Topics in Complex Analysis

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Chapter 1

Introduction

1 Complex numbers, complex plane

We define the following structure (=set + operations)

$$\mathbb{C} := \{z := (a,b) \mid a,b \in \mathbb{R}\} \ (= \mathbb{R}^2) \tag{1.1.1}$$

$$(a,b) + (c,d) := (a+c, b+d) \tag{1.1.2}$$

$$(a,b) \cdot (c,d) := (ac - bd, ad + bc) \tag{1.1.3}$$

Lemma 1.1.3 The strucuture $(\mathbb{C}, +, \cdot)$ satisfies the axioms of **field** with the multiplicative inverse given as follows:

$$z = (a,b) \neq 0 =: (0,0) \ , \quad z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \right) \ , \quad z \cdot z^{-1} = 1 =: (1,0). \tag{1.1.4}$$

Remark 1.1 For which n's we can endow the set $\mathbb{R}^n$ with a structure of division algebra? Only for $n = 1, 2, 4, 8$. This was proved by J. Adams in 1962 using methods of algebraic topology.

We will use the following shortccut for complex numbers

$$z = a + ib \ , \ i^2 = -1, \quad i := (0,1). \tag{1.1.5}$$

We also define the following *real-valued* functions

$$\Re(z) := a \ , \quad \Im(z) = b \tag{1.1.6}$$

and the following operation

$$\begin{array}{rcl} \bar{\ } : \mathbb{C} & \to & \mathbb{C} \qquad (1.1.7) \\ z & \mapsto & \overline{z} := a - ib \end{array} \tag{1.1.8}$$

We then have the following **properties**:

$$\Re(z) = \frac{z+\overline{z}}{2}\ , \qquad \Im(z) = \frac{z-\overline{z}}{2i} \tag{1.1.9}$$

$$\overline{z \pm w} = \overline{z} \pm \overline{w} \tag{1.1.10}$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \tag{1.1.11}$$

[From now on we omit the · to denote the multiplication of complex numbers]
The **modulus** and **argument** are defined by

$$|z| := \sqrt{z\overline{z}} = \overline{a^2+b^2} \geq 0 \ \text{ and } \ = 0 \text{ iff } z = 0. \tag{1.1.12}$$

$$\arg(z) = \phi(\text{mod } 2\pi)\ s.t.\ \cos\phi = \frac{\Re(z)}{|z|}\ ,\ \ \sin\phi = \frac{\Im(z)}{|z|}\ . \tag{1.1.13}$$

Lemma 1.1.13 [Exercise] Prove that

$$|\ |z| - |w|\ | \leq |z+w| \leq |z| + |w| \tag{1.1.14}$$

Exercise 1.1 *Find all complex numbers satisfying the following relations*

$$z = \overline{z} \tag{1.1.15}$$

$$z^{-1} = \frac{\overline{z}}{|z|^2},\ \ z \neq 0. \tag{1.1.16}$$

Examples 1.1 .

1. The map $z \mapsto \overline{z}$ is the reflection about the Real axis.
2. The set $\{|z - z_0| = R\}$ is the circle of radius R centered at z_0.
3. The set $\{|\arg(z) - \theta_0| < \epsilon\}$ is a wedge of width 2ϵ with bisecant forming an agle θ_0 with the positive real axis.

Lemma 1.1.16 [Exercise]

$$|zw| = |z|\,|w| \tag{1.1.17}$$

$$arg(zw) = arg(z) + arg(w) \text{ mod } 2\pi \tag{1.1.18}$$

Corollary 1.1.18 (Exercise) *The map $z \mapsto \lambda z$ with $\lambda \in \mathbb{C}^\times$ ($\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$) is a rotation of angle* $\arg(\lambda)$ *followed by a dilation by $|\lambda|$.*

Example 1.1 *Using the tautological identification of $\mathbb{C} \simeq \mathbb{R}^2$, find the matrix that represents the linear map $T(z) := \lambda z : \mathbb{C} \to \mathbb{C}$ as a linear transformation $\tilde{T} : \mathbb{R}^2 \to \mathbb{R}^2$. Characterize all linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ that can be represented by multiplication by a complex number.*

Lemma 1.1.18 [**Euler's formula**] Let $z = x + iy$, then

$$e^{x+iy} = e^x \bigg(\cos(y) + i\sin(y) \bigg) \tag{1.1.19}$$

Exercise 1.2 *Using Euler's formula prove that*

$$T_n(x) := \cos(n\theta) \ , \quad x := \cos(\theta) \tag{1.1.20}$$

is a **polynomial** *in x of degree n (Tchebicheff's polynomial).*
[Hint: $\cos(n\theta) = \Re(\exp(in\theta))$*].*

2 Sequences, series and convergence

The topology of $\mathbb{C}$ is inherited from the tautological identification with $\mathbb{R}^2$ (which is a **metric space**) i.e.

