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Large deviations for multi-scale jump-diffusion processes

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Received 20 March 2015; received in revised form 31 July 2016; accepted 31 July 2016

Abstract

We obtain large deviation results for a two time-scale model of jump-diffusion processes. The processes on the two time scales are fully inter-dependent, the slow process has small perturbative noise and the fast process is ergodic. Our results extend previous large deviation results for diffusions. We provide concrete examples in their applications to finance and biology, with an explicit calculation of the large deviation rate function.

MSC: 60F10; 60J75; 49L25

Keywords: Large deviation principle; Multi-scale asymptotics; Jump diffusions

1. Introduction

For a number of processes in finance and biology the appropriate stochastic modeling is done in terms of multi-scale Markov processes with fully dependent slow and fast fluctuating variables. The most common examples of such multi-scale processes (random evolutions, diffusions, state dependent Markov chains) are all particular cases of jump-diffusions. The law of large numbers limit, central limit theorem, and the corresponding large deviations behavior of these models are all of interest in applications.

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http://dx.doi.org/10.1016/j.spa.2016.07.016

Please cite this article in press as: R. Kumar, L. Popovic, Large deviations for multi-scale jump-diffusion processes, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.07.016

One case of their use in finance is in multi-factor stochastic volatility models, which are used to capture the smiles and skews of implied volatility. The separation of time scales is helpful for calibration, since it allows one to reduce the number of group parameters. The rate function from the large deviation principle for the stock price process can be used to obtain the price of short maturity options, as well as the limit of the at-the-money implied volatility. These have been explicitly calculated for models in which the logarithm of the stock price and the stochastic volatility are driven by diffusions [18,17]. However, much of the empirical evidence [6,26] suggests that mean-reverting jump-diffusions would be a more appropriate model for the problem.

In biology one case of their use is in models of intracellular biochemical reactions. Due to low copy numbers of various key molecular types and varying strengths in chemical bonds, normalized copy numbers of different types of molecules are processes on multiple time-scales (see [2,24] for references to the biology literature). Changes in molecular compositions are modeled by state-dependent Markov chains, and on the slower time scale are well approximated by diffusions with small noise or piecewise deterministic Markov chains [25]. The rate function from the large deviation principle for slowly fluctuating molecular species is used to calculate the propensity for switching in a network that has multiple stable equilibria. Since intracellular processes are also subject to other sources of 'extrinsic' noise, multiple time-scale diffusions may include jumps from additional sources. For example, there can be errors during cell division [22,21]; a stochastic model combining both reactions and cell division was analyzed in [27].

Large deviation results for multi-scale diffusions have been studied by Freidlin (see [20] Chapter 7), Veretennikov [30], Dupuis et al. [14], and Puhalskii [28]. For the multi-scale Markov chains where the slow process is a piecewise deterministic Markov processes and the fast process is a Markov chain on a finite state space explicit results were obtained by Faggionato et al. [16,15]. For jump-diffusions there are very few large deviation results. On a single time scale, there are results by Imkeller et al. [23] for first exit times for SDEs driven by symmetric stable and exponentially light-tail symmetric Lévy processes. An approach based on control theory and the variational representation was developed by Budhiraja et al. in [7] and extended to infinite dimensional versions [8] (that is, SPDEs rather than SDEs driven by a Poisson random measure). It is not easy to see how to use these results in a multi-scale model of jump-diffusions. A special case of a multi-scale process where the slow process is a diffusion and the fast process is a mean-reverting process driven by a Levy process was studied by Bardi et al. [4], and the authors use PDE methods to prove asymptotics of an optimal control problem.

A general method for Markov processes based on non-linear semigroups and viscosity methods was developed by Feng and Kurtz in [19]. However, verifying the abstract conditions needed to apply this method to multi-scale jump-diffusions is a non-trivial task. In this paper we give a proof of large deviations for two time-scale jump-diffusions, using a technique developed by Feng et al. in [18]. The advantage of this method is that it is constructive and, with some effort, can be tailored to different multi-scale processes. Our proof follows the steps of [18], extending it to processes with jumps and full dependence of the slow and fast components. It is based on viscosity solutions to the Cauchy problem for a sequence of partial integro-differential equations and uses a construction of the sub- and super-solutions to related Cauchy problems as in [18]. Our results hold for slow and fast jump-diffusions which are fully inter-dependent, and where the fast processes are ergodic but not necessarily symmetric. In case the evolution of both processes is spatially homogeneous in the slow variables, we can also provide a more explicit (than a solution to a variational problem) formula for the rate function.

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2. Two time-scale jump-diffusion

Consider the following system of stochastic differential equations:

$$\begin{aligned} dX_{\epsilon,t} &= b(X_{\epsilon,t-}, Y_{\epsilon,t-})dt + \epsilon b_0(X_{\epsilon,t-}, Y_{\epsilon,t-})dt + \sqrt{\epsilon}\sigma(X_{\epsilon,t-}, Y_{\epsilon,t-})dW_t^{(1)} \\ &\quad + \epsilon \int k(X_{\epsilon,t-}, Y_{\epsilon,t-}, z)\tilde{N}^{\frac{1}{\epsilon}.(1)}(dz, dt), \qquad (1a) \\ dY_{\epsilon,t} &= \frac{1}{\epsilon} b_1(X_{\epsilon,t-}, Y_{\epsilon,t-})dt + \frac{1}{\sqrt{\epsilon}}\sigma_1(X_{\epsilon,t-}, Y_{\epsilon,t-})\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right) \\ &\quad + \int k_1(X_{\epsilon,t-}, Y_{\epsilon,t-}, z)\tilde{N}^{\frac{1}{\epsilon}.(2)}(dz, dt), \qquad (1b) \\ X_{\epsilon,0} &= x_0, \ Y_{\epsilon,0} = \ y_0, \end{aligned}$$

where $N^{\frac{1}{\epsilon}.(1)}(\cdot,\cdot)$, $N^{\frac{1}{\epsilon}.(2)}(\cdot,\cdot)$ are independent Poisson random measures with intensity measures $\nu_1(dz) \times \frac{1}{\epsilon}dt$, $\nu_2(dz) \times \frac{1}{\epsilon}dt$; the Lévy measures ν_1 and ν_2 satisfy $\int_{\mathbb{R}}(1 \wedge z^2)\nu_2(dz) < \infty$ and $\int_{\mathbb{R}}(1 \wedge z^2)\nu_2(dz) < \infty$; the centered versions are defined as

$$\begin{aligned} \tilde{N}^{\frac{1}{\epsilon}.(1)}(\cdot,\cdot) &= N^{\frac{1}{\epsilon}.(1)}(\cdot,\cdot) - \nu_1(dz) \times \frac{1}{\epsilon}dt, \\ \tilde{N}^{\frac{1}{\epsilon}.(2)}(\cdot,\cdot) &= N^{\frac{1}{\epsilon}.(2)}(\cdot,\cdot) - \nu_2(dz) \times \frac{1}{\epsilon}dt \end{aligned}$$

and $W^{(1)}$, $W^{(2)}$ are independent Brownian motions independent of $N^{\frac{1}{\epsilon}.(1)}(\cdot,\cdot)$, $N^{\frac{1}{\epsilon}.(2)}(\cdot,\cdot)$.

To ensure existence and uniqueness of solutions to the system (1) we assume

Assumption 2.1 (*Lipschitz Condition*)**.** There exists $K_1 > 0$ such that $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$\begin{aligned} &|b(x_2, y_2) - b(x_1, y_1)|^2 + |b_0(x_2, y_2) - b_0(x_1, y_1)|^2 + |b_1(x_2, y_2) - b_1(x_1, y_1)|^2 \\ &\quad + |\sigma(x_2, y_2) - \sigma(x_1, y_1)|^2 + |\sigma_1(x_2, y_2) - \sigma_1(x_1, y_1)|^2 \\ &\quad + \int |k(x_2, y_2, z) - k(x_1, y_1, z)|^2 \nu_1(z)dz \\ &\quad + \int |k_1(x_2, y_2, z) - k_1(x_1, y_1, z)|^2 \nu_2(z)dz \\ &\leq K_1(|x_2 - x_1|^2 + |y_2 - y_1|^2). \end{aligned} \tag{2}$$

Assumption 2.2 (*Growth Condition*)**.** There exists $K_2 > 0$ such that $\forall (x, y) \in \mathbb{R}^2$

$$\begin{aligned} &|b(x,y)|^2 + |b_0(x,y)|^2 + |b_1(x,y)|^2 + |\sigma(x,y)|^2 + |\sigma_1(x,y)|^2 \\ &\quad + \int |k_1(x,y,z)|^2 \nu_2(z)dz + \int |k(x,y,z)|^2 \nu_1(z)dz \leq K_2(1 + x^2 + y^2). \end{aligned} \tag{3}$$

Define

$$V(y; x, p) := b(x,y)p + \frac{1}{2}\sigma^2(x,y)p^2 + \int \left(e^{pk(x,y,z)} - 1 - pk(x,y,z)\right)\nu_1(z)dz. \tag{4}$$

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For each x and p in $\mathbb{R}$ there exists $K_{x,p} > -\infty$ such that

$$V(y; x, p) \geq K_{x,p} \quad \forall y \in \mathbb{R}. \tag{5}$$

If existence and uniqueness of solutions to (1a) + (1b) can be established by other means, we will only assume the growth condition i.e. Assumption 2.2, that the coefficients are continuous, and the lower bound (5) on V.

The infinitesimal generator of (X_ϵ, Y_ϵ) is for $f \in C_b^2(\mathbb{R} \times \mathbb{R})$ defined by

$$\begin{aligned}\mathcal{L}_\epsilon f(x, y) &= b(x, y)\partial_x f(x, y) + \rho\sigma(x, y)\sigma_1(x, y)\partial^2_{xy} f(x, y)\\ &+ \epsilon b_0(x, y)\partial_x f(x, y) + \frac{\epsilon}{2}\sigma^2(x, y)\partial^2_{xx} f(x, y)\\ &+ \frac{1}{\epsilon}\int (f(x + \epsilon k(x, y, z), y) - f(x, y) - \epsilon k(x, y, z)\partial_x f(x, y))\, \nu_1(z)dz\\ &+ \frac{1}{\epsilon}\Big[b_1(x, y)\partial_y f(x, y) + \frac{1}{2}\sigma_1^2(x, y)\partial^2_{yy} f(x, y)\\ &+ \int \big(f(x, y + k_1(x, y, z)) - f(x, y) - k_1(x, y, z)\partial_y f(x, y)\big)\, \nu_2(z)dz\Big].\end{aligned} \tag{6}$$

Fix $x \in \mathbb{R}$ and let Y^x denote the process satisfying the SDE

$$\begin{aligned}dY_t &= b_1(x, Y_{t-})dt + \sigma_1(x, Y_{t-})\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\right)\\ &\quad + \int k_1(x, Y_{t-}, z)\tilde{N}^{(2)}(dz, dt), \qquad Y_0^x = y_0.\end{aligned} \tag{7}$$

This is the SDE (1b) where ϵ is set equal to 1 and $X_{\epsilon,t}$ is set equal to x. Let $\mathcal{L}_1^x$ denote the generator of Y^x, then, for $f \in C_b^2(\mathbb{R})$,

$$\begin{aligned}\mathcal{L}_1^x f(y) &:= b_1(x, y)\partial_y f(y) + \frac{1}{2}\sigma_1^2(x, y)\partial^2_{yy} f(y)\\ &\quad + \int \big(f(y + k_1(x, y, z)) - f(y) - k_1(x, y, z)\partial_y f(x, y)\big)\, \nu_2(z)dz.\end{aligned} \tag{8}$$

For fixed $p \in \mathbb{R}$ define the perturbed $\mathcal{L}_1^{x,p}$ generator for $f \in C_b^2(\mathbb{R}^2)$ by

$$\begin{aligned}\mathcal{L}_1^{x,p} f(y) &:= [\rho\sigma(x, y)\sigma_1(x, y)p + b_1(x, y)]\, \partial_y f(y) + \frac{1}{2}\sigma_1^2(x, y)\partial^2_{yy} f(y)\\ &\quad + \int \big(f(y + k_1(x, y, z)) - f(y) - k_1(x, y, z)\partial_y f(x, y)\big)\, \nu_2(z)dz,\end{aligned} \tag{9}$$

and let $Y^{x,p}$ be the process corresponding to the generator $\mathcal{L}_1^{x,p}$. For each $x, p \in \mathbb{R}$ we assume the following about $Y^{x,p}$.

