consider now the expression $M(p,\xi) := (X(p) - X(\xi))K(p,\xi)$; its divisor properties w.r.t. p,ξ follow from the properties of K:

$$(M(p,\xi))_p \geq -\Gamma \tag{3-4}$$

$$(M(p,\xi)_\xi \geq \Gamma - 2\mathfrak{X}\ , \tag{3-5}$$

where the pole on the diagonal is now absent because of the multiplication by $X(p) - X(\xi)$.

Again the Riemann–Roch theorem implies that generically

$$\mathbf{r}(-\Gamma) = \mathbf{i}(\Gamma) - g + \deg\Gamma + 1 = R + 1 \tag{3-6}$$

since (generically) $\mathbf{i}(\Gamma) = 0$.

Also $\mathbf{i}(\Gamma - 2\mathfrak{X}) = R+1$ (generically); to see this we note that the space of third-kind differentials with poles not exceeding $2\mathfrak{X}$ has dimension $\deg(2\mathfrak{X}) - 1 + g = 2R + 1 + g$. Imposing the vanishing at $g + R$ points gives as many linear constraints, hence reducing the dimension to $R + 1$.

Let $\psi_0(p),\dots,\psi_R(p)$ be any basis of the vector space of meromorphic functions with divisor exceeding $-\Gamma$ and let $\varphi_0(p),\dots,\varphi_R(p)$ be any basis of the vector space of differentials with divisor exceeding $\Gamma - 2\mathfrak{X}$. Let us introduce the notations

$$\boldsymbol{\psi} := \left[\begin{array}{c} \psi_0(p) \\ \vdots \\ \psi_R(p) \end{array}\right]\ , \qquad \boldsymbol{\varphi} := \left[\begin{array}{c} \varphi_0(p) \\ \vdots \\ \varphi_R(p) \end{array}\right]\ . \tag{3-7}$$

These vectors will be called the **pair of dual Baker–Akhiezer** vectors and we will show later on how this term is motivated by the Serre duality. Note that the notion of the **dual Baker-Akhiezer function** was first introduced in [11] where it was applied to construct algebro-geometric solutions to Gelfand-Dickii, NLS and sine-Gordon hierarchies.

Since the dual pair spans their respective spaces it follows that

$$M(p,\xi) \in \mathbb{C}\{\psi_0,\dots\psi_R\} \otimes \mathbb{C}\{\varphi_0,\dots,\varphi_R\} \tag{3-8}$$

In other words there is a $(R+1)\times(R+1)$ matrix $\mathbb{K}$ with constant coefficients such that

$$(X(p)-X(\xi))K(p,\xi)=\boldsymbol{\varphi}^t(\xi)\mathbb{K}\boldsymbol{\psi}(p) \tag{3-9}$$

which gives immediately

Proposition 3.2 *There is a constant matrix $\mathbb{K}$ of size $R+1$, depending on the choice of bases ψ_j, φ_j, such that*

$$K(p,\xi)=\frac{\boldsymbol{\varphi}^t(\xi)\mathbb{K}\boldsymbol{\psi}(p)}{X(p)-X(\xi)} \tag{3-10}$$

From this expression we derive the following identity.

Corollary 3.1 *Independently of the choices of the bases $\boldsymbol{\psi}, \boldsymbol{\varphi}$ we have the identity*

$$dX(p)=\boldsymbol{\varphi}^t(p)\mathbb{K}\boldsymbol{\psi}(p)\ . \tag{3-11}$$

Moreover, if $p,q\in\mathcal{L}\setminus(\Gamma\cup\mathfrak{X})$ are two points such that $X(p)=X(q)$ then $\boldsymbol{\varphi}^t(p)\mathbb{K}\boldsymbol{\psi}(q)=0$.

Proof. Taking the residue on the diagonal we have

$$1=\operatorname*{res}_{\xi=p}K(p,\xi)=\frac{\boldsymbol{\varphi}^t(p)\mathbb{K}\boldsymbol{\psi}(p)}{dX(p)}\ . \tag{3-12}$$

The second statement follows from the fact that $K(p,q)$ is regular and hence the numerator in its expression must vanish whenever $X(p)=X(q)$. **Q.E.D.**

The Lax matrix can now be constructed from these spectral data if $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}^t\mathbb{K}$ are viewed as the right/left eigenvectors with eigenvalue $Y(p)$ at the point(s) p above $x=X(p)$. The explicit residue formula will be given later in Sect. 3.2 in a generalized setting.

3.1 Using different tensor weights

In the above scheme we are using a pair of BA vectors with tensor weights 0 and 1 respectively, that is, functions and differentials. This is, in fact unnecessary and in some applications (typically to the asymptotics of ODEs) it may even be too restrictive.

In general, we could widen the scope of the construction so that $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ can be tensors of weight ν and $1-\nu$ respectively; similar considerations (motivated by applications to quantum field theory) were used in [22, 23, 24, 29]. The tensor weight ν can be typically integer or half–integer; the particularly useful case [10] is $\nu=\frac{1}{2}$ where both elements of the pair are spinors (half–differentials). Clearly some modifications in the way the Riemann–Roch theorem is applied are needed (i. p., more general Serre–duality arguments).

If $h_\nu(\mathcal{D})$ is the dimension of the space of ν–differentials with divisor exceeding $\mathcal{D}$ then we know that, for $\nu \geq \frac{1}{2}$, $h_\nu(\mathcal{D}) = 0$ if $\mathcal{D}$ is generic and $\deg \mathcal{D} \geq \delta_{\nu 1} + (2\nu - 1)(g-1)$.

Thus, let Γ be generic and of degree $\deg(\Gamma) = (2\nu - 1)(g-1) + R + 1$; then one finds from the Riemann–Roch theorem that, for $\nu \in \frac{1}{2}\mathbb{Z},\ j \geq \frac{1}{2}$ (the case $\nu = 1$ being the one discussed above)

$$h_{1-\nu}(-\Gamma) = h_\nu(\Gamma) + \deg(\Gamma) - (2\nu - 1)(g-1) = R + 1\ . \tag{3-13}$$

Thus we should use a divisor Γ of degree $(2\nu - 1)(g-1) + R + 1$ and choose $\boldsymbol{\psi}$ to be a basis in $H_{1-\nu}(-\Gamma)$ (of dimension $R+1$).

The dual BA vector $\boldsymbol{\varphi}$ would then span $H_\nu(\Gamma - 2\mathfrak{X})$ also of the same dimension.

The relevant "Cauchy" kernel is then a bidifferential of weights $(1-\nu, \nu)$ with divisor properties

$$(\mathfrak{K}(p,q))_{\left\{ {p \atop q} \right\}} \geq \left\{ \begin{array}{c} -\Gamma + \mathfrak{X} - q \\ \Gamma - \mathfrak{X} - p \end{array} \right. \tag{3-14}$$

and normalized by the requirement that the "ν–residue" is one, namely the expansion along the diagonal $p = q$ in local coordinate $z(p) = z,\ z(q) = z'$ is

$$\mathfrak{K}(p,q) = \frac{\mathrm{d}z^{1-\nu}\mathrm{d}z'^{\nu}}{z - z'}(1 + \mathcal{O}(z - z'))\ . \tag{3-15}$$

The leading coefficient of the expansion is invariant under changes of a local coordinate and hence it is a geometrical quantity.

Note that if ν is a half–integer then, for completeness, one should also choose a spinor bundle (i.e. signs with which the half-integer spinor changes along each handle).

Completely similar considerations as before show that

Proposition 3.3 *There exists a constant matrix* $\mathbb{K}^{(\nu)}$ *such that*

$$\mathfrak{K}(p,q) = \frac{\boldsymbol{\varphi}(q)\mathbb{K}^{(\nu)}\boldsymbol{\psi}(p)}{X(p) - X(q)} \tag{3-16}$$

and hence

$$\mathrm{d}X(p) = \boldsymbol{\varphi}(p)\mathbb{K}^{(\nu)}\boldsymbol{\psi}(p) \tag{3-17}$$

In the next sections we can now consider this more general case but remove the explicit reference to the tensor weight ν. Thus we will have

- Γ a generic divisor of degree $(2\nu - 1)(g-1) + R + 1$;
- $\boldsymbol{\psi} = [\psi_0, \ldots, \psi_R]$ a vector of basis $(1-\nu)$–differentials in $\mathcal{H}_{1-\nu}(-\Gamma)$ (of dimension $R+1$);
- $\boldsymbol{\varphi} = [\varphi_0, \ldots, \varphi_R]$ a vector of basis ν–differentials in $\mathcal{H}_\nu(\Gamma - 2\mathfrak{X})$;
- $\mathfrak{K}(p,q)$ the unique bi–tensor of bi–weight $(1-\nu, \nu)$ with the divisor properties and the normalization listed above.

3.1.1 Twisting by flat line bundles

In applications to isospectral dynamics and in several other applications it is necessary to consider a slight generalization of the above picture, in that one twists the line bundle implicitly associated to the divisor Γ by some other line bundle (which may depend on external parameters or "times"). Typically (as in the classical example of finite-gap integration of KP or KdV dynamics) the extra line-bundle data falls within the class to be described below.

Let η be an arbitrary meromorphic differential on the curve $\mathcal{L}$ with pole divisor

$$(\eta) \geq \sum_{i=1}^{K} d_i c_i, \ , \quad c_1, \cdots, c_K \in \mathcal{L} \ , \ d_1, \ldots, d_K \in \mathbb{N}. \tag{3-18}$$

Let us denote its residues by t_i

$$t_i := \operatorname*{res}_{c_i} \eta \ , \quad \sum_{i=1}^{K} t_i = 0 \ . \tag{3-19}$$

The Abelian integral $2i\pi \int^p \eta$ (where the base-point of integration affects only an overall normalization) has in general nontrivial periods around the $2g$ handles of the curve and around the punctures c_i. The exponential $\mathrm{e}^{2i\pi \int^p \eta}$ defines a homomorphism of $\pi_1(\mathcal{L} \setminus \{c_1, \ldots, c_K\}) \mapsto \mathbb{C}^\times$ and hence a certain flat line-bundle. Moreover this line-bundle has transition functions of exponential type[6] at the punctures c_i.

Twisting the previous description by this line-bundle $\mathfrak{L}_\eta$ is then equivalent to considering ν–differentials (resp. $(1-\nu)$–differentials) with essential singularities at the punctures $c_i, \ i = 1, \ldots, K$ of the same type as $\mathrm{e}^{\pm 2i\pi \int^p \eta}$.