Definition 2.1 *A sequence $z_n : \mathbb{N} \to \mathbb{C}$ converges $\lim_{n\to\infty} z_n = w$ iff $\lim_{n\to\infty} |z_n - w| = 0$.*

Since $\mathbb{R}^2$ is **complete**, so is $\mathbb{C}$, namely

Corollary 1.2.0 *Every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$, or, in more detail:*
A sequence $\{z_n\}_{n\in\mathbb{N}}$ **converges** *iff*

$$\forall \epsilon > 0 \ \ \exists N \ \ s.t. \ \forall n, m > N \ \ |z_n - z_m| < \epsilon \tag{1.2.1}$$

As for real numbers, convergence for **series** is defined according to the convergence of the **partial sums**, i.e. the series

$$\sum_{n=0}^{\infty} z_n \tag{1.2.2}$$

converges iff the sequence of its partial sums $s_n := \sum_{j=0}^{n} z_j$ converges.

Lemma 1.2.2 [Exercise] If the series

$$\sum_{n=0}^{\infty} |z_n| \tag{1.2.3}$$

converges then the series $\sum_{n=0}^{\infty} z_n$ converges as well and

$$\left| \sum_{n=0}^{\infty} z_n \right| \leq \sum_{n=0}^{\infty} |z_n| \tag{1.2.4}$$

Definition 2.2 *If the series $\sum_{n=0}^{\infty} |z_n|$ converges we say that the series $\sum_{n=0}^{\infty} z_n$ is* **absolutely convergent**.

Example 2.1 *Suppose we have* $\sum^{\infty} z_n$ *and* $\sum^{\infty} w_n$ *two absolutely convergent series, with sum respectively* Z *and* W*. Prove (with an* $\epsilon - \delta$ *proof) that*

$$\sum a z_n + b w_n = aZ + bW \tag{1.2.5}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z_n w_m = ZW \tag{1.2.6}$$

3 Functions of one complex variable

Definition 3.1 *A subset* $\mathcal{D}$ *of* $\mathbb{C}$ *is called a* **domain** *if it is* **open** *and* **connected**.

Recall that a set Y in a topological space (X, τ) is said to be connected if the following condition applies

$$Y = Y_1 \cup Y_2, \quad Y_1 \text{ and } Y_2 \text{ open subsets }, Y_j \neq \emptyset, Y, \; Y_1 \cap Y_2 = \emptyset. \tag{1.3.1}$$

We now begin the study of the properties of functions

$$f : \mathcal{D} \to \mathbb{C} \tag{1.3.2}$$

Limits and **continuity** are defined as for any maps between topological (actually metric) spaces. Differentiability can be defined using the tautological identification $\mathbb{C} \simeq \mathbb{R}^2$, however we will need a refinement of this notion which is "compatible" with the structure of the complex plane. We will use the notations

$$z = x + iy \quad f(z) = u(x, y) + iv(x, y). \tag{1.3.3}$$

Exercise 3.1 *Formulate the continuity requirement and prove that a function* $f = u + iv$ *is continuous (at a point/ on a domain) iff both its real and imaginary parts (*u *and* v*) are. Prove that the product, linear combination, ratio (iff the denominator does not vanish) of two continuous functions* f, g *is continuous.*

Examples 3.1 .

1. $f(z) = z$
2. $f(z) = \overline{z}$
3. $f(x) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} z^n \overline{z}^m$

Remark 3.1 [Exercise] If $\mathcal{D} \subset \mathbb{C}$ is closed and bounded (and hence **compact**) then a continuous function f is also **uniformly continous**, namely

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall z, w \in \mathcal{D}, \; |z - w| < \delta \;\Rightarrow\; |f(z) - f(w)| < \epsilon \; . \tag{1.3.4}$$

Chapter 2

Calculus

1 Holomorphic functions

Definition 1.1 *Given a function $f(z) : \mathcal{D} \to \mathbb{C}$ we say that it has* **complex derivative** *at z_0 if the following limit exists finite*

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \ . \tag{2.1.1}$$

Note that the ratio is the ratio of complex numbers and the limit is taken in $\mathbb{C}$.