Assumption 2.3 (*Ergodicity Condition*)**.** The process $Y^{x,p}$ is Feller continuous with transition probability $p_t^{x,p}(y_0, dy)$, which at $t = 1$ has a positive density $p_1^{x,p}(y_0, y)$ with respect to some reference measure $\alpha(dy)$.

Assumption 2.4 (*Lyapunov Condition*)**.** There exists a positive function $\zeta(\cdot) \in C^2(\mathbb{R})$, such that ζ has compact finite level sets, and for each compact set $\Gamma \subset \mathbb{R}$, $\theta \in (0, 1]$ and $l \in \mathbb{R}$, there exists a compact set $A_{l,\theta,\Gamma} \subset \mathbb{R}$ such that

$$\{y \in \mathbb{R} : -\theta e^{-\zeta} \mathcal{L}_1^{x,p} e^{\zeta}(y) - (|V(y;x,p)| + |b_0(x,y)p| + \sigma^2(x,y)) \leq l\} \subset A_{l,\theta,\Gamma}, \quad \forall p \in \Gamma,\ \forall x \in \mathbb{R}. \tag{10}$$

Remark 2.1. In the case where the domain of Y is compact, we can define $\zeta \equiv 0$ which will satisfy Assumption 2.4.

Remark 2.2. Some arguments are simpler in the special case $Y^{x,p}$ in addition has a unique invariant probability measure $\pi^p(x,\cdot)$ with respect to which $p_t^{x,p}(y_0, y)$ is symmetric and $\pi^p(x,\cdot)$ is reversible, that is

$$\int_{y\in\mathbb{R}} \mathcal{L}_1^{x,p} f(y)\pi^p(x,y)dy = 0, \quad \forall f \in C_c^\infty(\mathbb{R})$$

and

$$\int f(y)\mathcal{L}_1^{x,p} g(y)\pi^p(x,y)dy = \int g(y)\mathcal{L}_1^{x,p} f(y)\pi^p(x,y)dy, \quad \forall f, g \in C^2(\mathbb{R}).$$

2.1. Examples

We give some examples of Y that satisfy Assumption 2.3 as well as a multiplicative ergodicity condition of the form

$$e^{-\tilde{\zeta}} \mathcal{L}_1^{x,p} e^{\tilde{\zeta}}(y) \leq -\tilde{\zeta}(y) + d$$

for $\tilde{\zeta}$ with compact level sets and some constant $d > 0$. One needs to know the coefficients of the process X to know whether these examples also satisfy Assumption 2.4. Define $\tilde{V}^p(x,y) := V(y;x,p) + |b_0(x,y)| + \sigma^2(x,y)$. If $\tilde{V}^p(\cdot,\cdot)$ is a bounded function for bounded p, then the multiplicative ergodicity condition is sufficient for Assumption 2.4 to hold. If $\tilde{V}^p(x,y)$ is an unbounded function but has compact level sets, and if the $\tilde{V}$-multiplicative ergodicity condition of the form

$$e^{-\zeta} \mathcal{L}_1^{x,p} e^{\zeta}(y) \leq -c\tilde{V}^p(x,y) + d, \quad \text{or some } c > 1, d > 0$$

is met for ζ with compact finite level sets, then it may be possible to use this condition in place of Assumption 2.4 and obtain all the same results (see Example 4.1 and Remark 4.1).

Example 2.1. Let $\rho = 0$, $b_1(x,y) = -b_1(x)y$, $\sigma_1(x,y) = \sigma_1(x)$ and $k_1(x,y,z) = \frac{\sigma_1(x)}{\sqrt{b_1(x)}}z - y$, where $b_1(x), \sigma_1(x) > 0$ are continuous. Let $\nu_2(z) = \exp\{-z^2\}$. Since the intensity measure ν_1 is a bounded measure, we use $N^{(2)}$ instead of the compensated Poisson process $\tilde{N}^{(2)}$. For each $x \in \mathbb{R}$, the solution to

$$dY_t^x = -b_1(x)Y_t^x dt + \sigma_1(x)dW_t^{(2)} + \int_{\mathbb{R}-\{0\}} \left(\frac{\sigma_1(x)}{\sqrt{b_1(x)}}z - Y_t^x\right) N^{(2)}(dz, dt)$$

has unique invariant probability distribution $\pi(x,dy) = \sqrt{\frac{b_1(x)}{\pi\sigma_1^2(x)}}\exp\{-\frac{b_1(x)y^2}{\sigma_1^2(x)}\}dy$ and Y^x is symmetric with respect to it. Geometric ergodicity is satisfied by $\tilde{\zeta}(y) := \frac{b_1(x)}{2\sigma_1^2(x)}y^2$.

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Example 2.2. Take $\rho = 0$ and let $\alpha \in (1, 2)$. Let Z_t be a 1-dimensional symmetric Levy process whose Levy measure is $\nu_2(z)dz = |z|^{-(1+\alpha)}\mathbf{1}_{|z|>1}dz$. Its infinitesimal generator is the truncated fractional Laplacian $-(-\Delta)^{\alpha/2}_{>1}$ defined as

$$-(-\Delta)^{\alpha/2}_{>1} f(y) = \int_{|z|>1} (f(y+z) - f(y)) \frac{1}{|z|^{1+\alpha}} dz, \quad \text{for } f \in C_c^2(\mathbb{R}).$$

Let $\sigma_1(x, y) := a(x)\sigma_1(y)$ where $a(\cdot), \sigma_1(\cdot) > 0$ are such that $a(\cdot)$ is continuous and $\sigma_1(\cdot)$ is locally $1/\alpha$-Hölder continuous and $\liminf_{|y|\to\infty} \frac{\sigma_1(y)}{|y|} > 0$. Let

$$dY_t^x = \sigma_1(x, Y_{t-}^x)dZ_t.$$

Then from Theorem 1.7(i) in [9], $\pi(x, dy) := \frac{\sigma_1(y)^{-\alpha}dy}{\int \sigma_1(y)^{-\alpha}dy}$ is the unique invariant probability measure for the Y^x process and Y^x is $\pi(x, \cdot)$-reversible. From Lemma 3.2 in [9], we get $\tilde{\zeta}(y) := \ln(1 + |y|^\theta)$ for $\theta \in (0, 1)$ satisfies the geometric ergodicity condition. The special case of this example with $\sigma_1 \equiv 1$ is also considered in [4].

Example 2.3. Let $c(z, z')$ be a non-symmetric function such that $0 < c_0 \le c(z, z') \le c_1$, $c(z, z') = c(z, -z')$ and $|c(z, z'') - c(z', z'')| \le c_2|z - z'|^\beta$ for some $\beta \in (0, 1)$. Let $\alpha \in (0, 2)$, and Z_t be a 1-dimensional non-symmetric process whose infinitesimal generator is defined by

$$\mathcal{L}_c^\alpha f(y) = \lim_{\delta\to 0} \int_{|z|>\delta} (f(y+z) - f(y)) \frac{c(y, y+z)}{|z|^{1+\alpha}} dz, \quad \text{for } f \in C_c^2(\mathbb{R}).$$

Let

$$dY_t = -Y_t dt + dZ_t.$$

Heat kernel estimates from [10] imply this non-symmetric jump diffusion is Feller continuous with a positive transition density $p_t(y_0, y), \forall t > 0$.

Example 2.4. Let Y^x be a birth–death Markov chain with birth rate $r_+(y) = \lambda(x)$ and death rate $r_-(y) = \mu(x)y$, satisfying $\lambda(x), \mu(x) > 0$. Since its state space is countable its transition density is positive, with a unique reversible invariant distribution $\pi(x, y) = e^{-\lambda(x)/\mu(x)} \frac{(\lambda(x)/\mu(x))^y}{y!}$, $y \in \{0, 1, \ldots\}$.

3. Large deviation principle

We prove a large deviation principle for $\{X_{\epsilon,t}\}_{\epsilon>0}$ as $\epsilon \to 0$ using the viscosity solution approach to verify convergence of a sequence of exponential generators. Define

$$u_\epsilon^h(t, x, y) := \epsilon \ln E\left[e^{\frac{h(X_{\epsilon,t})}{\epsilon}} | X_{\epsilon,0} = x, Y_{\epsilon,0} = y\right], \tag{11}$$

where $h \in C_b(\mathbb{R})$, the space of bounded uniformly continuous functions on $\mathbb{R}$. It can be shown (see [19]) that for each $h \in C_b(\mathbb{R})$, u_ϵ^h is a viscosity solution of the Cauchy problem:

$$\begin{aligned} \partial_t u &= H_\epsilon u \quad \text{in } (0, T] \times \mathbb{R} \times \mathbb{R}, \\ u(0, x, y) &= h(x), \quad \text{for } (x, y) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{12}$$

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where the non-linear operator is the exponential generator:

$$
\begin{aligned}
H_\epsilon u(x,y) &:= \epsilon e^{-u/\epsilon}\mathcal{L}_\epsilon e^{u/\epsilon} \\
&= b(x,y)\partial_x u(x,y) + \rho\sigma(x,y)\sigma_1(x,y)\partial^2_{xy}u(x,y) + \frac{1}{2}\sigma^2(x,y)(\partial_x u(x,y))^2 \\
&\quad + \epsilon\left[b_0(x,y)\partial_x u(x,y) + \frac{1}{2}\sigma^2(x,y)\partial^2_{xx}u(x,y)\right] \\
&\quad + \int \left(e^{\frac{u(x+\epsilon k(x,y,z),y)-u(x,y)}{\epsilon}} - 1 - k(x,y,z)\partial_x u(x,y)\right)\nu_1(z)dz \\
&\quad + \frac{1}{\epsilon}\Big[\rho\sigma(x,y)\sigma_1(x,y)\partial_x u(x,y)\partial_y u(x,y) + b_1(x,y)\partial_y u(x,y) \\
&\quad + \frac{1}{2}\sigma_1^2(x,y)\partial^2_{yy}u(x,y)\Big] \\
&\quad + \int \left(e^{\frac{u(x,y+k_1(x,y,z))-u(x,y)}{\epsilon}} - 1 - \frac{k_1(x,y,z)}{\epsilon}\partial_y u(x,y)\right)\nu_2(z)dz \\
&\quad + \frac{1}{2\epsilon^2}\sigma_1^2(x,y)(\partial_y u(x,y))^2.
\end{aligned} \tag{13}
$$

In systems with averaging under the law of large number scaling we can identify the limiting non-linear operator $\overline{H}_0$ as the solution to an eigenvalue problem for the driving process Y^x obtained from Y_ϵ with $X_\epsilon = x$ and $\epsilon = 1$.