The *twisted* ν–Cauchy kernel is then a $(1-\nu, \nu)$–bidifferential with singularities of the form $\mathrm{e}^{2i\pi \int^p \eta}$ and $\mathrm{e}^{2i\pi \int_q \eta}$, such that (we still use the same symbol)

$$\exp\left(2i\pi \int_q^p \eta\right) \mathfrak{K}(p,q) \tag{3-20}$$

is locally a $(\nu, 1-\nu)$ bidifferential with divisor $\geq -\Gamma + \mathfrak{X} - q$ (in p) and $\geq \Gamma - \mathfrak{X} - p$ (in q), and with multiplicative multivaluedness along the homotopy group of $\mathcal{L} \setminus \{c_1, \ldots c_K\}$ given by the character

$$\begin{aligned} \chi_\eta : \pi_1(\mathcal{L} \setminus \{c_1, \ldots, c_K\}) \to \mathbb{C}^\times \\ \chi_\eta(\gamma) = \exp\left(2i\pi \oint_\gamma \eta\right) \ . \end{aligned} \tag{3-21}$$

and such that near c_j in a local coordinate z it has a singularity of type $z^{\pm 2i\pi t_j}$. Thus in general, unless the residues t_i are integers, this kernel has logarithmic branching at the points c_i.

[6]If $d_i = 1$ then the singularity may be a pole or power-like singularity with non-integer exponent and hence also branching singularity.

Correspondingly, the bases $\boldsymbol{\psi}$ ($\boldsymbol{\varphi}$) are $1-\nu$–differentials (resp. (ν)-differentials) with essential singularities of type $\mathrm{e}^{\pm 2i\pi\int\eta}$. The **uniqueness** of such kernel is a simple argument in function theory and Riemann–Roch theorem. On the existence we do not insist at this point (although it would not be difficult to prove it abstractly) since we are going to produce explicit expressions in terms of Theta functions in Sect. 3.5.

3.2 Residue formulæ for the Lax matrix

Let $\mathfrak{J}$ be the polar divisor of Y. We start with the observation that $Y(\xi)\boldsymbol{\psi}(\xi)\mathfrak{K}(p,\xi)$ is a 1–differential (in ξ) with poles only at $\mathfrak{J},\mathfrak{X}$ and a simple pole at p with residue $-Y(p)\boldsymbol{\psi}(p)$, therefore

$$Y(p)\boldsymbol{\psi}(p) = -\mathop{\mathrm{res}}_{\xi=p} Y(\xi)\boldsymbol{\psi}(\xi)\mathfrak{K}(p,\xi) = \sum_{q\in\mathfrak{J},\mathfrak{X}}\mathop{\mathrm{res}}_{\xi=q} Y(\xi)\boldsymbol{\psi}(\xi)\mathfrak{K}(p,\xi) =$$
$$= \sum_{q\in\mathfrak{J},\mathfrak{X}}\mathop{\mathrm{res}}_{\xi=q} Y(\xi)\boldsymbol{\psi}(\xi)\frac{\boldsymbol{\varphi}^t(\xi)\mathbb{K}\boldsymbol{\psi}(p)}{X(p)-X(\xi)} = \left[\sum_{q\in\mathfrak{J},\mathfrak{X}}\mathop{\mathrm{res}}_{\xi=q} Y(\xi)\frac{\boldsymbol{\psi}(\xi)\boldsymbol{\varphi}^t(\xi)\mathbb{K}}{X(p)-X(\xi)}\right]\boldsymbol{\psi}(p) \tag{3-22}$$

The expression

$$A(x) := \sum_{q\in\mathfrak{J},\mathfrak{X}}\mathop{\mathrm{res}}_{\xi=q} Y(\xi)\frac{\boldsymbol{\psi}(\xi)\boldsymbol{\varphi}^t(\xi)\mathbb{K}}{x-X(\xi)} \tag{3-23}$$

is –*a priori*– a rational expression in x; it has poles at the X–projection of the divisor $\mathfrak{J}$ of poles of Y and at $x=\infty$ (i.e. has a polynomial part).

In particular if $\mathfrak{y}$ is a pole of Y of order k with $\infty\neq x_o = X(\mathfrak{y})$ and does not coincide with any branch-point of X (i.e. $\mathrm{d}X|_{X^{-1}(x_o)}\neq 0$) then $A(x)$ has a pole of order k

$$A(x) = \mathcal{O}((x-x_o)^{-k}) \tag{3-24}$$

If $\mathfrak{y}$ is a branch-point of X and $\mu\geq 1$ is minimum order of branching of X ($\mu=2$ being the case of a simple branch-points) then

$$A(x) = \mathcal{O}\left((x-x_o)^{-[k/\mu]}\right) \tag{3-25}$$

If $\mathfrak{y}$ coincides with one of the poles of X then the Lax matrix will have polynomial parts of degree k or $[k/(d-1)]$ (if d is the order of the pole). Obviously the degree of A depends on the maximal degree amongst all poles of Y above $x=\infty$. If Y has no poles coinciding with any of the poles of X then $A(x)$ will be necessarily bounded at $x=\infty$.

By construction we have

$$Y(p)\boldsymbol{\psi}(p) = A(X(p))\boldsymbol{\psi}(p) \tag{3-26}$$

Therefore (as expected) $\boldsymbol{\psi}(p)$ is the right eigenvector of the matrix $A(X(p))$ with eigenvalue $Y(p)$; the different points p lying above the same values of $X(p)$ give the (generically distinct) eigenvalue/eigenvector pairs.

3.2.1 Left eigenvector

Consider now $Y(p)\mathfrak{K}(p,\xi)\boldsymbol{\varphi}(p)^t$: this is a 1–differential in p with poles in p at $\mathfrak{J},\mathfrak{X}$ and ξ. Since the ν–residue at $\xi = p$ of $\mathfrak{K}(p,\xi)$ is 1, it follows from a simple computation in a local coordinate that the residue at $p=\xi$ of this bidifferential is $Y(\xi)\boldsymbol{\varphi}^t(\xi)$. Therefore

$$Y(\xi)\boldsymbol{\varphi}^t(\xi) = \mathop{\mathrm{res}}_{p=\xi} Y(p)\boldsymbol{\varphi}^t(p)\mathfrak{K}(p,\xi) = -\sum_{q\in\mathfrak{J},\mathfrak{X}} \mathop{\mathrm{res}}_{p=q} Y(p)\frac{\boldsymbol{\varphi}^t(\xi)\mathbb{K}\boldsymbol{\psi}(p)}{X(p)-X(\xi)}\boldsymbol{\varphi}^t(p) =$$

$$= \boldsymbol{\varphi}^t(\xi)\sum_{q\in\mathfrak{J},\mathfrak{X}} \mathop{\mathrm{res}}_{p=q} Y(p)\frac{\mathbb{K}\boldsymbol{\psi}(p)\varphi^t(p)}{X(\xi)-X(p)} = \boldsymbol{\varphi}^t(\xi)\widetilde{A}(X(\xi)) \tag{3-27}$$

$$\widetilde{A}(x) := \sum_{q\in\mathfrak{J},\mathfrak{X}} \mathop{\mathrm{res}}_{p=q} Y(p)\frac{\mathbb{K}\boldsymbol{\psi}(p)\boldsymbol{\varphi}^t(p)}{x-X(p)} \tag{3-28}$$

It is clear from the defining formulæthat

$$\widetilde{A}(x)\mathbb{K} = -\mathbb{K}A(x) \tag{3-29}$$

Therefore the left eigenvector of $A(x)$ is $\boldsymbol{\varphi}^t(\xi)\mathbb{K}$:

$$Y(\xi)\boldsymbol{\varphi}^t(\xi)\mathbb{K} = \boldsymbol{\varphi}^t(\xi)\widetilde{A}(X(\xi))\mathbb{K} = \boldsymbol{\varphi}^t(\xi)\mathbb{K}A(x)\Big|_{x=X(\xi)} . \tag{3-30}$$

From Corollary 3.1 it follows that

$$\boldsymbol{\ell}(p) := \frac{\boldsymbol{\varphi}^t(p)\mathbb{K}}{dX(p)} \tag{3-31}$$

is the **normalized** left eigenvector (a $\nu-1$-differential) of $A(x)$, in the sense that

$$\boldsymbol{\ell}(p)\cdot\boldsymbol{\psi}(p) \equiv 1. \tag{3-32}$$

From the second part of Corollary 3.1 follows also that (as it should) the evaluation of $\ell(q)$ at the other points $q\in\mathcal{L}$ above $X(p)$ are orthogonal to $\boldsymbol{\psi}(p)$. Note that the dual left-eigenvector has poles at the branch-points of the X projection.

The (generically) rank–one projector on the eigenspace with eigenvalue $Y(p)$ is given by

$$\Pi(p) := \boldsymbol{\psi}(p)\otimes\boldsymbol{\ell}(p) = \frac{\boldsymbol{\psi}(p)\boldsymbol{\varphi}^t(p)\mathbb{K}}{dX(p)} \tag{3-33}$$

3.2.2 Structure of the Lax matrix near a branch-point of X

Suppose c is a critical value of X and $\sum\mu_i\xi_i = X^{-1}(c)$. Let us choose local coordinates near the point ξ_i as

$$z_i = (X-c)^{\frac{1}{\mu_i}} . \tag{3-34}$$

Then we have

$$L(c) = \operatorname*{res}_{\mathfrak{X},\mathfrak{I}} \frac{Y(\xi)\Pi(\xi)}{X(\xi)-c} = \sum_i \operatorname*{res}_{\xi_i} \frac{Y(\xi)\Pi(\xi)}{z_i(\xi)^{\mu_i}} \tag{3-35}$$

We see that each residue extracts the μ_j jet of Y and Π, contributing to a rank-μ_j Jordan block of the Lax matrix and this point.

Indeed, by expanding

$$Y(\xi) = \sum Y_{j,\ell} {z_j}^{\ell} \Pi(\xi) = \sum \Pi_{j,\ell} {z_j}^{\ell} \mathrm{d}z_j \operatorname*{res}_{z_j=0} \frac{Y(\xi)\Pi(\xi)}{{z_j}^{\mu_j}} = \sum_{\ell=0}^{\mu_j-1} Y_{j,\mu_j-1-\ell}\Pi_{j,\ell} \tag{3-36}$$

If $\Pi(z)dz = R(z)L(z)dz$, one sees easily by induction that the rank of any linear combination of the first k derivatives of Π is less or equal to k, and generically is precisely k.

3.2.3 Change of divisor

The matrix $A(x)$ depends implicitly on the divisor Γ but its characteristic polynomial does not (since the latter describes the algebraic relation between the rational functions X, Y on the spectral curve $\mathcal{L}$). We investigate how a change in the divisor Γ (within the same class of generic divisors) affects the Lax matrix $A(x)$.

