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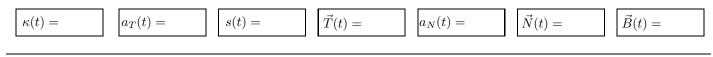
# February 11, 2005

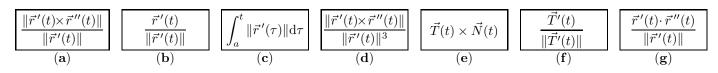
Instructor: Dr. Marco Bertola

Material allowed: non programmable, non-graphing calculators.

# Solve five problems of your choice

Self-serve formula sheet. For a vector-valued functions  $\vec{r}(t)$  in three dimension defining a smooth curve for  $t \in [a, b]$ 





# [10 points] Problem 1.

(i) Find the velocity and the acceleration vectors  $\vec{v}(t)$ ,  $\vec{a}(t)$  for a particle that moves as described by the vector-valued function

 $\vec{r}(t) = t\mathbf{i} + [t\sin(t) + \cos(t)]\mathbf{j} + [t\cos(t) - \sin(t)]\mathbf{k} .$ 

(ii) Compute the unit tangent  $\vec{T}(t)$ .

(iii) Compute the tangent component of the acceleration  $a_T$ .

# [10 points] Problem 2.

Using the chain rule, compute the partial derivatives  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  for the function f

$$f(u,v) = \cos(uv),$$

where the variables u, v depend on the variables x, y as follows

$$u = \ln(1 + x^2)$$
;  $v = x^2 y$ 

# [10 points] Problem 3.

For the given curve find the arclength parameter s(t) and reparametrize the curve in terms of it

$$\vec{r}(t) = t^2 \mathbf{i} + [t\sin(t) + \cos(t)]\mathbf{j} + [t\cos(t) - \sin(t)]\mathbf{k}, \quad t \ge 0$$

# [Advice: the ensuing integral is immediate if your derivatives are done properly]

# [10 points] Problem 4.

(i) Compute the gradient of the function

$$F(x, y, z) = xy^2 \cos(yz).$$

(ii) Find a direction along which the function F increases as fast as possible at the point  $(0, 1, \pi)$ .

(iii) Compute the directional derivative at the point  $(0, 1, \pi)$  along the direction of the vector  $\vec{u} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ .

# [10 points] Problem 5.

Find points on the surface  $x^2 + 3y^2 + 4z^2 - 2xy = 16$  at which the tangent plane is parallel to the zx-plane.

# [10 points] Problem 6.

Compute the divergence of the gradient of the following function of three variables

$$F(x, y, z) = x^2 y z - \frac{1}{3} y^3 z + z^2$$

### Solution to Problem 1

We have

$$\vec{v}(t) = \mathbf{i} + t\cos(t)\mathbf{j} - t\sin(t)\mathbf{k}$$
$$\vec{a}(t) = (\cos(t) - t\sin(t))\mathbf{j} + (-t\cos(t) - \sin(t))\mathbf{k}$$
$$\vec{T} = \frac{\vec{v}}{v} = \frac{1}{\sqrt{1+t^2}} (\mathbf{i} + t\cos(t)\mathbf{j} - t\sin(t)\mathbf{k})$$

The tangent component is

$$a_T(t) = \vec{T} \cdot \vec{a} = \frac{t \cos^2(t) - t^2 \sin(t) \cos(t) + t \sin^2(t) + t^2 \sin(t) \cos(t)}{\sqrt{1 + t^2}} = \frac{t}{\sqrt{1 + t^2}}$$

# Solution to Problem 2

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \\ &= -v \sin(uv) \frac{2x}{1+x^2} - u \sin(uv) 2xy = -\sin\left(x^2 y \ln(1+x^2)\right) \left(\frac{2x^3 y}{1+x^2} + 2xy \ln(1+x^2)\right) \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = \\ &= -v \sin(uv) 0 - u \sin(uv) x^2 = -\sin\left(x^2 y \ln(1+x^2)\right) \ln(1+x^2) x^2 \end{aligned}$$