We first identify u_0, the limit of u_ϵ as $\epsilon \to 0$, using heuristic arguments. Assume

$$
u_\epsilon(t,x,y) = u_0(t,x) + \epsilon u_1(t,x,y) + \epsilon^2 u_2(t,x,y) + \cdots . \tag{14}
$$

Using the ϵ expansion of u_ϵ, (14), in Eq. (12), and collecting terms of $O(1)$, we get

$$
\begin{aligned}
\partial_t u_0(t,x) = {} & b(x,y)\partial_x u_0(t,x) + \frac{1}{2}\sigma^2(x,y)(\partial_x u_0(t,x))^2 \\
& + \int \left(e^{\partial_x u_0(t,x)k(x,y,z)} - 1 - k(x,y,z)\partial_x u_0(t,x)\right)\nu_1(z)dz \\
& + \rho\sigma(x,y)\sigma_1(x,y)\partial_x u_0(t,x)\partial_y u_1(t,x,y) + b_1(x,y)\partial_y u_1(t,x,y) \\
& + \frac{1}{2}\sigma_1^2(x,y)\partial^2_{yy}u_1(t,x,y) \\
& + \int \left(e^{u_1(t,x,y+k_1(x,y,z))-u_1(t,x,y)} - 1 - k_1(x,y,z)\partial_y u_1(t,x,y)\right)\nu_2(z)dz \\
& + \frac{1}{2}\sigma_1^2(x,y)(\partial_y u_1(t,x,y))^2.
\end{aligned} \tag{15}
$$

Please note that as this is merely a formal derivation, we have ignored some technical details (such as justifying interchanging the limit and integral to get the second line in the above equation). The rigorous proof that follows shows that this formal derivation is indeed correct. Denote $\partial_x u_0(t,x)$ by p and $\partial_t u_0(t,x)$ by λ. Fix t, x and hence p and λ. Using the perturbed $\mathcal{L}_1$ generator (9), Eq. (15) can be written as an eigenvalue problem:

$$
\left(\mathcal{L}_1^{x,p} + V(y;x,p)\right)e^{u_1} = \lambda e^{u_1}, \tag{16}
$$

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where V is as defined in (4). Note that the eigenvalue λ depends on x and p, and that if we write $\overline{H}_0(x, p) := \lambda$ then u_0 satisfies

$$\partial_t u_0(t, x) = \overline{H}_0(x, \partial_x u_0(t, x)).$$

In the rigorous proof that follows, we identify the limiting operator $\overline{H}_0$ to be as defined in (18) which is shown in [12] to be the principal eigenvalue λ in (16). By the expansion (14), it is clear that $u_0(0, x) = h(x)$.

The approach of [19] for obtaining the large deviation principle is to prove convergence of nonlinear semigroups associated with the nonlinear operators H_ϵ. In [19] the first step is identifying the limit operator $\overline{H}_0$. Existence and uniqueness of the limiting semigroup is obtained by verifying the 'range condition' for the limit operator. This amounts to showing existence of solutions to the equation $(I - \alpha\overline{H}_0)f = h$ for small enough $\alpha > 0$ and sufficiently large class of functions h. Since the range condition is difficult to verify, a viscosity method approach is adopted and the range condition is replaced with a comparison principle condition for $(I - \alpha\overline{H}_0)f = h$. In the viscosity method, existence of the limiting semigroup is by construction, while uniqueness is obtained via the comparison principle.

The approach in this paper uses convergence of viscosity solutions to the Cauchy problem for PIDEs (12), and to show existence and uniqueness of the limit one then needs to verify the comparison principle for the Cauchy problem $\partial_t u_0(t, x) = \overline{H}_0(x, \partial_x u_0(t, x))$, with $u_0(0, x) = h(x)$.

In the proof of the comparison principle we will also use a Donsker–Varadhan variational representation [12] for $\overline{H}_0$ as follows. Let $\mathcal{P}(\mathbb{R})$ denote the space of probability measures on $\mathbb{R}$. Define the rate function $J(\cdot; x, p) : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$J(\mu; x, p) := - \inf_{g \in D^{++}(\mathcal{L}_1^{x,p})} \int_{\mathbb{R}} \frac{\mathcal{L}_1^{x,p} g}{g} d\mu, \tag{17}$$

where $D^{++}(\mathcal{L}_1^{x,p}) \subset C_b(\mathbb{R})$ denotes the domain of $\mathcal{L}_1^{x,p}$ with functions that are strictly bounded below by a positive constant. Then [12] implies that the principal eigenvalue $\overline{H}_0(x, p) = \lambda$ in (16) is also given by

$$\overline{H}_0(x, p) = \sup_{\mu \in \mathcal{P}(\mathbb{R})} \left(\int V(y; x, p) d\mu(y) - J(\mu; x, p) \right), \tag{18}$$

where $V(y; x, p) = b(x, y)p + \frac{1}{2}\sigma^2(x, y)p^2 + \int \left(e^{pk(x,y,z)} - 1\right) \nu_1(z)dz$.

Remark 3.1. In the special case $Y^{x,p}$ also has a reversible invariant measure $\pi^p(x, \cdot)$, we can use the Dirichlet form representation for J. Define the Dirichlet form associated with $Y^{x,p}$ by

$$\mathcal{E}^{x,p}(f, g) := - \int f(y)\mathcal{L}_1^{x,p} g(y) d\pi^p(x, dy).$$

Then, Theorem 7.44 in Stroock [29] implies that

$$J(\mu; x, p) = \begin{cases} \mathcal{E}^{x,p}\left(\sqrt{\dfrac{d\mu}{d\pi^p(x,\cdot)}}, \sqrt{\dfrac{d\mu}{d\pi^p(x,\cdot)}}\right) & \text{if } \mu(\cdot) \ll \pi^p(x, \cdot) \\ +\infty & \text{if } \mu(\cdot) \not\ll \pi^p(x, \cdot). \end{cases} \tag{19}$$

The variational formula (18) then reduces to the classical Rayleigh–Ritz formula

$$\overline{H}_0(x,p) = \sup_{f\in L^2(\pi^p),|f|^2=1} \left(\int V(y;x,p) f^2(y) d\pi^p(x,y)dy + \langle \mathcal{L}_1^{x,p} f, f\rangle \right). \tag{20}$$

To sum up, we will prove that:

Lemma 1. *Let $\overline{H}_0$ be as defined in (18), and suppose the comparison principle holds for the nonlinear Cauchy problem:*

$$\begin{aligned} &\partial_t u_0(t,x) = \overline{H}_0(x, \partial_x u_0(t,x)), \quad \text{for } (t,x)\in(0,T]\times\mathbb{R};\\ &u_0(0,x) = h(x). \end{aligned} \tag{21}$$

Under Assumptions 2.1–2.4, the sequence of functions $\{u_\epsilon^h\}_{\epsilon>0}$ defined in (11) converges uniformly over compact subsets of $[0,T]\times\mathbb{R}\times\mathbb{R}$ as $\epsilon\to 0$ to the unique continuous viscosity solution u_0^h of (21).

Lemma 2. *The sequence of processes $\{X_{\epsilon,t}\}_{\epsilon>0}$ is exponentially tight.*

Theorem 3. *Let $X_{\epsilon,0} = x_0$, and suppose all the assumptions from Lemma 1 hold. Then, $\{X_{\epsilon,t}\}_{\epsilon>0}$ satisfies a large deviation principle with speed $1/\epsilon$ and good rate function*

$$I(x,x_0,t) = \sup_{h\in C_b(\mathbb{R})} \{h(x) - u_0^h(t,x_0)\}. \tag{22}$$

Proof. By Bryc's theorem (Theorem 4.4.2 in [11]), Lemmas 1 and 2 give us a large deviation principle for $\{X_{\epsilon,t}\}_{\epsilon>0}$ as $\epsilon\to 0$ with speed $1/\epsilon$ and good rate function I given by (22). □

One of the key conditions for Lemma 1 requires one to check that the comparison principle holds for $\overline{H}_0$. This condition cannot be established using only the general Assumptions 2.1–2.4, and needs to be verified on a case by case basis. However, standard theory of comparison principles for viscosity solutions (Theorem 3.7 and Remark 3.8 in Chapter II of [3]) implies that it does hold for (21) as soon as $\bar{H}_0$ is uniformly continuous in x, p on compact sets (see Lemma 10 of the Appendix). In some cases $\bar{H}_0$ can be explicitly calculated (see Example 4.2) and continuity directly verified. In other cases one may need to resort to proving that the expression as on the right-hand side of (A.37) is non-positive, using the specifics for the case at hand.

Corollary 4. *Any of the following separate sets of conditions are sufficient for the comparison principle for the non-linear Cauchy problem (21) to hold:*

(i) *$\bar{H}_0$ is uniformly continuous in x, p on compact sets;*
(ii) *the coefficients $b_1(x,y), \sigma_1(x,y), k_1(x,y,z)$ are independent of x, the coefficients $b(x,\cdot), \sigma(x,\cdot)$ are bounded (bounded functions of y) for each x, and $\rho = 0$ i.e. the correlation between the Brownian motions driving X and Y is 0.*

Proof. For (i) see Lemma 10 of the Appendix which is based on Theorem 3.7 and Remark 3.8 in Chapter II of [3].

For (ii) we can directly verify that under these conditions $\overline{H}_0(x,p)$, given in (18), is uniformly continuous on compact sets of x and p. For this, first observe that under the conditions in (ii) the rate function J in (18) will be independent of x and p. Additionally, $\int V(y;x,p)d\mu(y)$

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is uniformly Lipschitz in x and p (uniform over all $\mu \in \mathcal{P}(\mathbb{R})$), over compact sets of x and p. Finally, since the supremum of uniformly Lipschitz functions is uniformly continuous over compact sets, we have the result. □

Note that in Corollary 4, condition (i) is a more general condition and (ii) is a sufficient condition (on the coefficients of the model) under which condition (i) holds.

In very special cases, we can further simplify the expression for the rate function:

Corollary 5. *If the coefficients in the SDE (1) are independent of x, then $\overline{H}_0(x,p)$ becomes $\overline{H}_0(p)$ and by Lemma D.1 in [18], we get*

$$I(x,;x_0,t) = t\overline{L}_0\left(\frac{x_0 - x}{t}\right), \tag{23}$$

where $\overline{L}_0(\cdot)$ is the Legendre transform of $\overline{H}_0(\cdot)$.