Proposition 3.4 *Let $A_\Gamma(x)$ the Lax matrix constructed as in Section 3.2 using a divisor Γ. Let $\widetilde{\Gamma}$ be another divisor of the same degree and with similar genericity properties; then*

$$A_{\widetilde{\Gamma}}(x) = C_{\Gamma,\widetilde{\Gamma}}(x) A_{\widetilde{\Gamma}}(x) C^{-1}_{\Gamma,\widetilde{\Gamma}}(x) \tag{3-37}$$

where

$$C_{\Gamma,\widetilde{\Gamma}}(x) = \operatorname*{res}_{\widetilde{\Gamma},\mathfrak{X}} \frac{\widetilde{\boldsymbol{\psi}}(\xi)\boldsymbol{\varphi}^t(\xi)\mathbb{K}}{x - X(\xi)} \tag{3-38}$$

$$C_{\widetilde{\Gamma},\Gamma}(x) = \operatorname*{res}_{\Gamma,\mathfrak{X}} \frac{\boldsymbol{\psi}(\xi)\widetilde{\boldsymbol{\varphi}}^t(\xi)\widetilde{\mathbb{K}}}{x - X(\xi)} = C_{\Gamma,\widetilde{\Gamma}}(x)^{-1} \tag{3-39}$$

Proof. The product $\widetilde{\boldsymbol{\psi}}(\xi)\mathfrak{K}(p,\xi)$ has poles in ξ only at $\widetilde{\Gamma}$, $\mathfrak{X}$ and at $\xi = p$ with residue $-\widetilde{\boldsymbol{\psi}}(p)$. Hence –as before–

$$\widetilde{\boldsymbol{\psi}}(p) = \operatorname*{res}_{\widetilde{\Gamma},\mathfrak{X}} \widetilde{\boldsymbol{\psi}}(\xi)\mathfrak{K}(p,\xi) = \operatorname*{res}_{\widetilde{\Gamma},\mathfrak{X}} \frac{\widetilde{\boldsymbol{\psi}}(\xi)\boldsymbol{\varphi}^t(p)\mathbb{K}}{X(p) - X(\xi)} \boldsymbol{\psi}(p) \tag{3-40}$$

from which the expression for $C_{\Gamma,\widetilde{\Gamma}}(x)$ follows. The expression for $C_{\widetilde{\Gamma},\Gamma}(x)$ and the fact that it is the inverse of $C_{\Gamma,\widetilde{\Gamma}}(x)$ follows from simply interchanging the rôles of Γ and $\widetilde{\Gamma}$. Analogous expressions can be found for the change of divisor for the matrix $\widetilde{A}_\Gamma(x)$ (using the basis of forms $\boldsymbol{\varphi}$). **Q.E.D.**

Note that the *transition matrices* $C_{\Gamma,\widetilde{\Gamma}}(x)$ are rational functions of x with poles only at the X–projection of the divisor $\widetilde{\Gamma}$ and with a polynomial part of degree equal to the degree of the subdivisor of $\widetilde{\Gamma}$ that coincides with some poles of X.

3.2.4 Change of line bundle

The matrix $A(x)$ also depends on the chosen third–kind differential η and we investigate what happens when we vary the differential within the same class.

Namely, let $\widetilde{\eta}$ be another third kind differential and let $\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\psi}}, \widetilde{\mathbb{K}}, \widetilde{\mathfrak{K}}$ be the same objects constructed before but using the differential $\widetilde{\eta}$ instead of η.

The one-form (in q)

$$\mathbb{K}(p,q)\widetilde{\boldsymbol{\psi}}(q) \tag{3-41}$$

has poles at $p=q$ with residue $-\widetilde{\boldsymbol{\psi}}(p)$, at $\mathfrak{X}$ and essential singularities at the poles of $\eta, \widetilde{\eta}$. The transition matrix is then given by deformation of contours

$$\widetilde{\boldsymbol{\psi}}(p) = \mathop{\mathrm{res}}_{q\in\mathfrak{X}\cup\{c_j,\widetilde{c}_j\}} \mathbb{K}(p,q)\widetilde{\boldsymbol{\psi}}(q) = \mathop{\mathrm{res}}_{q\in\mathfrak{X}\cup\{c_j,\widetilde{c}_j\}} \frac{\widetilde{\boldsymbol{\psi}}(q)\boldsymbol{\varphi}^t(q)\mathbb{K}}{X(p)-X(q)}\boldsymbol{\psi}(p) = M_{\eta,\widetilde{\eta}}(X(p))\boldsymbol{\psi}(p) \tag{3-42}$$

with

$$M_{\eta,\widetilde{\eta}}(x) = \mathop{\mathrm{res}}_{q\in\mathfrak{X}\cup\{c_j,\widetilde{c}_j\}} \frac{\widetilde{\boldsymbol{\psi}}(q)\boldsymbol{\varphi}^t(q)\mathbb{K}}{x-X(q)} \tag{3-43}$$

Here by the symbol res we simply mean the integral around a small loop encircling the point (the differential is not meromorphic there but has essential singularities). The matrix $M_{\eta,\widetilde{\eta}}(x)$ has thus essential singularities (in general) in the complex x–plane at the points $X(c_j), X(\widetilde{c}_j)$.

Consequently the Lax matrices are related by a simple conjugation.

$$A_\eta(x) = M_{\eta,\widetilde{\eta}}(x)^{-1}A_{\widetilde{\eta}}(x)M_{\eta,\widetilde{\eta}}(x) \tag{3-44}$$

Note that –by construction– the Lax matrices are still rational with the same pole structure even if related by a conjugation with a non-rational matrix. These formulæ provide the integration of any isospectral dynamics on rational matrix-valued functions available in the literature on integrable systems (see, e.g. [14, 1, 2, 28] and references therein) and are, in fact, even more general. Indeed if η depends (smoothly) on one or several a "time" parameters, the above formulæ would provide integration of the flow on the isospectral manifold induced by the dependence of η.

Note that the framework we are proposing is more general than the one in [1, 2] since the differential η need not have poles coinciding with any of the poles of X (which is the case in loc. cit.).

3.2.5 Linear (smooth) deformations of the line bundle χ_η

If η depends *linearly* (or even smoothly) on a set of times generically denoted by t then $\dot{M}_\eta M_\eta^{-1}$ is a **rational** matrix (i.e. without essential singularities) as long as the residues of η_t are independent of t.

To show this suppose $\eta = \eta_t$ depends smoothly on a parameter t; in the literature the dependence is taken to be linear in the sense that the coefficients of the singular parts at the poles (in some chosen and fixed local coordinate) evolve linearly in t but the statement we are making here is more general in that it may include **any** deformation, including a motion of the position of the poles.

Let $\boldsymbol{\psi}_t, \boldsymbol{\varphi}_t$ be the dual bases of sections evolving in a smooth way; the reader should realize that this evolution implies a "gauge arbitrariness" consisting in the freedom of (smooth) change of basis within the same vector spaces of sections. This arbitrariness makes no difference on the rational nature of the infinitesimal deformation.

Denoting by a dot the "time" derivative we note that $\dot{\boldsymbol{\psi}}$ (and $\dot{\boldsymbol{\varphi}}$) are then sections of the same tensor space with additional singularities. Indeed, near any of the poles c_j of η we have $\boldsymbol{\psi} = f_t(p)\mathrm{e}^{\int^p \eta_t}$, with $f_t(p)$ analytic near c_j

$$\dot{\boldsymbol{\psi}} = \left(\int^p \dot{\eta}_t f_t + \dot{f}_t\right)\mathrm{e}^{\int^p \eta_t} \ . \tag{3-45}$$

Note that $\int^p \dot{\eta}_t$ near a pole of η_t has a pole singularity (without logarithmic term) because the residues of η_t are independent of t and hence $\dot{\eta}_t$ is a second-kind Abelian differential.

Applying the argument that have already used several times before, we obtain

$$\dot{\boldsymbol{\psi}}_t(p) = -\operatorname*{res}_{q=p} \mathfrak{K}_t(p,q)\dot{\boldsymbol{\psi}}_t(q) \ . \tag{3-46}$$

The expression we are taking the residue of, is a differential with poles only along $\mathfrak{X}+p$ and along the divisor of poles of η_t –which we denote by $\mathfrak{C}$–, due to $\dot{\eta}_t$. Thus

$$\dot{\boldsymbol{\psi}}_t(p) = -\operatorname*{res}_{q=p} \mathfrak{K}_t(p,q)\dot{\boldsymbol{\psi}}_t(q) = \operatorname*{res}_{q\in\mathfrak{X}+\mathfrak{C}} \frac{\dot{\boldsymbol{\psi}}_t(q)\boldsymbol{\varphi}_t(q)\mathbb{K}_t}{X(p)-X(q)}\boldsymbol{\psi}_t(p). \tag{3-47}$$

Thus

$$\dot{M}_t(x)M_t(x)^{-1} = \operatorname*{res}_{q\in\mathfrak{X}+\mathfrak{C}} \frac{\dot{\boldsymbol{\psi}}_t(q)\boldsymbol{\varphi}_t(q)\mathbb{K}_t}{x-X(q)} \tag{3-48}$$

Since the differential in the numerator is **meromorphic** (without essential singularities) on the spectral curve $\mathcal{L}$, the latter expression is rational in $x \in \mathbb{C}$.

3.3 Spectral bidifferential

Suppose we are given a rational $(R+1)\times(R+1)$ matrix $A(x)$ and let us denote by $y_a(x)$ its (generically simple) $R+1$ eigenvalues; consider the following bidifferential

$$S((x,y_a(x));(x',y_b(x'))) := \frac{dx\,dx'}{(x-x')^2}\frac{\mathrm{Tr}(\widetilde{A-y_a})(x)(\widetilde{A-y_b})(x'))}{\mathrm{Tr}((\widetilde{A-y_a})(x))\mathrm{Tr}((\widetilde{A-y_b})(x'))} \tag{3-49}$$

where the tilde denotes the matrix of co-factors (the classical adjoint). Since

$$\Pi_a(x) := \frac{(\widetilde{A - y_a})(x)}{\mathrm{Tr}((\widetilde{A - y_a})(x))} \tag{3-50}$$

is the (generically rank–one) projection onto the eigenspace with eigenvalue $y_a(x)$, it is not difficult to see [9] that this bidifferential extends naturally to the spectral curve and, in fact, we are now going to prove this in general. This object has appeared in the context of isomonodromic deformations of rational connections in the sense of [17]; indeed in [9] it was shown that this bidifferential is the generating function of the Hessian of the logarithm of the isomonodromic tau function.