Example 1.1 *The function $f(z) = z\overline{z}$ does* **not** *have complex derivative in the above sense: however it is differentiable when seen as a function $\mathbb{R}^2 \to \mathbb{R}^2$.*

Exercise 1.1 *Prove that the functions $f(z) = z^n$, $n \in \mathbb{Z}$ have complex derivative at all points in $\mathbb{C}$ (excluding $z = 0$ for $n < 0$) and that*

$$f'(z) = nz^{n-1} \ . \tag{2.1.2}$$

Definition 1.2 *A function $f : \mathcal{D} \to \mathbb{C}$ defined on the domain $\mathcal{D}$ is called* **holomorphic** *if it has complex derivative at all points of the domain.*

We have the first fundamental

Theorem 2.1.2 (Cauchy–Riemann equations) *A holomorphicc function $f(z) = u(x, y) + iv(x, y)$ on the domain $\mathcal{D} \subset \mathbb{C}$ is* **differentiable** *(as a function $\mathbb{R}^2 \to \mathbb{R}^2$) at all points of $\mathcal{D}$. Moreover the partial derivatives satisfy:*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{2.1.3}$$

Viceversa, if $u(x, y), v(x, y)$ are differentiable functions $u, v : \mathcal{D} \to \mathbb{R}$ satisfying Cauchy-Riemann equations (2.1.3) then the function $f(z) = u + iv$ is a holomorphic function.

Proof. From the definition of complex derivative

$$f(z+h) - f(z) = f'(z)h + o(|h|) \tag{2.1.4}$$

where the standard symbol *small "o"* $o(|h|)$ means an infinitesimal quantity w.r.t. $|h|$,

$$\frac{o(|h|)}{|h|} \to 0 \ , \quad \text{as } |h| \to 0 \ . \tag{2.1.5}$$

We denote $z = x + iy$, $h = \Delta x + i\Delta y$, $f'(z) = \alpha + i\beta$ and we collect real and imaginary parts

$$u(x+\Delta x, y+\Delta y) - u(x,y) = \alpha\Delta x - \beta\Delta y + o(\sqrt{\Delta x^2 + \Delta y^2}) \tag{2.1.6}$$

$$v(x+\Delta x, y+\Delta y) - v(x,y) = \beta\Delta x + \alpha\Delta y + o(\sqrt{\Delta x^2 + \Delta y^2}) \tag{2.1.7}$$

These two equations precisely spell the definition of differentiability of functions $\mathbb{R}^2 \to \mathbb{R}$. In order to obtain the CR equations we first set $\Delta y = 0$ and $\Delta x \to 0$ and obtain

$$\alpha + i\beta = u_x + iv_x \ . \tag{2.1.8}$$

Setting $\Delta_x = 0$ and $\Delta y \to 0$ instead yields

$$\alpha + i\beta = v_y - iu_y \tag{2.1.9}$$

Equating these two identities gives (2.1.3).

The "viceversa" part of the theorem is left as exercise. Q.E.D.

Remark 1.1 The differential of a map

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \tag{2.1.10}$$

$$(x,y) \mapsto (u(x,y), v(x,y)) \tag{2.1.11}$$

at a point (x,y) is a linear map J (the "Jacobian") that gives the approximation of f up to $o(\sqrt{\Delta x^2 + \Delta y^2})$. The matrix representing this linear map is

$$\begin{pmatrix} u(x+\Delta x, y+\Delta y) - u(x,y) \\ v(x+\Delta x, y+\Delta y) - v(x,y) \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(\sqrt{\Delta x^2 + \Delta y^2}) \tag{2.1.12}$$

Cauchy–Riemann equations simply say that the Jacobian can be interpreted as multiplication by a complex number

$$J = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} \tag{2.1.13}$$

and hence corresponds (in the tangent spaces) to a dilation by $|f'(z)|$ followed by a rotation by $\arg f'(z)$. Such maps are called "conformal".