The proof of Lemma 1 takes up the bulk of the paper, and consists of the following steps.

(Sec 3.1)
- By taking appropriate limits of solutions u^h_ϵ to the Cauchy problem (12) we construct upper-semicontinuous and lower-semicontinuous functions $\overline{u}^h$ and $\underline{u}^h$, respectively;
- Using an indexing set $\alpha \in \Lambda$, we construct a family of operators $H_0(\cdot\,;\alpha)$ and $H_1(\cdot\,;\alpha)$, in such a way that the upper-semicontinuous function $\overline{u}^h$ is a subsolution to the Cauchy problem for the operator $\inf_{\alpha\in\Lambda}\{H_0(\cdot\,;\alpha)\}$, and the lower-semicontinuous function $\underline{u}^h$ is a supersolution to the Cauchy problem for the operator $\sup_{\alpha\in\Lambda}\{H_1(\cdot\,;\alpha)\}$.

(Sec 3.2)
- We prove a comparison principle between subsolutions of $\inf_{\alpha\in\Lambda}\{H_0(\cdot\,;\alpha)\}$ and supersolutions $\sup_{\alpha\in\Lambda}\{H_1(\cdot\,;\alpha)\}$ above;
- We show that this comparison principle implies convergence of solutions u^h_ϵ to the Cauchy problem (12) for H_ϵ to solutions u^h_0 to the Cauchy problem (21) for $\overline{H}_0$.

The proof of Lemma 2 uses the estimates obtained in the proof of Lemma 1 (Section 3.3).

3.1. Convergence of viscosity solutions of PIDEs

In Lemma 1 we use notions of viscosity solutions, subsolutions and supersolutions. For the standard meaning of these terms, as well as for the definition of the comparison principle, we refer the reader to Definition 4.1 in [18]. Their extension to partial integro-differential equations (PIDEs) was obtained already in [1] and can be found in [5].

The proof of convergence of u^h_ϵ to u^h_0 follows the same steps as Lemma 4.1 in [18] which carries over directly to viscosity solutions of PIDEs. Because we will need to verify that the conditions there are met, we restate Lemma 4.1 from [18] for viscosity solutions of PIDEs.

Let $\{H_\epsilon\}_{\epsilon>0}$ denote a family of integro-differential operators defined on the domain of functions $\bar{D}_+ \cup \bar{D}_-$ where

$$\bar{D}_+ := \{f : f \in C^2(\mathbb{R}^2),\ \lim_{r\to\infty}\inf_{|z|>r} f(z) = +\infty\}$$

$$\bar{D}_- := \{-f : f \in C^2(\mathbb{R}^2),\ \lim_{r\to\infty}\inf_{|z|>r} f(z) = +\infty\}.$$

Define domains D_+, D_- analogously replacing $\mathbb{R}^2$ by $\mathbb{R}$. Consider a class of compact sets in $\mathbb{R}\times\mathbb{R}$ defined by

$$\mathcal{Q} := \{K \times \tilde{K} : \text{compact } K, \tilde{K} \subset\subset \mathbb{R}\}.$$

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Let u_ϵ^h be the viscosity solution of the Cauchy problem $\partial_t u = H_\epsilon u$ for the above operator H_ϵ, with initial value h, and define

Definition 3.1.

$$u_\uparrow^h(t,x) := \sup\{\limsup_{\epsilon\to 0+} u_\epsilon^h(t_\epsilon, x_\epsilon, y_\epsilon) : \exists (t_\epsilon, x_\epsilon, y_\epsilon) \in [0,T]\times K \times \tilde{K},$$
$$(t_\epsilon, x_\epsilon) \to (t,x),\, K\times\tilde{K} \in \mathcal{Q}\},$$

$$u_\downarrow^h(t,x) := \inf\{\liminf_{\epsilon\to 0+} u_\epsilon^h(t_\epsilon, x_\epsilon, y_\epsilon) : \exists (t_\epsilon, x_\epsilon, y_\epsilon) \in [0,T]\times K \times \tilde{K},$$
$$(t_\epsilon, x_\epsilon) \to (t,x),\, K\times\tilde{K} \in \mathcal{Q}\}.$$

Define $\overline{u}^h$ to be the upper semicontinuous regularization of $u_\uparrow^h$, and $\underline{u}^h$ the lower semicontinuous regularization of $u_\downarrow^h$.

Finally, define the limiting operators (which will be first-order differential operators) H_0, H_1 on domains D_+ and D_- respectively, as follows. Let Λ be some indexing set, and

$$H_i(x,p;\alpha) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}, \quad \alpha\in\Lambda,\ i = 0,1.$$

Define $H_0 f(x) := H_0(x, \partial_x f(x))$, for $f \in D_+$ and $H_1 f(x) := H_1(x, \partial_x f(x))$, for $f \in D_-$, where

$$H_0(x,p) := \inf_{\alpha\in\Lambda} H_0(x,p;\alpha),$$
$$H_1(x,p) := \sup_{\alpha\in\Lambda} H_1(x,p;\alpha).$$

Henceforth, with slight abuse of notation, we will refer to $H_i(\cdot,\cdot)$ as operators.

Suppose the following conditions hold:

Condition 3.1 (*Limsup Convergence of Operators*). *For each $f_0 \in D_+$ and $\alpha \in \Lambda$, there exists $f_{0,\epsilon} \in \bar{D}_+$ (which may depend on α) such that*

1. *for each $c > 0$, there exists $K\times\tilde{K} \in \mathcal{Q}$ satisfying*

$$\{(x,y) : H_\epsilon f_{0,\epsilon}(x,y) \ge -c\} \cap \{(x,y) : f_{0,\epsilon}(x,y) \le c\} \subset K\times\tilde{K};$$

2. *for each $K\times\tilde{K}\in\mathcal{Q}$,*

$$\lim_{\epsilon\to 0} \sup_{(x,y)\in K\times\tilde{K}} |f_{0,\epsilon}(x,y) - f_0(x)| = 0; \tag{24}$$

3. *whenever $(x_\epsilon, y_\epsilon) \in K\times\tilde{K}\in\mathcal{Q}$ satisfies $x_\epsilon \to x$,*

$$\limsup_{\epsilon\to 0} H_\epsilon f_{0,\epsilon}(x_\epsilon, y_\epsilon) \le H_0(x, \nabla f_0(x); \alpha). \tag{25}$$

Condition 3.2 (*Liminf Convergence of Operators*). *For each $f_1 \in D_-$ and $\alpha \in \Lambda$, there exists $f_{1,\epsilon} \in \bar{D}_-$ (which may depend on α) such that*

1. *for each $c > 0$, there exists $K\times\tilde{K} \in \mathcal{Q}$ satisfying*

$$\{(x,y) : H_\epsilon f_{1,\epsilon}(x,y) \le c\} \cap \{(x,y) : f_{1,\epsilon}(x,y) \ge -c\} \subset K\times\tilde{K};$$

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2. *for each* $K \times \tilde{K} \in \mathcal{Q}$,

$$\lim_{\epsilon \to 0} \sup_{(x,y) \in K \times \tilde{K}} |f_1(x) - f_{1,\epsilon}(x, y)| = 0;$$

3. *whenever* $(x_\epsilon, y_\epsilon) \in K \times \tilde{K} \in \mathcal{Q}$, *and* $x_\epsilon \to x$,

$$\liminf_{\epsilon \to 0} H_\epsilon f_{1,\epsilon}(x_\epsilon, y_\epsilon) \geq H_1(x, \nabla f_1(x); \alpha).$$

In this case the following convergence results for u^h_ϵ as $\epsilon \to 0$ hold.

Lemma 6. *Suppose the viscosity solutions* u^h_ϵ *to the partial integro-differential equation*

$$\partial_t u = H_\epsilon u, \quad u(0, x) = h(x)$$

are uniformly bounded, $\sup_{\epsilon>0} \|u^h_\epsilon\| < \infty$. *Then, under Condition 3.1,* $\overline{u}^h$ *is a subsolution of*

$$\partial_t u(t, x) \leq H_0(x, \nabla u(t, x)) \tag{26}$$

and, under Condition 3.2, $\underline{u}^h$ *is a supersolution of*

$$\partial_t u(t, x) \geq H_1(x, \nabla u(t, x)) \tag{27}$$

with the same initial conditions.

As the proof is the same as the proof of Lemma 4.1 in [18] we omit it here. We do need to check Conditions 3.1 and 3.2 hold for our problem. This involves identifying the right indexing set Λ, the family of operators $H_0(\cdot; \alpha)$ and $H_1(\cdot; \alpha)$, and the appropriate test functions $f_{0,\epsilon}$ and $f_{1,\epsilon}$, for each given f_0 and f_1, respectively.

Verifying Condition 3.1: As in [18], we let

$$\Lambda := \{(\xi, \theta) : \xi \in C^2_c(\mathbb{R}), 0 < \theta < 1\}$$

and we define the sequence of operators H_ϵ as in (13) on the domain

$$D_+ := \{f \in C^2(\mathbb{R}) : f(x) = \phi(x) + \gamma \log(1 + x^2); \phi \in C^2_c(\mathbb{R}), \gamma > 0\}.$$

Define the family of operators $H_0(x, p; \xi, \theta)$ for $(\xi, \theta) \in \Lambda$ by

$$\begin{aligned} &H_0(x, p; \xi, \theta) \\ &:= \sup_{y \in \mathbb{R}} \left\{ b(x, y)p + \frac{1}{2}\sigma^2(x, y)p^2 + \int \left(e^{pk(x,y,z)} - 1 - pk(x, y, z)\right) \nu_1(z)dz \right. \\ &\quad \left. + (1 - \theta)e^{-\xi} \mathcal{L}^{x,p}_1 e^{\xi}(y) + \theta e^{-\zeta} \mathcal{L}^{x,p}_1 e^{\zeta}(y) \right\}. \end{aligned} \tag{28}$$

For any $f \in D_+$ and $(\xi, \theta) \in \Lambda$ define a sequence of functions

$$f_{0,\epsilon}(x, y) := f(x) + \epsilon g(y), \quad \text{where } g(y) := (1 - \theta)\xi(y) + \theta\zeta(y),$$

and ζ is the Lyapunov function on $\mathbb{R}$ satisfying Assumption 2.4. Then,

$$\begin{aligned} H_\epsilon f_{0,\epsilon}(x, y) = {}& b(x, y)\partial_x f(x) + \frac{1}{2}\sigma^2(x, y)(\partial_x f(x))^2 \\ &+ \epsilon \left(b_0(x, y)\partial_x f(x) + \frac{1}{2}\sigma^2(x, y)\partial^2_{xx} f(x) \right) \end{aligned}$$

$$
\begin{aligned}
&+ \int \left(e^{\frac{f(x+\epsilon k(x,y,z),y)-f(x,y)}{\epsilon}} - 1 - k(x,y,z)\partial_x f(x) \right) \nu_1(z)dz \\
&+ e^{-g} \mathcal{L}_1^{x,\partial_x f(x)} e^g(y) \\
\le\ & b(x,y)\partial_x f(x) + \frac{1}{2}\sigma^2(x,y)(\partial_x f(x))^2 \\
&+ \epsilon \left(b_0(x,y)\partial_x f(x) + \frac{1}{2}\sigma^2(x,y)\partial^2_{xx} f(x) \right) \\
&+ \int \left(e^{\frac{f(x+\epsilon k(x,y,z),y)-f(x,y)}{\epsilon}} - 1 - k(x,y,z)\partial_x f(x) \right) \nu_1(z)dz \\
&+ (1-\theta)e^{-\xi} \mathcal{L}_1^{x,\partial_x f(x)} e^{\xi}(y) + \theta e^{-\zeta} \mathcal{L}_1^{x,\partial_x f(x)} e^{\zeta}(y)
\end{aligned} \tag{29}
$$

so, for any sequence (x_ϵ, y_ϵ) such that $x_\epsilon \to x$

$$
\limsup_{\epsilon \to 0} H_\epsilon f_{0,\epsilon}(x_\epsilon, y_\epsilon) \le H_0(x, \partial_x f(x); \xi, \theta),
$$

thus verifying Condition 3.1.3 holds.