### Solution to Problem 3

The arclength parameter is

$$s(t) = \int_0^t v(\tau) \mathrm{d}\tau$$

Here the speed  $v(\tau)$  is

$$\vec{v}(\tau) = 2\tau \mathbf{i} + \tau \cos(\tau) \mathbf{j} - \tau \sin(\tau) \mathbf{k}$$
$$v(\tau) = \sqrt{4\tau^2 + \tau^2 \cos^2(\tau) + \tau^2 \sin^2(\tau)} = \sqrt{5\tau^2} = \tau \sqrt{5} \quad \text{(since } \tau \ge 0\text{)}$$

Therefore

$$s(t) = \int_0^t \tau \sqrt{5} \mathrm{d}\tau = \frac{\sqrt{5}}{2} t^2$$

To reparametrize the curve we have to express t in terms of s. Solving the above relation for t we have

$$t(s) = \sqrt{\frac{2}{\sqrt{5}}s}$$

We now plug this into the expression for  $\vec{r}(t)$ 

$$\frac{2}{\sqrt{5}}s\mathbf{i} + \left[\sqrt{\frac{2}{\sqrt{5}}s}\sin(\sqrt{\frac{2}{\sqrt{5}}s}) + \cos(\sqrt{\frac{2}{\sqrt{5}}s})\right]\mathbf{j} + \left[\sqrt{\frac{2}{\sqrt{5}}s}\cos(\sqrt{\frac{2}{\sqrt{5}}s}) - \sin(\sqrt{\frac{2}{\sqrt{5}}s})\right]\mathbf{k}$$

# Solution to problem 4

We have

$$\vec{\nabla}F(x,y,z) = y^2\cos(yz)\mathbf{i} + \left(2xy\cos(yz) - xy^2z\sin(yz)\right)\mathbf{j} - xy^3\sin(yz)\mathbf{k}$$

At the point  $(0, 1, \pi)$  the gradient is

$$\vec{\nabla}F(0,1,\pi) = \cos(\pi)\mathbf{i} = -\mathbf{i}.$$

Since it is already normalized (length one) the direction of max increase at this point is  $-\mathbf{i}$  (the gradient itself). To compute the directional derivative along  $\vec{u} = \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k})$  we have to compute

$$D_{\vec{u}}F|_{(0,1,\pi)} = \vec{\nabla}F(0,1,\pi) \cdot \vec{u} = (-\mathbf{i}) \cdot \left(\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})\right) = 0$$

#### Solution to Problem 5

For a plane to be parallel to the xz-plane it is necessary that the normal is -say- **j** (y-direction). That is the gradient of the function defining the surface must be parallel to **j** or equivalently must have zero components in the x and z direction. The gradient is

$$\vec{\nabla}F = (2x - 2y)\mathbf{i} + (6y - 2x)\mathbf{j} + 8z\mathbf{k}$$

Therefore we have to solve the system

$$\begin{cases} 2x - 2y = 0\\ 8z = 0\\ x^2 + 3y^2 + 4z^2 - 2xy = 16 \end{cases} \Rightarrow \begin{cases} x = y\\ z = 0\\ 4x^2 - 2x^2 = 16 \end{cases} \Rightarrow x = \pm\sqrt{8}$$

There are thus two points as required, namely  $(\sqrt{8}, \sqrt{8}, 0)$  and  $(-\sqrt{8}, -\sqrt{8}, 0)$ .

Solution to Problem 6 The gradient is

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abla} = V \hat{
abla} F(x,y,z) = 2xyz\mathbf{i} - y^2z\mathbf{j} + \left(x^2y - \frac{1}{3}y^3 + 2z\right)\mathbf{k}$$

$$\vec{\nabla}\cdot\vec{\nabla}F=\partial_x(\partial_xF)+\partial_y(\partial_yF)+\partial_z(\partial_zF)=2yz-2yz+2=2$$

Note that the end-result is a **scalar**