Notation We introduce the symbols of complex differentiation as follows

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{2.1.14}$$

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.1.15}$$

as well as the **complex differentials**

$$\mathrm{d}z := \mathrm{d}x + i\mathrm{d}y \tag{2.1.16}$$

$$\mathrm{d}\overline{z} := \mathrm{d}x - i\mathrm{d}y\ . \tag{2.1.17}$$

With these notations in place we define the differential of a function $f : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathrm{d}f := \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}\ . \tag{2.1.18}$$

Then we have

Lemma 2.1.18 [Exercise] Cauchy–Riemann equations (2.1.3) are equivalent to $\frac{\partial f}{\partial \overline{z}} \equiv 0$.

The *rationale* of the above definitions is that of a "change of variables" from the complex z to the real x, y

$$z = x + iy, \quad \overline{z} = x - iy \tag{2.1.19}$$

$$x = \frac{z + \overline{z}}{2}\ , \quad y = \frac{z - \overline{z}}{2i}\ . \tag{2.1.20}$$

Exercise 1.2 *Show that the polynomial*

$$P(z, \overline{z}) = \sum_{n=0}^{N} \sum_{m=0}^{M} c_{nm} z^n \overline{z}^m \tag{2.1.21}$$

is a **differentiable map**: $\mathbb{R}^2 \to \mathbb{R}^2$ *(identifying tautologically $\mathbb{C} \simeq \mathbb{R}^2$). Show that it is holomorphic in $\mathbb{C}$ iff $c_{nm} = 0$ for $m > 0$.*

Exercise 1.3 *Let f, g be two* holomorphic *functions on the domain $\mathcal{D}$. Prove the standard formulæ ($\epsilon - \delta$ proof)*

$$(af(z) + bg(z))' = af'(z) + bg'(z) \tag{2.1.22}$$

$$(f(z)g(z))' = f'(z)g(z) + f(z)g'(z) \tag{2.1.23}$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)} \qquad \textit{provided that } g(z) \neq 0\ ,\ z \in \mathcal{D}. \tag{2.1.24}$$

Lemma 2.1.24 [Exercise] Let $f : \mathcal{D} \to \mathbb{C}$ and $g : \mathcal{F} \to \mathbb{C}$ be holomorphic on their respective domains: le $f(\mathcal{D}) \supset \mathcal{F}$. Then $g \circ f : \mathcal{D} \to \mathbb{C}$ defined as $g(f(z))$, is holomorphic and

$$\frac{\mathrm{d}}{\mathrm{d}z} g(f(z)) = g'(f(z))\, f'(z). \tag{2.1.25}$$

2 Power series

Consider a sequence $\{f_n : \mathcal{D} \to \mathbb{C}\}_{n\in\mathbb{N}}$ of functions defined on the same domain $\mathcal{D} \subset \mathbb{C}$. We say that the sequence converges **uniformly** to $f(z)$, $z \in \mathcal{D}$ if

$$\forall \epsilon > 0\ \exists N \text{ s.t. } \forall n > N, z \in \mathcal{D}\ \ |f_n(z) - f(z)| < \epsilon \tag{2.2.1}$$

Uniform convergence of series of functions is defined as the uniform convergence of the sequence of its partial sums. We have

Theorem 2.2.1 (Weierstrass) *Consider the series*

$$\sum_{n=0}^{\infty} u_n(z) \tag{2.2.2}$$

where each u_n is defined for $z \in \mathcal{D}$ and is such that

$$|u_n(z)| \le A_n\ \forall n \ge 0,\ \ \forall z \in \mathcal{D} \tag{2.2.3}$$

where the series $\sum A_n$ is convergent. Then the series $\sum u_n$ is uniformly and absolutely convergent on $\mathcal{D}$ and

$$\left|\sum u_n(z)\right| \le \sum A_n \tag{2.2.4}$$

The particular case of power series is very important

$$f(z) := \sum_{n=0}^{\infty} c_n (z-a)^n \tag{2.2.5}$$

We recall some facts that follow immediately from analgous facts valid for power series of one real variable.

1. For any series $\sum c_n(z-a)^n$ there exists a $0 \le R \le \infty$ such that the series is absolutely convergent in the **disk**
$$D_a(R) := \{|z-a| < R\} \cup \{a\} \tag{2.2.6}$$
and divergent in $\mathbb{C} \setminus \overline{D_a(R)}$ (the complement of the closure). (In the case $R = 0$ the disk degenerates to the center $z = a$ and for $R = \infty$ the disk degenerates to the whole complex plane).