By choice of D_+, $f \in D_+$ has compact level sets in $\mathbb{R}$. Also note that $\|\partial_x f\| + \|\partial^2_{xx} f\| < \infty$. Assumption 2.4 ensures that $-H_\epsilon f_{0,\epsilon}(x,\cdot)$ has compact level sets for all x in compact sets. This proves Condition 3.1.1 holds. Condition 3.1.2 is obvious by choice of functions $f_{0,\epsilon}$.

Verifying Condition 3.2: is exactly the same as verifying Condition 3.1, except that the sequence of operators H_ϵ are now defined on the domain

$$
D_- := \{f \in C^2(\mathbb{R}) : f(x) = \phi(x) - \gamma \log(1+x^2);\ \phi \in C^2_c(\mathbb{R}), \gamma > 0\};
$$

the family of operators $H_1(x,p;\xi,\theta)$ for $(\xi,\theta) \in \Lambda$ is defined by

$$
\begin{aligned}
&H_1(x,p;\xi,\theta) \\
&:= \inf_{y\in\mathbb{R}} \Big\{ b(x,y)p + \frac{1}{2}\sigma^2(x,y)p^2 + \int \left(e^{pk(x,y,z)} - 1 - pk(x,y,z) \right) \nu_1(z)dz \\
&\quad + (1+\theta)e^{-\xi} \mathcal{L}_1^{x,p} e^{\xi}(y) - \theta e^{-\zeta} \mathcal{L}_1^{x,p} e^{\zeta}(y) \Big\};
\end{aligned} \tag{30}
$$

and for any $f \in D_-$ and $\xi, \theta \in \Lambda$ the sequence $f_{1,\epsilon}$ is defined as

$$
f_{1,\epsilon}(x,y) := f(x) + \epsilon g(y), \quad \text{for } g(y) := (1+\theta)\xi(y) - \theta\zeta(y),
$$

so that for any sequence (x_ϵ, y_ϵ) such that $x_\epsilon \to x$ we now have

$$
\liminf_{\epsilon \to 0} H_\epsilon f_{1,\epsilon}(x_\epsilon, y_\epsilon) \ge H_1(x, \partial_x f(x); \xi, \theta)
$$

verifies Condition 3.2.3 holds. Conditions 3.2.1, 3.2.2 hold by the same arguments as above.

3.2. *Comparison principle*

The rest of the claim of Lemma 1 requires proving uniqueness of solutions to $\partial_t u = \overline{H}_0 u$, with initial value h. This can be verified using the comparison principle on the subsolutions and supersolutions of the constructed limiting operators H_0 and H_1, and the variational representation of $\overline{H}_0$ from (18). We use the following Lemma 4.2 from [18].

Lemma 7. *Let $\underline{u}^h$ and $\overline{u}^h$ be defined as in Definition 3.1. If a comparison principle between subsolutions of (26) and supersolutions of (27) holds, that is, if every subsolution v_1 of (26) and every supersolution v_2 of (27) satisfy $v_1 \le v_2$, then $\underline{u}^h = \overline{u}^h$ and $u^h_\epsilon(t,x,y) \to u^h_0(t,x)$, where $u^h_0 := \underline{u}^h = \overline{u}^h$, as $\epsilon \to 0$, uniformly over compact subsets of* $[0,T]\times\mathbb{R}\times\mathbb{R}$.

Proof. The comparison principle gives $\overline{u}^h \le \underline{u}^h$, while by construction we have $\underline{u}^h \le \overline{u}^h$. This gives uniform convergence of $u^h_\epsilon \to u_0 := \overline{u}^h = \underline{u}^h$ over compact subsets of $[0,T]\times\mathbb{R}\times\mathbb{R}$. □

We next prove the comparison principle for subsolutions of (26) and supersolutions of (27), that is every subsolution of

$$\partial_t u(t,x) \le H_0(x,p) := \inf_{0<\theta<1,\xi\in C_c^2(\mathbb{R})} H_0(x,p;\xi,\theta),$$

where H_0 is as defined in (28), is less than or equal to every super solution of

$$\partial_t u(t,x) \ge H_1(x,p) := \sup_{0<\theta<1,\xi\in C_c^2(\mathbb{R})} H_1(x,p;\xi,\theta)$$

where H_1 is as defined in (30). We follow the steps in Section 5.2 in [18] with some modifications. The key step is proving

Operator inequality:

$$\inf_{0<\theta<1,\xi\in C_c^2(\mathbb{R})} H_0(x,p;\theta,\xi) \le \overline{H}_0(x,p) \le \sup_{0<\theta<1,\xi\in C_c^2(\mathbb{R})} H_1(x,p;\theta,\xi), \tag{31}$$

where $\overline{H}_0(x,p)$ is as defined in (18).

Recall the definition of the rate function J from (17) and variational representation of $\overline{H}_0$ as

$$\overline{H}_0(x,p) = \sup_{\mu\in\mathcal{P}(\mathbb{R})}\left(\int V(y;x,p)d\mu(y) - J(\mu;x,p)\right).$$

Following steps of Lemma 11.35 of [19] (which relies on Assumption 2.3) we get that

$$\inf_{0<\theta<1,\xi\in C_c^2(\mathbb{R})} H_0(x,p:\theta,\xi) \le \overline{H}_0(x,p).$$

From the proof of Lemma B.10 in [19], we have

$$\sup_{0<\theta<1,\xi\in C_c^2(\mathbb{R})} H_1(x,p:\theta,\xi) \ge \inf_{\mu\in\mathcal{P}(\mathbb{R})} \liminf_{t\to\infty} t^{-1}\ln E^\mu\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right].$$

Thus, we need to show that, irrespective of the initial distribution,

$$\liminf_{t\to\infty} t^{-1}\ln E\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right] \ge \overline{H}_0(x,p).$$

The proof of this claim depends on Assumption 2.3. We define the occupation measures of the $Y^{x,p}$ process:

$$\mu_t^{x,p}(\cdot) := \frac{1}{t}\int_0^t 1_{Y_s^{x,p}}(\cdot)ds.$$

Recall that $\mathcal{P}(\mathbb{R})$ is a separable metric space under the Prokhorov metric and that weak convergence of measures is equivalent to convergence in the Prokhorov metric. Let $\mathcal{Q}_{t,y_0}$ denote

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the probability measure on $\mathcal{P}(\mathbb{R})$ induced by the occupation measure μ_t of Y when $Y_0 = y_0$. In other words, for $A \in \mathcal{B}(\mathcal{P}(\mathbb{R}))$ (the borel sigma-algebra on $\mathcal{P}(\mathbb{R})$),

$$\mathcal{Q}_{t,y_0}(A) = P(\mu_t(\cdot) \in A | Y_0 = y_0).$$

Lemma 8. $\inf_{\mu \in \mathcal{P}(\mathbb{R})} \liminf_{t\to\infty} t^{-1} \ln E^{\mu}\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right] \geq \overline{H}_0(x,p)$.

Proof. Define $\phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ by $\phi(\mu) = \int V(y;x,p)\mu(dy)$. Take $\tilde{\nu_1} \in \mathcal{P}(\mathbb{R})$, and let $B(\tilde{\nu_1}, r)$ denote the open ball in $\mathcal{P}(\mathbb{R})$ of radius r, centered at $\tilde{\nu_1}$. Fix $\nu_1 \in \mathcal{P}(\mathbb{R})$, then there exists a compact set K in $\mathbb{R}$ such that $\nu_1(K) > 0$. The key ingredient in the proof is the uniform LDP lower bound for the occupation measures:

$$\liminf_{t\to\infty} \frac{1}{t} \log\left[\inf_{y_0 \in K} \mathcal{Q}_{t,y_0}(B(\tilde{\nu_1}, r))\right] \geq -J(\tilde{\nu_1}; x, p). \tag{32}$$

This is obtained from Theorem 5.5 in [13] under Assumption 2.3. While the statement of Theorem 5.5 in [13] is in terms of a process level LDP, by the contraction principle it ensures the uniform LDP lower bound (32) for the occupation measures $\mu_t^{x,p}$.

We now compute

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{t} \log E_1^{\nu}\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right] &= \liminf_{t\to\infty} \frac{1}{t} \log E_1^{\nu}\left[e^{t\phi(\mu_t^{x,p})}\right] \\
&\geq \liminf_{t\to\infty} \frac{1}{t} \log E_1^{\nu}\left[e^{t\phi(\mu_t^{x,p})} 1_{\{Y_0 \in K\}}\right] \\
&\geq \liminf_{t\to\infty} \frac{1}{t} \log\left[\inf_{y_0\in K} E^{y_0}\left(e^{t\phi(\mu_t^{x,p})}\right)\right] + \liminf_{t\to\infty} \frac{1}{t} \log \nu_1(K) \\
&= \liminf_{t\to\infty} \frac{1}{t} \log\left[\inf_{y_0\in K} \int_{\mu\in\mathcal{P}(\mathbb{R})} e^{t\phi(\mu)} d\mathcal{Q}_{t,y_0}(\mu)\right] \\
&\geq \liminf_{t\to\infty} \frac{1}{t} \log\left[\inf_{y_0\in K} \int_{\mu\in B(\tilde{\nu_1},r)} e^{t\phi(\mu)} d\mathcal{Q}_{t,y_0}(\mu)\right] \\
&\geq \inf_{\mu\in B(\tilde{\nu_1},r)} \phi(\mu) + \liminf_{t\to\infty} \frac{1}{t} \log\left[\inf_{y_0\in K} \mathcal{Q}_{t,y_0}(B(\tilde{\nu_1}, r))\right] \\
&\geq \inf_{\mu\in B(\tilde{\nu_1},r)} \phi(\mu) - J(\tilde{\nu_1}; x, p)
\end{aligned}$$

by (32). By Lemma 9 (see Appendix), ϕ is a lower semi-continuous function, and so $\phi(\tilde{\nu_1}) \leq \lim_{r\to 0} \inf_{\mu\in B(\tilde{\nu_1},r)} \phi(\mu)$. Thus taking limit as $r \to 0$ we get

$$\liminf_{t\to\infty} \frac{1}{t} \log E_1^{\nu}\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right] \geq \phi(\tilde{\nu_1}) - J(\tilde{\nu_1})$$