Here we are not considering such deformations but we can relate easily this bidifferential to the ν–Cauchy kernel introduced above. Let p, q be the abstract points on $\mathcal{L}$ with coordinates $(x, y_a(x))$ and $(x', y_b(x'))$. Note that the expression (3-49) seems to have at first sight a double pole whenever two points project to the same x-value, but in fact this occurs only when the branch of the eigenvalue is the same, meaning that it is a double pole only on the diagonal of the symmetric product of the spectral curve with itself.

Indeed it follows immediately that since $\Pi_a(x)\mathrm{d}x = \boldsymbol{\psi}(p)\boldsymbol{\varphi}^t(p)\mathbb{K}$ (and letting $x = X(p), x' = X(q)$)

$$S(p,q) = \mathrm{Tr}\left(\frac{\boldsymbol{\psi}(p)\boldsymbol{\varphi}^t(p)\mathbb{K}}{X(p)-X(q)}\frac{\boldsymbol{\psi}(q)\boldsymbol{\varphi}^t(q)\mathbb{K}}{X(p)-X(q)}\right) = \frac{\boldsymbol{\varphi}^t(p)\mathbb{K}\boldsymbol{\psi}(q)}{X(p)-X(q)}\frac{\boldsymbol{\varphi}^t(q)\mathbb{K}\boldsymbol{\psi}(p)}{X(p)-X(q)} = \mathfrak{K}(p,q)\mathfrak{K}(q,p) \tag{3-51}$$

so that the spectral bidifferential is nothing but the symmetric square of the ν–Cauchy kernel. It will be shown that $S(p,q)$ is the square of the Szegö kernel in Cor. 3.3; notice that, for the time being, the symmetric square has only a double pole on the diagonal $p = q$ and no other singularities (this follows from the divisor properties of $\mathfrak{K}(p,q)$, (3-14) and the type of essential singularities.

3.4 Elementary twisting lattice and dual wave functions

Suppose that we specify a sequence of **elementary** divisors $\mathfrak{T}_n$ of degree 0, $n \in \mathbb{Z}$; by "elementary" we mean that they are of the form

$$\mathfrak{T}_n = \infty_n^{(+)} - \infty_n^{(-)} \tag{3-52}$$

where $\infty_n^{(+)}, \infty_n^{(-)}$ are two sequences of points arbitrarily (but generically) chosen. We will assume that, for any m and n, $\infty_m^{(+)} \neq \infty_n^{(-)}$'s are distinct from each other (but points within the same sequence may be repeated).

If we twist the "initial" divisor Γ

$$\Gamma_0 := \Gamma$$

$$\Gamma_n := \left\{ \begin{array}{ll} \Gamma_{n-1} + \mathfrak{T}_n & n \geq 1 \\ \Gamma_{n+1} - \mathfrak{T}_{n+1} & n \leq -1 \end{array} \right. \tag{3-53}$$

we obtain a sequence of divisors Γ_n of degree $(2\nu-1)(g-1)+R+1$. Same strategy as before can still be applied (generically), namely we have a corresponding sequence of bases $\boldsymbol{\psi}_n$ and $\boldsymbol{\varphi}_n$ and of Christoffel–Darboux kernels $\mathfrak{K}_n(p,\xi)$ all satisfying

$$\mathfrak{K}_n(p,q) = \frac{\boldsymbol{\varphi}_n^t(q)\mathbb{K}_n\boldsymbol{\psi}_n(p)}{X(p)-X(q)}\ , \qquad (\mathfrak{K}_n(p,q))_{\substack{p\\q}} \geq \left\{ \begin{array}{l} -\Gamma_n + \mathfrak{X} - q \\ \Gamma_n - \mathfrak{X} - p \end{array} \right. \tag{3-54}$$

Let us define $\boldsymbol{\infty}^{(+)}$ as the **set** of all points $\infty_n^{(+)}$ (counted without multiplicity with which the may appear in our sequence).

Definition 3.1 *The divisor $\boldsymbol{\infty}^{(+)} = \bigcup\{\infty_n^{(+)}\}$ will be called the* **dualization divisor**.

Proposition 3.5 *The sequence of kernels $\mathfrak{K}_n(p,q)$ satisfies*

$$\mathop{\mathrm{res}}_{\xi\in\boldsymbol{\infty}^{(+)}} \mathfrak{K}_n(p,\xi)\mathfrak{K}_m(\xi,q) = \left\{ \begin{array}{ll} 0 & m \leq n \\ \mathfrak{K}_m(p,q) - \mathfrak{K}_n(p,q) & m > n \end{array} \right. \tag{3-55}$$

The proof is a simple inspection of the residues; the product of the two kernels is a differential in ξ that has no poles on $\boldsymbol{\infty}^{(+)}$ if $n \geq m$; and if $n < m$ then it has only poles at a finite number of points of $\boldsymbol{\infty}^{(+)}$ and at $\xi = p, \xi = q$, where it has the indicated residues.

Consider now the difference $\mathfrak{K}_{n+1}(p,q) - \mathfrak{K}_n(p,q)$; since both kernels have ν–residue 1 on the diagonal, this difference is regular there. Moreover

$$(\mathfrak{K}_{n+1}(p,q) - \mathfrak{K}_n(p,q))_p \geq -\Gamma_n - \infty_{n+1}^{(+)} + \mathfrak{X} \tag{3-56}$$

$$(\mathfrak{K}_{n+1}(p,\xi) - \mathfrak{K}_n(p,q))_q \geq \Gamma_n - \infty_{n+1}^{(-)} - \mathfrak{X} \tag{3-57}$$

The two divisors on the right hand side of these inequalities have degree $-(2\nu-1)(g-1)-1$ and $(2\nu-1)(g-1)-1$ respectively; it follows that (generically) there is a unique meromorphic $(1-\nu)$–differential π_n and a unique meromorphic ν–differential ρ_n in the respective spaces specified by these divisors. We have proved that

Corollary 3.2 *The difference of two consecutive Christoffel–Darboux kernels in the generalized Toda sequence factors*

$$\mathfrak{K}_{n+1}(p,q) - \mathfrak{K}_n(p,q) = \pi_n(p)\rho_n(q), \tag{3-58}$$

with $\pi_n(p), \rho_n(q)$ defined (up to multiplicative constants) by the requirements

$$(\pi_n(p)) \geq -\Gamma_n - \infty_{n+1}^{(+)} + \mathfrak{X} \tag{3-59}$$

$$(\rho_n(q)) \geq \Gamma_n - \infty_{n+1}^{(-)} - \mathfrak{X} \tag{3-60}$$

By induction,

$$\mathfrak{K}_{n+L}(p,q) - \mathfrak{K}_n(p,q) = \sum_{j=n}^{n+L-1} \pi_j(p)\rho_j(q). \tag{3-61}$$

Remark 3.2 *The name of "generalized Toda lattice" is due to the fact that if X is the projection of a hyperelliptic curve (with two simple poles $\mathcal{L} \ni \infty_\pm$ above $x = \infty$) and we use the sequence of elementary divisors $\mathfrak{T}_n = \infty_+ - \infty_-$ then we recover a setting of the standard Toda lattice theory by looking at suitable isospectral evolution.*

Using the Christoffel–Darboux kernels $\mathfrak{K}_n$ one can therefore reconstruct a sequence of Lax matrices $A_n(x)$ all sharing the same spectral curve and connected by conjugation by the transition matrices $C_{n,m}(x) := C_{\Gamma_n,\Gamma_m}(x)$ introduced in Prop. 3.4; these **ladder** matrices satisfy the obvious relations (which entail discrete integrability)

$$C_{n,m}(x)C_{m,\ell}(x) = C_{n,\ell}(x)\ , \qquad \forall n,m,\ell \in \mathbb{Z}. \tag{3-62}$$

$$C_{n,m}(X(p))\boldsymbol{\psi}_n(p) = \boldsymbol{\psi}_m(p) \tag{3-63}$$

$$\boldsymbol{\varphi}^t_n(p)\mathbb{K}_n C_{n,m}(X(p)) = \boldsymbol{\varphi}^t_m(p)\mathbb{K}_m \tag{3-64}$$

Remark 3.3 *The previous construction is a generalization of the "discrete variable Baker-Akhiezer function", an idea originally formulated in [21] which is the hinge of the theory of commuting difference operators.*

3.4.1 Dualization

Note that

$$\operatorname*{res}_{\boldsymbol{\infty}^{(+)}} \rho_m\pi_n = \delta_{mn} \tag{3-65}$$

since if $m \neq n$ the product is a differential with a polar divisor of degree 2 supported only at the points $\infty_n^{(+)}$'s or only at the points $\infty_n^{(-)}$'s; only if $m = n$ one pole is at $\infty_n^{(+)}$ and one at $\infty_n^{(-)}$ so that the residue over all $\boldsymbol{\infty}^{(+)}$ is nonzero. The fact that this residue is actually 1 follows from Prop. 3.5

$$\operatorname*{res}_{\boldsymbol{\infty}^{(+)}} \left(\mathfrak{K}_{n+1}(p,\xi) - \mathfrak{K}_n(p,\xi)\right)\left(\mathfrak{K}_{m+1}(\xi,q) - \mathfrak{K}_m(\xi,q)\right) = \delta_{mn}\left(\mathfrak{K}_{n+1}(p,q) - \mathfrak{K}_n(p,q)\right) \tag{3-66}$$

which implies that $\operatorname*{res}_{\boldsymbol{\infty}^{(+)}} \rho_n\pi_n = 1$.

In addition we have

$$\operatorname*{res}_{\boldsymbol{\infty}^{(+)}} \rho_m(p)\mathfrak{K}_n(p,q) = \left\{ \begin{array}{cc} 0 & m \geq n \\ \rho_m & m < n \end{array}\right. \qquad \operatorname*{res}_{\boldsymbol{\infty}^{(+)}} \pi_m(q)\mathfrak{K}_n(p,q) = \left\{ \begin{array}{cc} -\pi_m & m \geq n \\ 0 & m < n \end{array}\right. \tag{3-67}$$

3.5 Expressions in terms of Theta functions

We now present explicit expressions of all the objects introduced so far. Let us decompose the divisor Γ (of degree $(2\nu-1)(g-1)+R+1$) as

$$\Gamma = \gamma_1 + \ldots + \gamma_{R+1} + \sum_{\ell=1}^{2\nu-1} \Gamma_0^{(\ell)} \tag{3-68}$$

where $\Gamma_0^{(\ell)}$ are divisors of degree $g-1$ and $\gamma_1, \ldots, \gamma_{R+1}$ are R points singled out arbitrarily.

Let $\xi_1, \ldots \xi_{2\nu-1}$ be arbitrary fixed points (the final formulæ will only have a fictitious dependence on them).