2. The power series converges **uniformly and absolutely** in any closed disk $\overline{D_a(r)}$ of radius $r < R$.

3. The radius of convergence is uniquely characterized by **Cauchy–Hadamard**'s formula
$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}} \tag{2.2.7}$$

4. The sum of the series is a **holomorphic function** $f(z)$ defined in the disk $D_a(R)$. where we understand that if the limit is ∞ the radius is defined as zero, if the limit is zero, the radius is defined as infinity.

Remark 2.1 We recall the definition of lim sup: given a sequence a_n of real numbers we say that

$$\limsup_{n\to\infty} a_n = A \tag{2.2.8}$$

if the sequence $S_n := \sup\{a_j,\ j \geq n\}$ (which is **weakly decreasing**) has limit A, namely iff

$$A = \lim_{n\to\infty} \sup\{a_j,\ j \geq n\} = \inf_{n\in\mathbb{N}} \sup\{a_j,\ j \geq n\} \tag{2.2.9}$$

Exercise 2.1 *Prove the last statement, namely that the sum of the series is holomorphic. Specifically you need to prove that the complex derivative exists in the open disk of convergence (using Weierstrass theorem).*

Apart from the issue of convergence which is addressed in the previous exercise, it is clear that the formula for the derivative is

$$f'(z) = \sum nc_n(z-a)^n \ . \tag{2.2.10}$$

Since the radius of convergence of the above series is the same, by iteration we conclude also that $f(z)$ has infinitely many derivatives and hence it is $\mathcal{C}^\infty(D_a(R))$. Thus one has (we shift $z \to z-a$ so as to have a series centered at $z=0$)

Corollary 2.2.10 *The sum*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{2.2.11}$$

converges uniformly and absolutely in any closed disk contained in $|z| < R$ to an infinitely differentiable function, with complex derivatives given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n z^{n-k} \tag{2.2.12}$$

The coefficients c_n are uniquely determined by Taylor's formula

$$c_n = \frac{1}{n!} f^{(n)}(0) \ . \tag{2.2.13}$$

3 Integration

We consider the one-form

$$\omega := P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y \ , \tag{2.3.1}$$

where P, Q are at least $\mathcal{C}^0$ and complex valued. Given a curve $\gamma : [0,1] \to \mathbb{C}$ (piecewise $\mathcal{C}^1$) we define the integral of ω along γ by

$$\int_\gamma \omega := \int_0^1 (P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t))\mathrm{d}t \tag{2.3.2}$$

We may need to split this integral in a finite sum of integrals on the subsegments of $[0,1]$ where the curve admits tangent, if necessary. Also it is understood that ω is defined in a domain containing the curve so that the formula makes sense at all.

Let $\mathcal{D}$ be a domain such that the closure $\overline{\mathcal{D}}$ is compact (i.e. *relatively compact*) and such that the boundary

$$\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D} \tag{2.3.3}$$

is a **closed, piecewise smooth curve**. The orientation of $\partial\mathcal{D}$ is defined so that in each connected components of $\partial\mathcal{D}$ the positive orientation is the one such that "walking" along the curve, the interior of $\mathcal{D}$ is on your left.

Theorem 2.3.3 (Stokes' thm.) *We have the formula*

$$\int_{\partial\mathcal{D}} \omega = \int_{\mathcal{D}} \mathrm{d}\omega \tag{2.3.4}$$

where

$$\mathrm{d}\omega := (\partial_x Q - \partial_y P)\, \mathrm{d}x \wedge \mathrm{d}y. \tag{2.3.5}$$

Here we assume that ω is at least $\mathcal{C}^1(\mathcal{D})$ and $\mathcal{C}^0(\overline{\mathcal{D}})$ (i.e. differentiable inside and continuous up to the boundary). The orientation of the boundary must be the positive one.