(note that since V is bounded below, $\phi(\mu) > -\infty$, and so $\phi(\tilde{\nu_1}) - J(\tilde{\nu_1}; x, p)$ is well defined and not $-\infty + \infty$). Since $\tilde{\nu_1}$ is arbitrary, we get

$$\liminf_{t\to\infty} \frac{1}{t} \log E_1^{\nu}\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right] \geq \sup_{\tilde{\nu_1}\in\mathcal{P}(\mathbb{R})} \{\phi(\tilde{\nu_1}) - J(\tilde{\nu_1}; x, p)\}.$$

This holds for every $\nu_1 \in \mathcal{P}(\mathbb{R})$ and so

$$\inf_{\nu_1 \in \mathcal{P}(\mathbb{R})} \liminf_{t\to\infty} \frac{1}{t} \log E_1^{\nu}\left[e^{\int_0^t V(Y_s^{x,p};x,p)ds}\right] \geq \sup_{\tilde{\nu_1}\in\mathcal{P}(\mathbb{R})} \{\phi(\tilde{\nu_1}) - J(\tilde{\nu_1};x,p)\}.$$

This concludes the proof of the **Operator Inequality** (31). □

Remark 3.2. In the special case $Y^{x,p}$ also has a reversible invariant measure $\pi^p(x,\cdot)$ we could also follow the arguments for Lemma 5.4 in [18] using the Dirichlet form representation of J (19).

Proof of Lemma 1. By Lemma 6 and Operator Inequality (31), it follows that $\overline{u}^h$ is a subsolution and $\underline{u}^h$ a supersolution of the Cauchy problem (21): $\partial_t u(t,x) = \overline{H}_0(x, \partial_x u(t,x))$ with $u(0,x) = h(x)$. If the comparison principle holds for the Cauchy problem (21), then Lemma 7 gives us $\underline{u}^h = \overline{u}^h$ and that $u_\epsilon^h \to u_0^h \equiv \underline{u}^h = \overline{u}^h$ uniformly over compact subsets of $[0,T]\times\mathbb{R}\times\mathbb{R}$. □

3.3. Exponential tightness

Proof of Lemma 2. We prove exponential tightness using the convergence of H_ϵ and appealing to supermartingale arguments (see Section 4.5 of [19]).

Let $f(x) := \ln(1+x^2)$, so $f(x) \to \infty$ as $|x| \to \infty$, and also $\|f'\| + \|f''\| < \infty$. Define $f_\epsilon(x,y) := f(x) + \epsilon\zeta(y)$ where ζ is the positive Lyapunov function satisfying Assumption 2.4 (with $\theta = 1$). Then, for any $c > 0$, there exists a compact $K_c \subset \mathbb{R}$ such that $f_\epsilon(x,y) > c$, $\forall y \in \mathbb{R}, \forall x \notin K_c$.

Observe that by (29) (with $\theta = 1$)

$$\begin{aligned} H_\epsilon f_\epsilon(x,y) &= \epsilon e^{-f_\epsilon/\epsilon}\mathcal{L}_\epsilon e^{f_\epsilon/\epsilon} \\ &\leq b(x,y)\partial_x f(x) + \frac{1}{2}\sigma^2(x,y)(\partial_x f(x))^2 + \epsilon\left(b_0(x,y)\partial_x f(x) + \frac{1}{2}\sigma^2(x,y)\partial_{xx}^2 f(x)\right) \\ &\quad + \int (e^{\frac{f(x+\epsilon k(x,y,z),y)-f(x,y)}{\epsilon}} - 1 - pk(x,y,z))\nu_1(z)dz + e^{-\zeta}\mathcal{L}_1^{x,\partial_x f(x)} e^{\zeta}(y). \end{aligned}$$

By choice of f, growth conditions on the coefficients and Assumption 2.4, we get there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R},y\in\mathbb{R}} H_\epsilon f_\epsilon(x,y) \leq C < \infty, \qquad \forall \epsilon > 0.$$

Since $e^{(f_\epsilon(X_{\epsilon,t},Y_{\epsilon,t})-f_\epsilon(X_{\epsilon,0},Y_{\epsilon,0}))/\epsilon - \int_0^t H_\epsilon f_\epsilon(X_{\epsilon,s},Y_{\epsilon,s})ds}$ is a non-negative local martingale, by optional stopping

$$\begin{aligned} &P(X_{\epsilon,t} \notin K_c)e^{(c-f_\epsilon(x_0,y_0)-tC)/\epsilon} \\ &\leq E\left[\exp\left\{\frac{f_\epsilon(X_{\epsilon,t},Y_{\epsilon,t})}{\epsilon} - \frac{f_\epsilon(x_0,y_0)}{\epsilon} - \int_0^t H_\epsilon f_\epsilon(X_{\epsilon,s},Y_{\epsilon,s})ds\right\}\right] \leq 1. \end{aligned}$$

Therefore for each $c > 0$

$$\epsilon \ln P(X_{\epsilon,t} \notin K_c) \leq tC - f_\epsilon(x_0,y_0) - c.$$

As C is fixed and independent of c (which we can choose), $\{X_{\epsilon,t}\}_{\epsilon>0}$ is exponentially tight.

Remark 3.3. A similar argument can be used to verify the exponential compact containment condition in Corollary 4.17 in [19], which would give us $\{X_{\epsilon,\cdot}\}_{\epsilon>0}$ is exponentially tight. □

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4. Examples

4.1. Model for stock price with stochastic volatility

We consider the stochastic volatility model for stock price suggested by Barndorff-Nielson and Shephard [6]. Let X_t denote the logarithm of stock price and Y_t the stochastic volatility.

$$dX_t = \left(r - \frac{1}{2}Y_t\right)dt + \sqrt{Y_t}dW_t$$
$$dY_t = -\frac{Y_t}{\delta}dt + dZ_t^{1/\delta},$$

where W_t is a standard Brownian motion and $Z_t^{1/\delta}$ is an independent non-Gaussian Lévy process with intensity $\frac{1}{\delta}\nu(dz)$; the parameter $0 < \delta \ll 1$ denotes the mean-reversion time scale in stochastic volatility. The process Z is often referred to as the *background driving Lévy process*(BDLP). If we are interested in pricing options on the stock which are close to maturity, we will only be interested in small-time asymptotics of the model. We thus scale time by a parameter $0 < \epsilon \ll 1$, to get

$$\begin{aligned} dX_{\epsilon,t} &= \epsilon\left(r - \frac{1}{2}Y_{\epsilon,t}\right)dt + \sqrt{\epsilon}\sqrt{Y_{\epsilon,t}}dW_t \\ dY_{\epsilon,t} &= -\frac{\epsilon}{\delta}Y_{\epsilon,t}dt + dZ_{\epsilon t}^{1/\delta}, \end{aligned} \tag{33}$$

The multi scale structure comes from the fast mean reversion in stochastic volatility and the small time to maturity. We are interested in the situation where time to maturity (ϵ) is small, but large compared to mean-reversion time (δ) of stochastic volatility. The interesting regime as seen in [18] is when $\delta = \epsilon^2$. The generator of (X_ϵ, Y_ϵ) is given by:

$$\begin{aligned} \mathcal{L}_\epsilon f(x, y) &= \epsilon\left((r - \frac{1}{2}y)\partial_x f(x, y) + \frac{1}{2}y\partial_{xx}^2 f(x, y)\right) \\ &\quad + \frac{1}{\epsilon}\left(-y\partial_y f(x, y) + \int (f(x, y + z) - f(x, y))\,\nu(dz)\right), \end{aligned}$$

for $f \in C_b^2(\mathbb{R}^2)$.

For this example, since the coefficients are x-independent, the perturbed operator $\mathcal{L}_1^{x,p}$ is the same as $\mathcal{L}_1$, the generator of Y:

$$\mathcal{L}_1 f(y) = -yf'(y) + \int (f(y + z) - f(y))\,\nu(dz), \quad \text{for } f \in C_b^2(\mathbb{R}).$$

We can obtain the limiting Hamiltonian $\overline{H}_0$ by solving the eigenvalue problem (16). Here $V(y; x, p) \equiv V(y; p) = \frac{1}{2}yp^2$. $\overline{H}_0(p)$ is the eigenvalue λ of the eigenvalue problem

$$-yf'(y) + \int (f(y + z) - f(y))\,\nu(dz) + \frac{1}{2}yp^2 f(y) = \lambda f(y).$$

Note that $f(y) = e^{\frac{p^2}{2}y}$ and $\lambda(p) = \int \left(e^{\frac{p^2}{2}z} - 1\right)\nu(dz)$ satisfy the eigenvalue problem. So $\overline{H}_0(p) = \lambda(p) = \int \left(e^{\frac{p^2}{2}z} - 1\right)\nu(dz)$. In this example, in the absence of a Lyapunov function

ζ satisfying Assumption 2.4, we give a slightly altered proof as follows. The following proof assumes $\overline{H}_0(p)$ is finite.

To verify Condition 3.1, for $f \in D_+$ and $0 < \theta < 1$, we define $f_{0,\epsilon} := f(x) + \epsilon((1-\theta)g(x,y) + \theta\tilde{\zeta}(y))$, where $g(x,y) := \frac{1}{2}(f'(x))^2 y$ (logarithm of the eigenfunction) and $\tilde{\zeta}(y) := C^2 y$, $C := \sup_x |f'(x)|$. Then we get

$$\begin{aligned} H_\epsilon f_{0,\epsilon}(x,y) &\le \frac{1}{2}y(f'(x))^2 + \epsilon\left(\left(r - \frac{y}{2}\right)f'(x) + \frac{1}{2}yf''(x)\right) + (1-\theta)e^{-g}\mathcal{L}_1^{f'(x)}e^g \\ &\quad + \theta e^{-\tilde{\zeta}}\mathcal{L}_1^{f'(x)}e^{\tilde{\zeta}} \\ &= (1-\theta)\lambda(f'(x)) + \theta\left[-y(C^2 - \frac{1}{2}(f'(x))^2) + \int (e^{C^2 z} - 1)\nu(dz)\right] \\ &\quad + \epsilon\left(\left(r - \frac{y}{2}\right)f'(x) + \frac{1}{2}yf''(x)\right). \end{aligned}$$

Thus $H_\epsilon f_{0,\epsilon}$ satisfies Condition 3.1.1. Condition 3.1.2 is immediate and

$$\begin{aligned} \limsup_{\epsilon\to 0} H_\epsilon f_{0,\epsilon} &\le \inf_{0<\theta<1}\Bigg[(1-\theta)\lambda(f'(x)) \\ &\quad + \theta\sup_y\left(-y(C^2 - \frac{1}{2}(f'(x))^2) + \int (e^{C^2 z} - 1)\nu(dz)\right)\Bigg] \\ &\le \limsup_{\theta\to 0}\Bigg[(1-\theta)\lambda(f'(x)) \\ &\quad + \theta\sup_y\left(-y(C^2 - \frac{1}{2}(f'(x))^2) + \int (e^{C^2 z} - 1)\nu(dz)\right)\Bigg] \\ &= \lambda(f'(x)) =: \overline{H}_0(f'(x)). \end{aligned}$$

Similarly, to verify Condition 3.2, define $f_{1,\epsilon} := f(x) + \epsilon((1+\theta)g(x,y) - \theta\tilde{\zeta}(y))$. It is unnecessary to verify any operator inequality as the limiting operators H_0 and H_1 coincide and equal $\overline{H}_0$.