Theorem 3.1 *The twisted ν–Cauchy kernels of Sect. 3.1 are given by*

$$\mathfrak{K}_n(p,q) = \frac{T_n(q)}{T_n(p)} \frac{F_{\eta,n}(p,q)}{E(p,q)} \mathrm{e}^{-2i\pi \int_q^p \eta} \tag{3-69}$$

where

$$F_{\eta,n}(p,q) := \frac{\Theta \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} (p - q - \Gamma_n - (2\nu-1)\mathcal{K} + \mathfrak{X})}{\Theta \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} (\mathfrak{X} - \Gamma_n - (2\nu-1)\mathcal{K})} \prod_{j=1}^{K} \left(\frac{\Theta_\Delta(q - c_j)}{\Theta_\Delta(p - c_j)} \right)^{-2i\pi t_j} \tag{3-70}$$

$$T_n(q) = \prod_{\ell=1}^{2\nu-1} \frac{\Theta(q - \Gamma_0^{(\ell)} - \xi_\ell - \mathcal{K})}{E(q,\xi_\ell) h_\Delta(\xi_\ell)} \prod_{j=1}^{R+1} \frac{\Theta_\Delta(q-\gamma_j)}{\Theta_\Delta(q-\infty_j)} \times \left\{ \begin{array}{ll} \displaystyle\prod_{k=1}^{n} \frac{\Theta_\Delta(q-\infty_k^{(+)})}{\Theta_\Delta(q-\infty_k^{(-)})} & n \geq 0 \\ \displaystyle\prod_{k=1}^{-n} \frac{\Theta_\Delta(q-\infty_k^{(-)})}{\Theta_\Delta(q-\infty_k^{(+)})} & n < 0 \end{array} \right. \tag{3-71}$$

$$\mathcal{A} = 2 \left[\oint_{a_1} \eta, \ldots, \oint_{a_g} \eta \right]^t \in \mathbb{C}^g \tag{3-72}$$

$$\mathcal{B} = 2 \sum_{j=1}^{K} t_j \mathfrak{u}(c_j) - 2 \left[\oint_{b_1} \eta, \ldots, \oint_{b_g} \eta \right]^t \in \mathbb{C}^g \tag{3-73}$$

Correspondingly the dual $(\nu, 1-\nu)$–differentials ρ_n, π_n are given by

$$\begin{aligned} \rho_n(q) &= \frac{T_n(q) \prod_{j=1}^K \Theta_\Delta(q-c_j)^{-2i\pi t_j}}{C_n \Theta_\Delta(q - \infty_{n+1}^{(-)})} \Theta \begin{bmatrix} -\mathcal{A} \\ -\mathcal{B} \end{bmatrix} \left(q + \Gamma_n - (2\nu-1)\mathcal{K} - \mathfrak{X} - \infty_{n+1}^{(-)} \right) \mathrm{e}^{2i\pi \int_{p_0}^q \eta} h_\Delta(q) \\ \pi_n(p) &= \frac{{T_n}^{-1}(p) \prod_{j=1}^K \Theta_\Delta(p-c_j)^{2i\pi t_j}}{\Theta_\Delta(p - \infty_{n+1}^{(+)})} \Theta \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} \left(p - \Gamma_n + (2\nu-1)\mathcal{K} + \mathfrak{X} - \infty_{n+1}^{(+)} \right) \mathrm{e}^{-2i\pi \int_{p_0}^p \eta} h_\Delta(p) \end{aligned} \tag{3-74}$$

$$C_n := \frac{\Theta\left[\begin{smallmatrix}\mathcal{A}\\ \mathcal{B}\end{smallmatrix}\right]((2\nu-1)\mathcal{K}+\mathfrak{X}-\Gamma_n)\,\Theta\left[\begin{smallmatrix}\mathcal{A}\\ \mathcal{B}\end{smallmatrix}\right]((2\nu-1)\mathcal{K}+\mathfrak{X}-\Gamma_{n+1})}{\Theta_\Delta\left(\infty^{(+)}_{n+1}-\infty^{(-)}_{n+1}\right)}\ . \tag{3-75}$$

They are defined up to rescaling by λ, λ^{-1} respectively (corresponding to different choices of the common basepoint p_0 in the integrations above) and satisfy

$$\operatorname*{res}_{\infty^{(+)}} \rho_n \pi_m = \delta_{mn} \tag{3-76}$$

Proof. The proof is a straightforward check that the proposed expression satisfies the defining properties in Eqs. (3-54). The expression $T_n(q)$ in eq. 3-71 has the following properties;

- it has a tensor-weight of $\nu - \frac{1}{2}$ in the variable q, i.e. can be written in a local coordinate as $f(z)(\mathrm{d}z)^{\nu-\frac{1}{2}}$ and some multivaluedness around nontrivial cycles;
- it has zeroes at Γ_n and poles at $\mathfrak{X}$;
- it is **projectively-independent** (i.e. independent up to a multiplicative constant) of ξ_j as long as they are not chosen so that the divisors appearing in the numerator are special. Indeed two different choices give functions with the same multivaluedness and the same divisor properties, hence proportional by a constant.

The expression $G_n(p,q) = \frac{T_n(q)}{T_n(p)}\frac{1}{E(p,q)}$ then has tensor weight ν in q and $1-\nu$ in p and the divisor properties

$$(G_n(p,\bullet)) \geq \Gamma_n - \mathfrak{X} - p\ , \qquad (G_n(\bullet,q)) \geq -\Gamma_n + \mathfrak{X} - q \tag{3-77}$$

where Γ_n are defined in eq. (3-53). The remaining pieces of the formula make the final expression single valued and with the correct essential singularities. **Q.E.D.**

As a corollary we derive the promised relation between the spectral bidifferential and the Szegö kernels

Corollary 3.3 *The spectral bidifferential defined in Eq. 3-49 is given by*

$$\begin{aligned} S(p,q) &= \mathfrak{K}_n(p,q)\mathfrak{K}_n(q,p) = \\ &= \frac{\Theta\left[\begin{smallmatrix}\mathcal{A}\\ \mathcal{B}\end{smallmatrix}\right](p-q-\mathbf{e}_n)\,\Theta\left[\begin{smallmatrix}\mathcal{A}\\ \mathcal{B}\end{smallmatrix}\right](q-p-\mathbf{e}_n)}{\Theta\left[\begin{smallmatrix}\mathcal{A}\\ \mathcal{B}\end{smallmatrix}\right](-\mathbf{e}_n)^2\,E^2(p,q)} \end{aligned} \tag{3-78}$$

$$\mathbf{e}_n := \mathfrak{X} - \Gamma_n - (2\nu-1)\mathcal{K} \tag{3-79}$$

In [15], p. 26 we find that

$$S(p,q) = \Omega(p,q) + \sum_{j,k=1}^{g} \frac{\partial^2 \ln\Theta}{\partial_{\mathrm{u}_j}\partial_{\mathrm{u}_k}}(\mathbf{e}_n)\omega_j(p)\omega_k(q) \tag{3-80}$$

where Ω is the **normalized** fundamental bidifferential (also known as Bergman kernel), such that $\oint_{a_j} \Omega \equiv 0$.

Moreover, $S(p,q)$ is the square of the Szegö kernel with complex characteristics; specifically, if ρ, ϵ are the (half)-characteristics of $\mathbf{e} = \Gamma + (2\nu - 1)\mathcal{K} - \mathfrak{X} + \mathcal{A} + \tau\mathcal{B}$

$$\mathbf{e} = 2\rho + 2\tau\epsilon \tag{3-81}$$

then the above can also be rewritten as

$$S(p,q) = \frac{\Theta\left[\begin{smallmatrix}\rho\\ \epsilon\end{smallmatrix}\right](p-q)\Theta\left[\begin{smallmatrix}\rho\\ \epsilon\end{smallmatrix}\right](q-p)}{\Theta\left[\begin{smallmatrix}\rho\\ \epsilon\end{smallmatrix}\right](0)^2 E^2(p,q)} = \mathcal{S}_{\rho,\epsilon}(p,q)\mathcal{S}_{\rho,\epsilon}(q,p)\ , \tag{3-82}$$

where the Szegö kernel with characteristics is defined by

$$\mathcal{S}_{\rho,\epsilon}(p,q) = \frac{\Theta\left[\begin{smallmatrix}\rho\\ \epsilon\end{smallmatrix}\right](p-q)}{\Theta\left[\begin{smallmatrix}\rho\\ \epsilon\end{smallmatrix}\right](0)E(p,q)}\ . \tag{3-83}$$

4 Finite band recurrence relations and commuting (pseudo) difference operators

Let us partition the polar divisor of X into two **disjoint** subdivisors $\mathfrak{X} = \mathfrak{X}^{(+)} + \mathfrak{X}^{(-)}$ of degree d and $R+1-d$ respectively :

$$(X)_- = -\sum_{j=0}^{d-1} \infty_j^{(+)} - \sum_{j=0}^{R-d} \infty_j^{(-)} =: -\mathfrak{X} \tag{4-1}$$

$$\deg(\mathfrak{X}) = R+1\ . \tag{4-2}$$

We will choose the divisor $\mathfrak{X}^{(+)}$ as our **dualization divisor**.

This means that according to the general scheme in Sect. 3.4 we will choose all the points $\infty_j^{(+)}$ within $\mathfrak{X}^{(+)}$ (hence the same symbol is used). Correspondingly, all the points $\infty_j^{(-)}$ are chosen within $\mathfrak{X}^{(-)}$.

In this expression the points $\infty_j^{(\pm)}$ are **not supposed** to be necessarily distinct (within the same subset), so that we can consider poles of arbitrary order for X.

We will postulate that

$$\infty_{j+r}^{(-)} \equiv \infty_j^{(-)}\ , \qquad r := R-d+1 \tag{4-3}$$

$$\infty_{j+d}^{(+)} \equiv \infty_j^{(+)} \tag{4-4}$$

and assume that X has at least two distinct poles ($r \geq 1$); the modifications for the case of a single pole are left to the reader. We also fix a third kind differential η as in Sect. 3.1.1.