We do not prove this theorem but the proof is not (horribly) difficult. Recall also

Definition 3.1 *A one-form $\omega = P\mathrm{d}x + Q\mathrm{d}y$ continuous on the domain $\mathcal{D}$, $P, Q \in \mathcal{C}^1(\mathcal{D})$ is said to be* **closed** *if $\mathrm{d}\omega \equiv 0$. It is said to be* **exact** *if there exists a (complex valued) function $f(x, y)$ such that $\mathrm{d}f = \omega$.*

Recall that every exact one-form is closed, but the converse is not necessarily true. Moreover recall

Exercise 3.1 *Suppose ω is exact, i.e. $\omega = df$ for some ($\mathcal{C}^1$) function $f : \mathcal{D} \to \mathbb{C}$; let $\gamma : [0,1] \to \mathcal{D}$ be a piecewise smooth curve. Then*

$$\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0)) \ . \tag{2.3.6}$$

Using the operators $\partial_z, \partial_{\overline{z}}$ introduced in (2.1.15) we can rewrite a one-form in complex notation

$$\omega = P\mathrm{d}x + Q\mathrm{d}y = \tilde{P}\mathrm{d}z + \tilde{Q}\mathrm{d}\overline{z} \tag{2.3.7}$$

$$\tilde{P} := \frac{1}{2}(P - iQ) \ , \qquad \tilde{Q} := \frac{1}{2}(P + iQ) \tag{2.3.8}$$

and introduce the total differential $\mathrm{d} = \partial + \overline{\partial}$

$$\mathrm{d}\omega := \partial\omega + \overline{\partial}\omega = \partial_z\omega \wedge \mathrm{d}z + \partial_{\overline{z}}\omega \wedge \mathrm{d}\overline{z}. \tag{2.3.9}$$

The condition of ω being closed is then written as

$$0 \equiv \mathrm{d}\omega = \left(\partial_z\tilde{Q} - \partial_{\overline{z}}\tilde{P}\right)\mathrm{d}z \wedge \mathrm{d}\overline{z}. \tag{2.3.10}$$

We also recall the following definitions

Definition 3.2 *Two (piecewise smooth) curves $\gamma_1, \gamma_2 : [0,1] \to \mathcal{D}$ such that*

$$\gamma_1(0) = \gamma_2(0) \tag{2.3.11}$$

$$\gamma_1(1) = \gamma_2(1) \tag{2.3.12}$$

are said to be **$\mathcal{D}$-homotopic at fixed enpoints** *if there exists a* **homotopy**, *namely a continous function*

$$\Gamma := [0,1] \times [0,1] \to \mathcal{D} \tag{2.3.13}$$

such that

$$\Gamma(0,t) = \gamma_1(t), \quad \Gamma(1,t) = \gamma_2(t) \tag{2.3.14}$$

We can think of $\Gamma(s, \bullet)$ as a continuous deformation (parametrized by $s \in [0,1]$) of the curve γ_1 into the curve γ_2. The homotopy relation is an **equivalence** relation amongst all parametrized curves with the same starting and ending points. If $\mathcal{D}$ is the whole complex plane, we usually omit reference to the domain. It can be proved that if two curves are $\mathcal{D}$-homotopic, with $\mathcal{D}$ a domain, then the homotopy can be chosen of class $\mathcal{C}^1$ in $(0,1) \times [0,1]$.

Corollary 2.3.14 *Let ω be a $\mathcal{C}^1(\mathcal{D})$ closed one-form and γ_i, $i = 1, 2$ be two $\mathcal{D}$–homotopic curves at fixed endpoints; then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{2.3.15}$$

Exercise 3.2 *Prove the above corollary.*

We can define homotopy also for closed curves

Definition 3.3 *Two closed curves $\gamma_i : [0,1] \to \mathcal{D}$, ($\gamma_i(0) = \gamma_i(1)$, $i = 1, 2$) are said to be $\mathcal{D}$–homotopic if there exists a continous (smooth) $\Gamma : [0,1] \times [0,1]$ such that*

$$\Gamma(0,t) = \gamma_1(t) \ , \qquad \Gamma(1,t) = \gamma_2(t) \ , \quad \forall t \in [0,1] \Gamma(s,0) = \Gamma(s,1) \ , \quad \forall s \in [0,1] \ . \tag{2.3.16}$$

Also this notion of homotopy is an equivalence relation among all closed curves in $\mathcal{D}$.