Remark 4.1. Recall the definition of $\tilde{V}$ at the beginning of Section 2.1, $\tilde{V}^p(x,y) := V(y; x,p) + |b_0(x,y)| + \sigma^2(x,y)$. In general, in case we have a solution to the eigenvalue problem defining the Hamiltonian $\overline{H}_0$, then the exact same proof as above using $f_{0,\epsilon} = f(x) + \epsilon((1-\theta)g(x,y) + \theta\tilde{\zeta}(y))$, with $g(x,y)$ the logarithm of the eigenfunction and $\tilde{\zeta}$ satisfying the $\tilde{V}$-multiplicative ergodicity condition

$$e^{-\tilde{\zeta}}\mathcal{L}_1^{x,p}e^{\tilde{\zeta}}(y) \le -cV^p(x,y) + d, \quad \text{for } c > 1, d > 0$$

is enough to conclude our large deviation results (provided $\tilde{V}$ has compact finite level sets, as it was above).

In Barndorff-Nielsen and Shephard [6], the BDLP, Z, is assumed to have only positive increments. A simple example of such a Lévy process is a jump process taking finitely many jumps that is the Lévy measure is $\nu(z_i) > 0$ where $z_i > 0$, $i = 1,2,\ldots,k$. We can then explicitly compute $\overline{H}_0(p)$ and its Legendre transform $\bar{L}(p)$. As seen in [18] (Lemma D.1 in [18]), since $\overline{H}_0(p)$ is not state dependent, we get the rate function to be $I(x,x_0,t) = t\bar{L}\left(\frac{x_0 - x}{t}\right)$. In finance, a common example is where Z is a gamma process, in which case $\nu(dz) = \frac{a}{z}e^{-bz}dz$,

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$a, b > 0$. Then

$$\overline{H}_0(p) = \begin{cases} a \ln\left(1 + \frac{p^2}{2b - p^2}\right) & \text{if } -\sqrt{2b} < p < \sqrt{2b} \\ \infty & \text{if } p^2 > 2b, \end{cases}$$

and the rate function is given by $I(x; x_0, t) = t\bar{L}\left(\frac{x_0 - x}{t}\right)$, where

$$\bar{L}(q) = \begin{cases} -a + \sqrt{a^2 + 2bq^2} - a \ln 2b + a \ln\left(\frac{-2a^2}{q^2} + \frac{2a}{q^2}\sqrt{a^2 + 2bq^2}\right) & \text{if } q > 0 \\ 0 & \text{if } q = 0 \\ -a - \sqrt{a^2 + 2bq^2} - a \ln 2b + a \ln\left(\frac{-2a^2}{q^2} - \frac{2a}{q^2}\sqrt{a^2 + 2bq^2}\right) & \text{if } q < 0. \end{cases}$$

This rate function then gives the asymptotic behavior of a European Call option on the stock. Let K denote the strike price and $S_{\epsilon,t} = e^{X_{\epsilon,t}}$, then for $S_0 = e^{x_0} < K$ (out-of-the-money call),

$$\lim_{\epsilon \to 0} \epsilon \log E\left[S_{\epsilon,t} - K\right]^+ = -I(\log K; x_0, t),$$

where maturity time $T = \epsilon t$. This follows from Corollary 1.3 in [17].

4.2. Model for self-regulating protein production

The simplest model for translation of protein from DNA is the system below, with a gene that is either in its "on" state G_1, or in its "off" state G_0, and in which the protein activates the changes from "off" to "on" state:

$$\begin{array}{llllllll} (1) & G_0 + P & \overset{\kappa_1'}{\rightharpoonup} & G_1 + P & (3) & G_1 & \overset{\kappa_2'}{\rightharpoonup} & G_1 + P \\ (2) & G_1 + P & \overset{\kappa_{-1}'}{\rightharpoonup} & G_0 + P & (4) & P & \overset{\kappa_3'}{\rightharpoonup} & \emptyset. \end{array}$$

Suppose the amount of protein P is of order $1/\epsilon$, whose rate of production $\kappa_2' = 1/\epsilon\, \kappa_2$, while its rate of degradation $\kappa_3' = \kappa_3$; where κ_2, κ_3 are of $O(1)$. The amount of genes in the "on"- and "off"-state is $\in \{0, 1\}$, their total amount always equaling 1, and suppose the rates of changes of the gene from the "on"-state to the "off"-state and back are very rapid due to its regulation by the large amounts of protein $\kappa_1' = \kappa_1, \kappa_{-1}' = \kappa_{-1}$, where κ_1, κ_{-1} are of $O(1)$. This system is characteristic of eukaryotes, where the gene switching noise dominates over the transcriptional and translational noise. We can represent the changes in the system using the process X_ϵ for the count of protein molecules normalized by ϵ, and Y_ϵ for the (unnormalized) count of "on"-gene molecules. A diffusion process is a good approximation for the evolution of X_ϵ as long as the count of proteins is not too small, that is, the unnormalized count is $\gg \epsilon$ and $X_\epsilon \sim O(1)$ ([25] gives a rigorous justification of diffusion approximations for Markov chain models that apply in stochastic reaction kinetics). This diffusion solves $dX_{\epsilon,t} = b(X_{\epsilon,t}, Y_{\epsilon,t})dt + \sqrt{\epsilon}\sigma(X_{\epsilon,t}, Y_{\epsilon,t})dW_t$ with drift $b(x, y) = \kappa_2 y - \kappa_3 x$ (protein production has only two possible values: it will be 0 when $y = 0$, or κ_2 when $y = 1$), with diffusion coefficient $\sigma^2(x, y) = \kappa_2 y + \kappa_3 x$, and initial value $X_{\epsilon,0} = x_0 > 0$. Changes in the amount of proteins due to other independent sources of noise, such as errors after cell splitting, can be modeled by an additional jump term for X_ϵ where the jump measure $\nu_1(dx)$ can be as simple as $\nu_1(z) = \frac{1}{2}\delta_{-1}(z) + \frac{1}{2}\delta_{+1}(z)$, producing

$$dX_{\epsilon,t} = (\kappa_2 Y_{\epsilon,t} - \kappa_3 X_{\epsilon,t})dt + \sqrt{\epsilon(\kappa_2 Y_{\epsilon,t} + \kappa_3 X_{\epsilon,t})}dW_t + \epsilon \int \mathbf{1}_{X_\epsilon > \epsilon} z \tilde{N}^{\frac{1}{\epsilon}}(dz, dt).$$

The amount of genes G_1 in the "on"-state is a rapidly fluctuating two-state Markov chain Y on $\{0, 1\}$ with rates $r_{0\to1}(x) = \frac{1}{\epsilon}\kappa_1 x$ and $r_{1\to0}(x) = \frac{1}{\epsilon}\kappa_{-1} x$ that depend on the normalized amount of protein (note that the amount of genes G_0 in the "off"-state is $1 - Y$). This chain is reversible, and for each $x > 0$ it has a unique stationary distribution $\pi^x(1) = 1 - \pi^x(0) = \kappa_1/(\kappa_1 + \kappa_{-1})$.

Signaling proteins such as morphogens have to be in the right range of concentrations to avoid triggering the expression of genes at the wrong times. The probabilities of their amounts being out of range are given by the Large Deviation Principle for X_ϵ as $\epsilon \to 0$, for which we need to obtain the solution to the eigenvalue problem for the operator $V(y; x, p) + \mathcal{L}^x$ where $\mathcal{L}^x f(y) = r_{0\to1}(x)\big(f(y+1) - f(y)\big)1_{y=0} + r_{1\to0}(x)\big(f(y-1) - f(y)\big)1_{y=1}$.

In order to solve $(V(y; x, p) + \mathcal{L}^x)e^{u_1} = \lambda e^{u_1}$ for λ, let $e^{u_1(x,1)} = a_1(x)$, $e^{u_1(x,0)} = a_0(x)$, for some a_1, a_0 strictly positive functions. Then

$$\begin{aligned}&(\kappa_2 - \kappa_3 x)pa_1(x) + (\kappa_2 + \kappa_3 x)p^2 a_1(x)\\ &\qquad + \frac{1}{2}(e^p + e^{-p} - 2)a_1(x) + \kappa_{-1}x(a_0(x) - a_1(x)) = \lambda a_1(x)\\ &-\kappa_3 xpa_0(x) + \kappa_3 xp^2 a_0(x) + \frac{1}{2}(e^p + e^{-p} - 2)a_0(x) + \kappa_1 x(a_1(x) - a_0(x)) = \lambda a_0(x)\end{aligned}$$

equivalently, with $a(x) = a_1(x)/a_0(x)$,

$$(\kappa_2 - \kappa_3 x)p + (\kappa_2 + \kappa_3 x)p^2 + \kappa_{-1}x\left(\frac{1}{a(x)} - 1\right) = -\kappa_3 xp + \kappa_3 xp^2 + \kappa_1 x(a(x) - 1)$$

which, since $a(x)$ has to be positive, gives

$$\begin{aligned}&a(x) = \frac{-B + \sqrt{B^2 - 4AC}}{2A},\\ &\quad A = \kappa_1 x,\ B = -\kappa_2 p - \kappa_2 p^2 + (\kappa_{-1} - \kappa_1)x,\ C = -\kappa_{-1}x\end{aligned}$$

and consequently, using notation above,

$$\bar{H}_0(x, p) = -\kappa_3 xp + \kappa_3 xp^2 + \kappa_1 x(a(x) - 1) + \frac{1}{2}(e^p + e^{-p} - 2).$$

Note that when $\kappa_{-1} = \kappa_1$ then

$$a(x) = \frac{\kappa_2 p(1+p) + \sqrt{(\kappa_2 p(1+p))^2 + (2\kappa_1 x)^2}}{2\kappa_1 x}$$

and

$$\begin{aligned}\bar{H}_0(x, p) = &-\kappa_3 p(1-p)x + \frac{1}{2}\kappa_2 p(1+p) + \frac{1}{2}\sqrt{(\kappa_2 p(1+p))^2 + (2\kappa_1 x)^2} - \kappa_1 x\\ &+ \frac{1}{2}(e^p + e^{-p} - 2).\end{aligned}$$

Note that $\bar{H}_0(x, p)$ is both convex in p and continuous in x.