Define the divisors

$$\mathfrak{U}_n := \begin{cases} -\Gamma + \mathfrak{X} - \sum_{j=1}^{n+1} \infty_j^{(+)} + \sum_{\ell=1}^{n} \infty_\ell^{(-)} & n \geq 0 \\ -\Gamma + \mathfrak{X} + \sum_{j=n+1}^{0} \infty_j^{(+)} - \sum_{\ell=n}^{0} \infty_\ell^{(-)} & n < 0 \end{cases} \tag{4-5}$$

$$\deg(\mathfrak{U}_n) = -(2\nu - 1)(g-1) - 1 \tag{4-6}$$

$$\mathfrak{V}_n := \begin{cases} \Gamma - \mathfrak{X} + \sum_{j=1}^{n} \infty_j^{(+)} - \sum_{\ell=1}^{n+1} \infty_\ell^{(-)} & n \geq 0 \\ \Gamma - \mathfrak{X} - \sum_{j=-n}^{0} \infty_j^{(+)} + \sum_{\ell=-n+1}^{0} \infty_\ell^{(-)} & n < 0 \end{cases} \tag{4-7}$$

$$\deg(\mathfrak{V}_n) = (2\nu - 1)(g-1) - 1 \tag{4-8}$$

In the formulæ above it is understood that the sums are zero if the ranges are empty. The main point of these definitions is that

$$\mathfrak{U}_n = -\Gamma_n + \mathfrak{X} - \infty_{n+1}^{(+)}\ , \qquad \mathfrak{V}_n = \Gamma_n - \mathfrak{X} - \infty_{n+1}^{(-)}\ . \tag{4-9}$$

and these, in view of Sect. 3.4, are precisely the divisors characterizing (up to scalar multiplication)[7] π_n and ρ_n

$$\begin{array}{ll} (\pi_n) \geq \mathfrak{U}_n & \text{a } (1-\nu)\text{–differential} \\ (\rho_n) \geq \mathfrak{V}_n & \text{a } \nu\text{–differential} \end{array} \tag{4-10}$$

These two sequences span vector spaces dual to each other under **Serre's** duality;

$$\operatorname*{res}_{\mathfrak{X}^{(+)}} \pi_n \rho_m = \delta_{mn} \tag{4-11}$$

Another crucial point that motivates the choice of twisting divisors is that now

$$\mathbb{C}\{\pi_j\}_{d-R\leq j\leq d} = \mathcal{H}_{1-\nu}(-\Gamma_0) \tag{4-12}$$

$$\mathbb{C}\{\rho_j\}_{-d\leq j\leq R-d} = \mathcal{H}_{\nu}(\Gamma_0 - 2\mathfrak{X}) \tag{4-13}$$

and more generally

$$\mathbb{C}\{\pi_j\}_{d-R+n\leq j\leq n+d} = \mathcal{H}_{1-\nu}(-\Gamma_n) \tag{4-14}$$

$$\mathbb{C}\{\rho_j\}_{n-d\leq j\leq n+R-d} = \mathcal{H}_{\nu}(\Gamma_n - 2\mathfrak{X}) \tag{4-15}$$

and hence we can conveniently choose them as components of the vectors $\boldsymbol{\psi}_n, \boldsymbol{\varphi}_n$ used in the general construction.

[7] The essential singularities described in Sect. 3.1.1 are implied.

In this fashion, the vectors $\boldsymbol{\psi}_n$ and $\boldsymbol{\varphi}_n$ are **windows** of consecutive $R+1$ elements within a pair of infinite vectors

$$\Psi := \begin{bmatrix} \vdots \\ \pi_{n-R+d} \\ \vdots \\ \vdots \\ \pi_n \\ \vdots \\ \pi_{n+d} \\ \vdots \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} \vdots \\ \vdots \\ \rho_{n-d} \\ \vdots \\ \rho_n \\ \vdots \\ \vdots \\ \rho_{n+R-d} \\ \vdots \end{bmatrix} =: \Phi$$

Definition 4.1 *The vectors Ψ, Φ will be called the **wave–vectors**. The dual **windows** are defined by*

$$\boldsymbol{\psi}_n := [\pi_{n-R+d}, \ldots, \pi_{n+d}]^t \ ,$$
$$\boldsymbol{\varphi}_n := [\rho_{n-d}, \ldots, \rho_{n+R-d}]^t \ ,$$

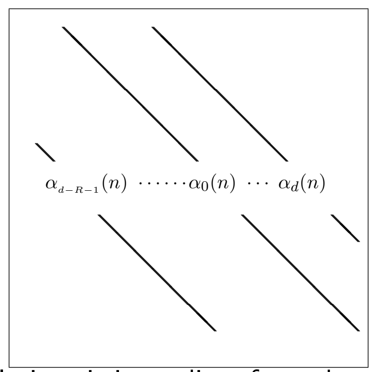
where the entries, depending on the base divisor $\Gamma = \Gamma_0$ and the line bundle associated to the differential η are given by the expressions in Thm. 3.1.

Proposition 4.1 *The wave vectors satisfy a finite band recurrence relation*

$$X\Psi = \mathbf{X}\Psi \ , \qquad X\Phi^t = \Phi^t\mathbf{X} \tag{4-16}$$

where the doubly infinite matrix $\mathbf{X}$ has a finite band structure of the form indicated on the right, and $\alpha_j(n) := \operatorname*{res}_{\mathfrak{X}^{(+)}} \rho_{n+j} X\pi_n$.

$\mathbf{X} =$



Proof. The fact that Ψ, Φ solve transposed recurrence relations is immediate from the residue–pairing

$$\operatorname*{res}_{\mathfrak{X}^{(+)}} \Psi\Phi^t = \mathbf{1} \ , \qquad \operatorname*{res}_{\mathfrak{X}^{(+)}} X\Psi\Phi^t = \mathbf{X} \tag{4-17}$$

where $\mathbf{1}$ is the infinite identity matrix. The shape of the matrix X follows from inspection of the divisor properties of π_n and $X\pi_n$. **Q.E.D.**

Keeping this in mind we can prove

Proposition 4.2 *The Christoffel–Darboux pairing $\mathbb{K}_n$ is given by the non-zero $(R+1)\times(R+1)$ block in*

$$\mathbb{K}_n := [\Pi_n, \mathbf{X}] \ , \tag{4-18}$$

where $\Pi_n = \operatorname{diag}(\ldots, \ldots, 1, 0, \ldots)$ is the projector up to n (i.e. the zero entries on the diagonal start at the entry $(n+1, n+1)$).

Proof. Using the definition of the matrix $\mathbb{K}_n$

$$(X(p) - X(q))\mathfrak{K}_n(p,q) = \boldsymbol{\varphi}_n^t(q)\mathbb{K}_n\boldsymbol{\psi}_n(p) \tag{4-19}$$

we find that the entries of $\mathbb{K}_n$ are given by

$$(\mathbb{K}_n)_{a,b} = \mathop{\mathrm{res}}_{p\in\mathfrak{X}^{(+)}} \mathop{\mathrm{res}}_{q\in\mathfrak{X}^{(+)}} (X(p) - X(q))\mathfrak{K}_n(p,q)\rho_a(p)\pi_b(q)$$
$$a = n-d, \ldots, n+R-d \ ; \quad b = n-R+d, \ldots, n+d \qquad (4\text{-}20)$$

Note that the order of the residues is irrelevant because the integrand is regular on the diagonal $p = q$. Using relations (3-67) we conclude that the nonzero entries are

$$\mathbb{K}_{ab} = \begin{cases} \mathop{\mathrm{res}}_{\mathfrak{X}^{(+)}} X\rho_a\pi_b & b \geq n+1 \ , \ a \leq n-1 \\ -\mathop{\mathrm{res}}_{\mathfrak{X}^{(+)}} X\rho_a\pi_b & b \leq n-1 \ , \ a \geq n+1 \end{cases} \qquad (4\text{-}21)$$

Explicitly, using the notation $\mathbf{X}_{nm} = \mathop{\mathrm{res}}_{\mathfrak{X}^{(+)}} \varphi_{n+k} X \psi_n = \alpha_k(n)$, we have (recall also that $\alpha_k(n) \equiv 0$ for $k > d$ and $k < d-R$)

$$\mathbb{K} = \mathbb{K}_n = -\left[\begin{array}{cccc|ccc} & & & & \alpha_d(n-d) & & \\ & & & & \vdots & \ddots & \\ & & & & \cdots & \cdots & \alpha_d(n) \\ \hline -\alpha_{d-R-1}(n+1) & \cdots & & -\alpha_{-1}(n+1) & & & \\ & \ddots & \ddots & \vdots & & & \\ & & \ddots & \vdots & & & \\ & & & \ddots & & & \\ & & & -\alpha_{d-R-1}(n+d-R) & & & \end{array}\right] \qquad (4\text{-}22)$$

This concludes the proof. **Q.E.D.**

Since all basis elements ψ_n, φ_n and the coefficients of $\mathbb{K}_n$ have been expressed in the most direct way as Theta functions or residues thereof, we have achieved our goal of providing a completely explicit expression for the solution of the inverse spectral problem described by formula (3-23) and Section 3.2.

Of course one should consider only one member of the sequence, without reference to the full sequence: so, for example, we can identify the matrix in (3-23) with the zeroth term.

4.1 Lax and ladder matrices

As it was explained in Sect. 3.4, we have a **sequence of Lax matrices** $\{A_n(x)\}_{n\in\mathbb{Z}}$ and an intertwining sequence of **ladder matrices** $\{C_n(x)\}_{n\in\mathbb{Z}}$ such that

$$A_{n+1}(x) = C_n(x)^{-1} A_n(x) C_n(x) \ . \qquad (4\text{-}23)$$

The ladder matrices C_n are linear in x as follows from the explicit formula in Sect. 3.2.3. In this context the formula reads

$$C_n(x) = \operatorname*{res}_{\mathfrak{X}^{(+)}} \frac{\boldsymbol{\psi}_{n+1}(p)\boldsymbol{\varphi}_n(p)^t\mathbb{K}_n}{X(p)-x} \tag{4-24}$$

$$\boldsymbol{\psi}_{n+1}(p) = C_n(X(p))\boldsymbol{\psi}_n(p)\ . \tag{4-25}$$

The reader should check that they have companion–like form as in [5], where the coefficients of $\mathbf{X}$ appear in the nontrivial row (column) of the ladder matrix.

The situation is very similar to the recurrence relation satisfied by orthogonal polynomials and generalization thereof [5]: in that case the recurrence relations are typically of Hessenberg form ($d=1$ in our setting).

However in the case of (generalized) orthogonal polynomials, the ladder matrices induce Schlesinger transformations for the associated Riemann–Hilbert problem, whereas here they have simply the meaning of an elementary twisting of the divisor. Note that the matrix representing multiplication by X in the infinite basis of the wave–functions Ψ (or, dually, Φ) is a finite–band matrix by construction; on the other hand multiplication by Y does not result in a finite band matrix, namely the matrix

$$\mathbf{Y}_{nm} := \operatorname*{res}_{\mathfrak{X}^{(+)}} Y\pi_m\rho_n \tag{4-26}$$

is in general a "full" doubly-infinite matrix; it has a finite number of nonzero supradiagonals (corresponding to the degree of the subdivisor of $\mathfrak{Y}$ supported at $\mathfrak{X}^{(+)}$) but in general the part below the diagonal is not finite-band.

Nonetheless the following matricial identity holds

$$[\mathbf{X},\mathbf{Y}] = 0 \tag{4-27}$$

which makes sense entry-wise since, X being finite band, the commutator involves only a finite number of terms. The commutativity clearly follows from the fact that the two matrices represent commuting multiplication operators.Thus we are looking at a pair of **commuting pseudo-difference operators**, where $\mathbf{X}$ is a bona-fide difference operator, while $\mathbf{Y}$ is not. The only case in which $\mathbf{Y}$ is of finite band structure as well is if the divisor of poles of Y coincides with the polar divisor of X (although of different multiplicity in general). We remark however that in the case of pseudo-difference operators, even if the matrix $\mathbf{Y}$ has no obvious shape it has nonetheless a hidden **rank condition**.

Indeed, under some additional genericity assumptions we can factor $\mathbf{Y}$ into the inverse of a lower-triangular matrix and a finite–band matrix.

To see this, let us separate the poles of Y into the poles $\mathfrak{Y}_x$ that are also poles of X and the "other" poles $\mathfrak{Y}_o$ of degrees d_x and d_o respectively.

In a generic situation, we can find a linear combination $\widetilde{\pi}_n \in \mathbb{C}\{\pi_n, \ldots, \pi_{n-d_o}\}$ whose divisor exceeds $\mathfrak{Y}_o$ and hence $Y\widetilde{\pi}_n$ will be a linear combination of π_{n+j}, for $|j| < d_x$; the actual shape depends on how the poles $\mathfrak{Y}_x$ are distributed into the divisors $\mathfrak{X}^{(\pm)}$.

Setting $\mathfrak{Y}_x^{(\pm)} = \mathfrak{Y}_x \cap \mathfrak{X}^{(\pm)}$ and letting $m_\pm$ be the respective degrees, we see that there is a lower–triangular matrix L with d_o subdiagonals and a finite–band matrix H with m_+ supradiagonals and m_- subdiagonals such that

$$YL\Psi = H\Psi \quad \Rightarrow \quad \mathbf{Y} = L^{-1}H \ . \tag{4-28}$$

This implies that all the submatrices below the main diagonal have rank at most d_o and this is our "hidden" rank condition.

Remark 4.1 *A more general construction is needed in application to large degree asymptotics of certain biorthogonal polynomials (we will pursue this in a different publication [8]). In this case the twisting of the base divisor is performed in a different way so that* **both** $\mathbf{X}$ *and* $\mathbf{Y}$ *are pseudo difference operators subject to similar rank conditions.*

4.2 Riemann–Hilbert problems

Although it is outside of the scope of this paper, we would like to explain what makes this investigation of potential relevance in a study of large degree asymptotics of (multi)–orthogonal polynomials.

We start with the observation that the windows $\boldsymbol{\psi}_n, \boldsymbol{\varphi}_n^t$ are eigenvectors of a matrix $A_n(x)$ which depends only on the value $x = X(p)$; since there are $R+1 = \deg(X)$ other points $p_1(x), \ldots, p_{R+1}(x) \in \mathcal{L}$ (generically distinct) with the same X–projection, the evaluation at those points provides a basis of eigenvectors. The points (and so the eigenvectors) are distinct away from the ramification divisor of the map $X : \mathcal{L} \to \mathbb{C}$ and hence the sections $p_i : \mathbb{C} \to \mathcal{L}$ are well defined only on a suitable simply connected domain obtained by removing some cuts originating at the branch-points of the X–projection.

The matrices

$$P_R(x) := \left[\frac{\boldsymbol{\psi}_n(p_1(x))}{\mathrm{d}X(p_1(x))^{1-\nu}}\mathrm{e}^{-2i\pi\int^{p_1(x)}\eta}, \ldots, \frac{\boldsymbol{\psi}_n(p_{R+1}(x))}{\mathrm{d}X(p_{R+1}(x))^{1-\nu}}\mathrm{e}^{-2i\pi\int^{p_1(x)}\eta}\right] \tag{4-29}$$

$$P_L(x) := \left[\begin{array}{c} \frac{\boldsymbol{\varphi}_n^t(p_1(x))\mathbb{K}_n}{\mathrm{d}X(p_1(x))^{\nu}}\mathrm{e}^{2i\pi\int^{p_1(x)}\eta} \\ \vdots \\ \frac{\boldsymbol{\varphi}_n^t(p_{R+1}(x))\mathbb{K}_n}{\mathrm{d}X(p_{R+1}(x))^{\nu}}\mathrm{e}^{2i\pi\int^{p_1(x)}\eta} \end{array}\right] \tag{4-30}$$

are inverses of each other. They solve a Riemann–Hilbert problem with quasipermutation monodromies around the branch-points of X (due to the permutation of columns and to the multivaluedness of columns as functions on the spectral curve itself) and diagonal multivaluedness around the X–projection of the poles of η (due to logarithmic singularities). It would not be difficult, but too long, to spell out in detail the Riemann-Hilbert data, as they include some growth conditions at infinity and at the branch-points. Note in general that we can expect singularities at a branch-point x_o of order k of the form $(x-x_o)^{\frac{\nu-1}{k}}$ for P_R and of the form $(x-x_o)^{-\frac{\nu}{k}}$ for P_L, or combination of singularities of this type if there are more than one ramification points on the spectral curve above the same branch-point.

Riemann–Hilbert problems of this sort have been used in [10, 8] in the asymptotic analysis of certain (bi)orthogonal polynomials (for the case $\nu = \frac{1}{2}$).

The point of contact between the above RHP and the ones satisfied by (multi)orthogonal polynomials in the asymptotic regime is that such problems with quasi–permutation monodromies appear when the original RHP is "normalized" by the use of a suitable collection of g–functions [12].

5 Commuting (pseudo)–difference operators in duality related to the two–matrix model

We consider now a particular case which is of relevance for the asymptotic analysis of the biorthogonal polynomials for the so–called "two–matrix model" [18, 5]; we will remark later on what are the choices of the tensor weights and divisors which are more strictly relevant to that situation.

The restriction will be that X and Y share the same polar divisor in the specific form [4]

$$(X) \geq -\infty_x - d_2\infty_y \ , \qquad (Y) \geq -d_1\infty_x - \infty_y \ . \tag{5-1}$$

We use $\infty^{(+)} = \infty_x$, $\infty^{(-)} = \infty_y$ and the same general framework used earlier, with a (generic) divisor Γ of degree $(2\nu-1)(g-1)+d_2$ and the third–kind differential η.

The wave-vectors are then characterized (up to constants) by the formulæ in Thm. 3.1 suitably specialized; for reader's convenience we recall the divisor properties

$$(\pi_n) \geq -\Gamma - n\infty_x + (n+d_2)\infty_y \ , \quad (\rho_n) \geq \Gamma + (n-1)\infty_x - (n+1+d_2)\infty_y \tag{5-2}$$

$$\mathop{\mathrm{res}}_{\infty_x} \pi_n\rho_m = \delta_{mn} \ . \tag{5-3}$$

Since the functions X, Y share the polar divisor, the sequence of wave-functions also satisfies a finite band Y–recurrence relation and hence the matrices $\mathbf{X}, \mathbf{Y}$ can be thought of as **two commuting difference operators**[25]; denoting

$$\mathbf{Y}_{m,n} := -\mathop{\mathrm{res}}_{\infty_x} Y\rho_m\pi_n \tag{5-4}$$

$$Y\Psi = -\mathbf{Y}\Psi \ , \qquad Y\Phi^t = -\Phi^t\mathbf{Y} \tag{5-5}$$

we see that the two matrices $\mathbf{X}, \mathbf{Y}$ are finite–band Hessenberg matrices with d_2 subdiagonals for $\mathbf{X}$ and d_1 supradiagonals for $\mathbf{Y}$.

On the other hand we could have switched the rôles of X and Y, used a (generic) divisor $\widetilde{\Gamma}$ of degree $(2\widetilde{\nu}-1)(g-1)+d_1$ and a differential $\widetilde{\eta}$ and repeated the whole construction so as to get another pair of sequences of wave–functions

$$(\widetilde{\pi}_n) \geq -\widetilde{\Gamma} - n\infty_y + (n+d_1)\infty_x \ , \quad (\widetilde{\rho}_n) \geq \widetilde{\Gamma} + (n-1)\infty_y - (n+1+d_1)\infty_x \tag{5-6}$$

$$\operatorname*{res}_{\infty_y} \widetilde{\pi}_n \widetilde{\rho}_m = \delta_{mn} \tag{5-7}$$

$$\widetilde{\mathbf{Y}}_{nm} = \operatorname*{res}_{\infty_y} Y \widetilde{\pi}_n \widetilde{\rho}_m \ , \qquad \widetilde{\mathbf{X}}_{nm} = -\operatorname*{res}_{\infty_y} X \widetilde{\pi}_n \widetilde{\rho}_m \tag{5-8}$$

It is clear that in general the matrices $\mathbf{X}, \mathbf{Y}$ and their tilde-counterparts have nothing in common except the shape. In the formal asymptotics of biorthogonal polynomials the two matrices should however be the same [5]: we will see below that this implies certain constraints on the divisors $\Gamma, \widetilde{\Gamma}$ and the differentials $\eta, \widetilde{\eta}$.

Suppose that

$$\eta + \widetilde{\eta} = dF = \text{exact differential} \ , \qquad \Gamma + \widetilde{\Gamma} - \mathfrak{X} - \mathfrak{Y} \equiv (\nu + \widetilde{\nu} - 1)\mathcal{C} \tag{5-9}$$

where $\mathcal{C}$ is a canonical divisor. This means that there exists a $(\nu+\widetilde{\nu}-1)$–differential whose divisor is the one above. Also we assume that $\eta, \widetilde{\eta}$ have the same polar divisor (in particular opposite residues). Under these conditions we have

Proposition 5.1 *If the divisors $\Gamma, \widetilde{\Gamma}$ and differentials $\eta, \widetilde{\eta}$ are* **dual** *in the sense of eq. (5-9) then there exists a $(\nu, \widetilde{\nu})$ –bidifferential $\mathfrak{L}$ which we call the* **Laplace kernel** *and a $(1-\nu, 1-\widetilde{\nu})$–bidifferential $\widehat{\mathfrak{L}}$ which we call the* **co-Laplace kernel** *with the properties*

$$(\mathfrak{L}(p,q))_p \geq \Gamma - \infty_x - d_2\infty_y - q \ , \qquad (\mathfrak{L}(p,q))_q \geq \widetilde{\Gamma} - d_1\infty_x - \infty_y - p \tag{5-10}$$

$$(\widehat{\mathfrak{L}}(p,q))_q \geq -\Gamma + \infty_x + d_2\infty_y - p \ , \quad (\widehat{\mathfrak{L}}(p,q))_p \geq -\widetilde{\Gamma} + d_1\infty_x + \infty_y - q \tag{5-11}$$

Along the diagonal $p=q$ they behave as

$$\mathfrak{L} \sim \mathrm{d}z^{\nu}\mathrm{d}z'^{\widetilde{\nu}} \frac{f(z)}{z-z'} + \ldots \tag{5-12}$$

$$\widehat{\mathfrak{L}} \sim \mathrm{d}z^{1-\nu}\mathrm{d}z'^{1-\widetilde{\nu}} \frac{\widehat{f}(z)}{z-z'} + \ldots \tag{5-13}$$

where $\omega := f(z)\mathrm{d}z^{\nu+\widetilde{\nu}-1}$ and $\widehat{\omega} := \widehat{f}(z)\mathrm{d}z^{1-\nu-\widetilde{\nu}}$ are invariantly defined differentials of the indicated weights.