If one were to use an approximation of the evolution of the normalized protein amount X_ϵ by a piecewise deterministic process then (without additional noise)

$$dX_{\epsilon,t}^{\text{PDMP}} = (\kappa_2 Y_{\epsilon,t} - \kappa_3 X_{\epsilon,t})dt$$

while Y_ϵ is the same fast Markov chain on $\{0, 1\}$. In this case $V(y; x, p) = (\kappa_2 - \kappa_3 x)p$ and the Hamiltonian (when $\kappa_1 = \kappa_{-1}$) becomes

$$\bar{H}_0^{\text{PDMP}}(x, p) = -\kappa_3 px + \frac{1}{2}\kappa_2 p + \frac{1}{2}\sqrt{(\kappa_2 p)^2 + (2\kappa_1 x)^2} - \kappa_1 x$$

which is easy to compare to the Hamiltonian $\bar{H}_0$ of the diffusion process X_ϵ taking into account the small perturbative noise arising from randomness in the timing of chemical reactions and from randomness in the outcomes of cell splitting.

Acknowledgments

First author's work was partially supported by National Science Foundation grant DMS 1209363.

Appendix

Lemma 9. *Fix $x, p \in \mathbb{R}$ and let $\phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be defined by $\phi(\mu) = \int V(y; x, p)\mu(dy)$. Then, ϕ is a lower semi-continuous (l.s.c.) function on $\mathcal{P}(\mathbb{R})$.*

Proof. For the rest of the proof, we will write $V(y)$ for $V(y; x, p)$. Let $V_M := V \cdot 1_{V \le M} + M \cdot 1_{V \ge M}$, for $M \ge \inf_y V(y)$. To show that $\phi(\mu)$ is l.s.c, it is sufficient to show that if $\mu_n \longrightarrow \mu$ weakly, then $\phi(\mu) \le \liminf_{n\to\infty} \phi(\mu_n)$. Assume $\mu_n \longrightarrow \mu$ weakly. Then

$$\int V_M d\mu = \lim_{n\to\infty} \int V_M d\mu_n,$$

by definition of weak convergence of measures, since V_M is a bounded function. By the monotone convergence theorem we get

$$\begin{aligned}
\phi(\mu) = \int V d\mu &= \lim_{M\to\infty} \int V_M d\mu \\
&= \lim_{M\to\infty} \lim_{n\to\infty} \int V_M d\mu_n \\
&= \sup_M \lim_{n\to\infty} \int V_M d\mu_n \\
&\le \liminf_{n\to\infty} \sup_M \int V_M d\mu_n \\
&= \liminf_{n\to\infty} \int V d\mu_n
\end{aligned}$$

by Monotone convergence theorem

$$= \liminf_{n\to\infty} \phi(\mu_n). \quad \square$$

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Lemma 10. *Let u_1 be a bounded, upper semicontinuous (u.s.c.), viscosity subsolution and u_2 a bounded, lower semicontinuous (l.s.c.), viscosity supersolution of $\partial_t u(t,x) = \overline{H}_0(x, \partial_x u(t,x))$ respectively. If $u_1(0,\cdot) \le u_2(0,\cdot)$, and $\bar{H}_0$ is uniformly continuous on compact sets, then $u_1 \le u_2$ on $[0,T]\times\mathbb{R}$ for any $T > 0$.*

Proof. Suppose

$$\sup_{t\le T,x} \{u_1(t,x) - u_2(t,x)\} > A \ge \delta > 0. \tag{A.34}$$

Let $g(t,x) = \ln(1+x^2) + t^2$. Define

$$\psi(t,x,s,y) = u_1(t,x) - u_2(s,y) - \frac{1}{2}\ln\left(1 + \frac{|x-y|^2 + |t-s|^2}{\epsilon}\right) - \beta\left(g(t,x) + g(s,y)\right) - At.$$

Fix $\beta > 0$ and let $(\bar{t}_\epsilon, \bar{x}_\epsilon, \bar{s}_\epsilon, \bar{y}_\epsilon)$ denote the point of maximum of ψ in $([0,T]\times\mathbb{R}\times[0,T]\times\mathbb{R})$ for $\epsilon > 0$. Since u_1, u_2 are bounded, for fixed $\beta > 0$, there exists an $R_\beta > 0$ such that $|\bar{x}_\epsilon|, |\bar{y}_\epsilon| \le R_\beta$ for all $\epsilon > 0$.

Using

$$\psi(\bar{t}_\epsilon, \bar{x}_\epsilon, \bar{t}_\epsilon, \bar{x}_\epsilon) + \psi(\bar{s}_\epsilon, \bar{y}_\epsilon, \bar{s}_\epsilon, \bar{y}_\epsilon) \le 2\psi(\bar{t}_\epsilon, \bar{x}_\epsilon, \bar{s}_\epsilon, \bar{y}_\epsilon),$$

we get

$$\begin{aligned}&\frac{1}{2}\ln\left(1 + \frac{|\bar{x}_\epsilon - \bar{y}_\epsilon|^2 + |\bar{t}_\epsilon - \bar{s}_\epsilon|^2}{\epsilon}\right)\\ &\quad\le A(\bar{s}_\epsilon - \bar{t}_\epsilon) + u_1(\bar{t}_\epsilon, \bar{x}_\epsilon) - u_1(\bar{s}_\epsilon, \bar{y}_\epsilon) + u_2(\bar{t}_\epsilon, \bar{x}_\epsilon) - u_2(\bar{s}_\epsilon, \bar{y}_\epsilon)\\ &\quad\le 2AT + 2\|u_1\| + 2\|u_2\| =: C < \infty,\end{aligned}$$

which gives us

$$|\bar{x}_\epsilon - \bar{y}_\epsilon|^2 + |\bar{t}_\epsilon - \bar{s}_\epsilon|^2 \le \epsilon e^{2C}.$$

Therefore $|\bar{x}_\epsilon - \bar{y}_\epsilon|, |\bar{s}_\epsilon - \bar{t}_\epsilon| \to 0$ as $\epsilon \to 0$.

Let

$$\phi_1(t,x) := u_2(\bar{s}_\epsilon, \bar{y}_\epsilon) + \frac{1}{2}\ln\left(1 + \frac{|x - \bar{y}_\epsilon|^2 + |t - \bar{s}_\epsilon|^2}{\epsilon}\right) + \beta\left(g(t,x) + g(\bar{s}_\epsilon, \bar{y}_\epsilon)\right) + At$$

and

$$\phi_2(s,y) := u_1(\bar{t}_\epsilon, \bar{x}_\epsilon) - \frac{1}{2}\ln\left(1 + \frac{|\bar{x}_\epsilon - y|^2 + |\bar{t}_\epsilon - s|^2}{\epsilon}\right) - \beta\left(g(\bar{t}_\epsilon, \bar{x}_\epsilon) + g(s,y)\right) - A\bar{t}_\epsilon.$$

Then $(\bar{t}_\epsilon, \bar{x}_\epsilon)$ is a point of maximum of $u_1(t,x) - \phi_1(t,x)$ and $(\bar{s}_\epsilon, \bar{y}_\epsilon)$ is a point of minimum of $u_2(s,y) - \phi_2(s,y)$. Since u_1 and u_2 are sub and super solutions respectively, by the definition of sub and super solutions we get

$$\frac{\frac{\bar{t}_\epsilon - \bar{s}_\epsilon}{\epsilon}}{1 + \frac{|\bar{x}_\epsilon - \bar{y}_\epsilon|^2 + |\bar{t}_\epsilon - \bar{s}_\epsilon|^2}{\epsilon}} + A + 2\beta\bar{t}_\epsilon \le \overline{H}_0\left(\bar{x}_\epsilon, \frac{\frac{\bar{x}_\epsilon - \bar{y}_\epsilon}{\epsilon}}{1 + \frac{|\bar{x}_\epsilon - \bar{y}_\epsilon|^2 + |\bar{t}_\epsilon - \bar{s}_\epsilon|^2}{\epsilon}} + \frac{2\beta\bar{x}_\epsilon}{1 + \bar{x}_\epsilon^2}\right), \tag{A.35}$$

and

$$\frac{\frac{\bar{t}_\epsilon-\bar{s}_\epsilon}{\epsilon}}{1+\frac{|\bar{x}_\epsilon-\bar{y}_\epsilon|^2+|\bar{t}_\epsilon-\bar{s}_\epsilon|^2}{\epsilon}}-2\beta\bar{s}_\epsilon \geq \overline{H}_0\left(\bar{y}_\epsilon, \frac{\frac{\bar{x}_\epsilon-\bar{y}_\epsilon}{\epsilon}}{1+\frac{|\bar{x}_\epsilon-\bar{y}_\epsilon|^2+|\bar{t}_\epsilon-\bar{s}_\epsilon|^2}{\epsilon}}-\frac{2\beta\bar{y}_\epsilon}{1+\bar{y}_\epsilon^2}\right). \tag{A.36}$$

Subtracting (A.36) from (A.35), we get

$$\begin{aligned} A+2\beta(\bar{t}_\epsilon+\bar{s}_\epsilon) \leq\ & \overline{H}_0\left(\bar{x}_\epsilon, \frac{\frac{\bar{x}_\epsilon-\bar{y}_\epsilon}{\epsilon}}{1+\frac{|\bar{x}_\epsilon-\bar{y}_\epsilon|^2+|\bar{t}_\epsilon-\bar{s}_\epsilon|^2}{\epsilon}}+\frac{2\beta\bar{x}_\epsilon}{1+\bar{x}_\epsilon^2}\right) \\ & -\overline{H}_0\left(\bar{y}_\epsilon, \frac{\frac{\bar{x}_\epsilon-\bar{y}_\epsilon}{\epsilon}}{1+\frac{|\bar{x}_\epsilon-\bar{y}_\epsilon|^2+|\bar{t}_\epsilon-\bar{s}_\epsilon|^2}{\epsilon}}-\frac{2\beta\bar{y}_\epsilon}{1+\bar{y}_\epsilon^2}\right). \end{aligned} \tag{A.37}$$

Since $\overline{H}_0(\cdot,\cdot)$ is uniformly continuous over compact sets, and since $|\bar{x}_\epsilon-\bar{y}_\epsilon| \to 0$ as $\epsilon \to 0$ (for fixed β), the right-hand side of the above inequality goes to 0 as $\epsilon \to 0$ and $\beta \to 0$ (note that the terms $\frac{\frac{\bar{x}_\epsilon-\bar{y}_\epsilon}{\epsilon}}{1+\frac{|\bar{x}_\epsilon-\bar{y}_\epsilon|^2+|\bar{t}_\epsilon-\bar{s}_\epsilon|^2}{\epsilon}}$, $\frac{2\bar{x}_\epsilon}{1+\bar{x}_\epsilon^2}$ and $\frac{2\bar{y}_\epsilon}{1+\bar{y}_\epsilon^2}$ are bounded and that $|\bar{x}_\epsilon|, \bar{y}_\epsilon| \leq R_\beta$ for each β).

Taking $\epsilon \to 0$ and then $\beta \to 0$, we get

$$A \leq 0,$$

which contradicts (A.34). Therefore we must have

$$\sup_{t,x}\{u_1(t,x)-u_2(t,x)\} \leq 0$$

which gives us $u_1 \leq u_2$. $\square$

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