Proof Similarly to the construction of the ν–Cauchy kernel, we split the divisors $\Gamma, \widetilde{\Gamma}$ into

$$\Gamma = \sum_{j=1}^{d_2+1} \gamma_j + \sum_{a=1}^{2\nu-1} \overbrace{\Gamma^{(a)}}^{\deg = g-1} \tag{5-14}$$

$$\widetilde{\Gamma} = \sum_{j=1}^{d_1+1} \widetilde{\gamma}_j + \sum_{a=1}^{2\widetilde{\nu}-1} \overbrace{\widetilde{\Gamma}^{(a)}}^{\deg=g-1} \tag{5-15}$$

and choose $2\nu - 1$ points ξ_a and $2\widetilde{\nu} - 1$ points $\widetilde{\xi}_b$ (again, the formulæ will depend only projectively on those). We rewrite formulæ (3-70, 3-71, 3-74) specializing them twice:

- the first time with $\infty_n^{(+)} \equiv \infty_x,\ \infty_n^{(-)} \equiv \infty_y, \mathfrak{X} = \infty_x + d_2\infty_y$;
- the second time with $\widetilde{\infty}_n^{(+)} \equiv \infty_y,\ \widetilde{\infty}_n^{(-)} \equiv \infty_x, \mathfrak{Y} = \infty_y + d_1\infty_x$.

Duality (5-9) implies that

$$\widetilde{\mathcal{A}} = -\mathcal{A}\ ,\ \ \widetilde{\mathcal{B}} = -\mathcal{B}\ ,\ \widetilde{t}_j = -t_j\ ,\ \mathfrak{u}\left(\Gamma + (2\nu-1)\mathcal{K} - \mathfrak{X}\right) = \mathfrak{u}\left(-\widetilde{\Gamma} - (2\widetilde{\nu}-1)\mathcal{K} + \mathfrak{Y}\right) \tag{5-16}$$

so that

$$F_{\eta,0}(p,q) = F_{\widetilde{\eta},0}(q,p) \tag{5-17}$$

with the quantities being defined as in eq. (3-70) and the above specializations. Then a direct inspection shows that (T_0 defined as part of (3-71))

$$\mathfrak{L}(p,q) = \frac{T_0(p)\widetilde{T}_0(q)}{E(p,q)} F_{\widetilde{\eta},0}(p,q) \exp\overbrace{\left(-2i\pi\int_p^q \eta + 2i\pi F(q)\right)}^{2i\pi F(p) - 2i\pi\int_q^p \widetilde{\eta}} \tag{5-18}$$

$$\widehat{\mathfrak{L}}(p,q) = \frac{F_{\eta,0}(p,q)}{T_0(p)\widetilde{T}_0(q)E(p,q)} \mathrm{e}^{2i\pi\int_p^q \eta - 2i\pi F(q)} \tag{5-19}$$

The $(\nu + \widetilde{\nu} - 1)$–differential and $(1 - \nu - \widetilde{\nu})$–differential advocated for in the proposition are (up to multiplicative constant)

$$\omega(p) := T_0(p)\widetilde{T}_0(p)\mathrm{e}^{2i\pi F(p)}\ , \qquad \widehat{\omega}(p) = \frac{1}{\omega(p)}\ , \tag{5-20}$$

where the fact that these expressions are single–valued follows once more from (5-9). **Q.E.D.**

Corollary 5.1 *The normalizations of the four sequences of wave–functions can be chosen in such a way that*

$$\mathop{\mathrm{res}}_{\xi=\infty_{x,y}} \mathfrak{L}(\xi,q)\pi_n(\xi) = \omega(q)\pi_n(q) = \widetilde{\rho}_n(q) \tag{5-21}$$

$$\mathop{\mathrm{res}}_{\xi=\infty_{x,y}} \widehat{\mathfrak{L}}(\xi,p)\rho_n(\xi) = \widehat{\omega}(p)\rho_n(p) = \widetilde{\pi}_n(p) \tag{5-22}$$

$$\mathop{\mathrm{res}}_{\xi=\infty_{x,y}} \mathfrak{L}(p,\xi)\widetilde{\pi}_n(\xi) = \omega(p)\widetilde{\pi}_n(p) = \rho_n(p) \tag{5-23}$$

$$\mathop{\mathrm{res}}_{\xi=\infty_{x,y}} \widehat{\mathfrak{L}}(q,\xi)\widetilde{\rho}_n(\xi) = \widehat{\omega}(p)\widetilde{\rho}_n(q) = \pi_n(q) \tag{5-24}$$

Proof. We first note that the differentials appearing in all the residues above are **meromorphic** since the essential singularities (by construction) cancel out. The only poles are *a priori* at $\infty_{x,y}$ and hence the sum over the two residues is (minus) the residue along the diagonal. Thus the statement of the theorem, keeping into account that $\omega\widehat{\omega} \equiv 1$, amounts to checking that

$$\pi_n(p) = \widehat{\omega}(p)\widetilde{\rho}_n(p) \ , \qquad \rho_n(p) = \omega(p)\widetilde{\pi}_n(p) \ . \tag{5-25}$$

The check is a straightforward computation using the explicit expressions (5-20) and the expressions for the two dual sequences derived from the suitable specializations of eqs. (3-74): the tilded sequences defined in (5-25) coincide with those obtained by specialization of (3-74) up to a rescaling $\widetilde{\rho}_n \to \lambda_n\widetilde{\rho}_n, \ \widetilde{\pi}_n \to \frac{1}{\lambda_n}\widetilde{\pi}_n$ (which leaves invariant the duality (3-76)). **Q.E.D.**

Corollary 5.2 *If the duality (5-9) is satisfied then the matrices representing multiplication by X and Y are the same (up to a transposition and a sign) in the two dual bases.*

Proof. Indeed

$$\mathbf{X}_{nm} = \mathop{\rm res}_{\infty_x} X\pi_n\rho_m = \mathop{\rm res}_{\infty_x} X\omega\pi_n\widehat{\omega}\rho_m = \mathop{\rm res}_{\infty_x} X\widetilde{\rho}_n\widetilde{\pi}_m = -\mathop{\rm res}_{\infty_y} X\widetilde{\rho}_n\widetilde{\pi}_m = -\widetilde{\mathbf{X}}_{mn} \tag{5-26}$$

$$\mathbf{Y}_{nm} = \mathop{\rm res}_{\infty_x} Y\pi_n\rho_m = \mathop{\rm res}_{\infty_x} Y\omega\pi_n\widehat{\omega}\rho_m = \mathop{\rm res}_{\infty_x} Y\widetilde{\rho}_n\widetilde{\pi}_m = -\mathop{\rm res}_{\infty_y} Y\widetilde{\rho}_n\widetilde{\pi}_m = -\widetilde{\mathbf{Y}}_{mn} \tag{5-27}$$

Q.E.D.

There are two Lax-matrices (see [5] for the analysis for biorthogonal polynomials): $A_n(x)$ of size $(d_2+1)\times(d_2+1)$ and $B_n(y)$ of size $(d_1+1)\times(d_1+1)$ which share the characteristic polynomial (by construction).

In addition, due to Corollary 3.3 and the duality 5-9 the two spectral bidifferential coincide.

This means that if we denote by $A(x)$ the Lax matrix reconstructed using $X(p)$ as spectral parameter (of dimension d_2+1) and $B(y)$ the $(d_1+1)^2$ Lax matrix obtained by using instead $Y(p)$ as spectral parameter and $X(p)$ as eigenvalue, we have the identity

$$\det(x\mathbf{1}_{d_1+1} - B(y)) = c\det(y\mathbf{1}_{d_2+1} - A(x)) \tag{5-28}$$

$$\frac{\mathrm{d}x\mathrm{d}x'}{(x-x')^2}\frac{\mathrm{Tr}\left(\widetilde{(A-y)}(x)\widetilde{(A-y)}(x')\right)}{\mathrm{Tr}\left(\widetilde{(A-y)}(x)\right)\mathrm{Tr}\left(\widetilde{(A-y)}(x')\right)} = = \frac{\mathrm{d}y\mathrm{d}y'}{(y-y')^2}\frac{\mathrm{Tr}\left(\widetilde{(B-x)}(y)\widetilde{(B-x)}(y')\right)}{\mathrm{Tr}\left(\widetilde{(B-x)}(y)\right)\mathrm{Tr}\left(\widetilde{(B-x)}(y')\right)} \tag{5-29}$$

Conclusion. We conclude the section by pointing out the specialization that will be of use in the study of the asymptotics for biorthogonal polynomials appearing in the two–matrix model; indeed, the choice of the differentials $\eta, \widetilde{\eta}$ above was too generic.

The relevant case would be $\nu = \widetilde{\nu} = \frac{1}{2}$; in this case $\Gamma, \widetilde{\Gamma}$ have degrees d_2, d_1 respectively and one should choose them as $\Gamma = d_2 \infty_y,\ \widetilde{\Gamma} = d_1 \infty_x$. The differential η is then defined by $2i\pi\eta = NY\mathrm{d}X$ and the dual one by $2i\pi\widetilde{\eta} = NX\mathrm{d}Y$ so that in (5-9), $2i\pi F = NXY$.

The parameter N in the biorthogonal polynomial context is a large parameter (corresponding to the size of the underlying matrix model) and the degree of the polynomials whose asymptotics we are interested in, is $n = N + r$ with $r \in \mathbb{Z}$ arbitrary but not scaling with N, i.e. bounded.

In the spirit of this paper we should think of N as a deformation of the *line bundle* η while the "fluctuations around the Fermi level" (using a common analogy in the physically oriented literature) should be identified with r and give rise to the generalized Toda lattice (in this case it is the 2–Toda lattice).

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