Definition 3.4 *A closed curve in $\mathcal{D}$ is* **contractible** *if it is homotopic to a constant curve. If every closed curve in $\mathcal{D}$ is contractible, we say that $\mathcal{D}$ is* **simply connected***.*

Definition 3.5 *A domain $\mathcal{D}$ is said to be* **star-shaped** *if there is a point $P_0 \in \mathcal{D}$ such that the segment from P_0 to any other point $P \in \mathcal{D}$ is entirely contained in D.*

Exercise 3.3 *Prove that every star-shaped domain is simply-connected.*

Lemma 2.3.16 [Exercise] If $\mathcal{D}$ is simply connected then any closed one-form ω is exact.

Exercise 3.4 *Suppose that $\omega \in \mathcal{C}^1(\mathcal{D})$ has the property that $\oint_\gamma \omega = 0$ for every closed curve $\gamma \subset \mathcal{D}$. Prove that ω is exact.*

Exercise 3.5 *Prove that $\omega = \frac{\mathrm{d}z}{z}$ is closed but not exact in $\mathcal{D} = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.*

3.1 Cauchy's Theorem

We now consider a particular class of closed differentials

$$\omega = f(z)\mathrm{d}z = f(x+iy)\mathrm{d}x + if(x+iy)\mathrm{d}y \tag{2.3.17}$$

where $f(z)$ is holomorphic.

Exercise 3.6 *Verify that for $f(z)$ holomorphic in $\mathcal{D}$ then the one-form $\omega = f(z)\mathrm{d}z$ is closed in $\mathcal{D}$.*

We now prove

Theorem 2.3.17 *Let $\mathcal{D}$ be simply connected and $f(z)$ holomorphic in $\mathcal{D}$. Then*

$$\oint_\gamma f(z)\mathrm{d}z = 0 \tag{2.3.18}$$

for any closed contour $\gamma \subset \mathcal{D}$

Proof. The proof is an exercise **if** we assume $f(z) \in \mathcal{C}^1(\mathcal{D})$ (because then we can use homotopy arguments).

We prove it without assumptions first for a **rectangle** $\Pi \subset \mathcal{D}$ namely

$$I(\Pi) := \int_{\partial\Pi} f(z)\mathrm{d}z = 0 \ , \quad \forall \Pi \subset \mathcal{D}, \quad \text{rectangle} \tag{2.3.19}$$

We use a dicotomy argument: suppose there is $\Pi \subset \mathcal{D}$ such that $I(\Pi) \neq 0$. Divide Π into four equal rectangles Π_j and

$$I(\Pi) = \sum_{j=1}^{4} I(\Pi_j) \ . \tag{2.3.20}$$

For at least one of them (say Π_1) we have

$$|I(\Pi_1)| \geq \frac{1}{4} I(\Pi) > 0 \tag{2.3.21}$$

We repeat the argument on Π_1 so that we obtain a sequence of nested rectangles Π_n of areas $\mathcal{A}(\Pi_n) = \mathcal{A}(\Pi)4^{-n}$. Let z_0 be in the intersection of all Π_n's

$$\{z_0\} \subset \bigcap_n \Pi_n \tag{2.3.22}$$

Since $f(z)$ is holomorphic at $z = z_0$ we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|). \tag{2.3.23}$$

Hence

$$I(\Pi_n) = f(z_0) \int_{\partial\Pi_n} 1\mathrm{d}z + f'(z_0) \int_{\partial\Pi_n} (z - z_0)\mathrm{d}z + \int_{\partial\Pi_n} o(|z - z_0|)\mathrm{d}z \tag{2.3.24}$$

The first two terms are identically zero: indeed they are the loop integrals of **exact** differentials $\mathrm{d}z = dh(z)$ with $h(z) = z$ and $(z - z_0)\mathrm{d}z = dg(z)$ with $g(z) = \frac{1}{2}(z - z_0)^2$ (see Exercise 3.1 applied to a closed loop). For the third we can estimate

$$\left| \int_{\partial\Pi_n} o(|z - z_0|)\mathrm{d}z \right| \leq o(\ \mathrm{diam}(\Pi_n))\ \mathrm{diam}(\Pi_n) = o(\mathcal{A}(\Pi_n)) \tag{2.3.25}$$

From this we find that

$$|I(\Pi)| \leq 4^n |I(\Pi_n)| = 4^n o(\mathcal{A}(\Pi_n)) = o(1) \tag{2.3.26}$$

and hence $I(\Pi) = 0$.

In order to prove it for a general loop we use only an intuitive argument, without going into details.

There are two steps:

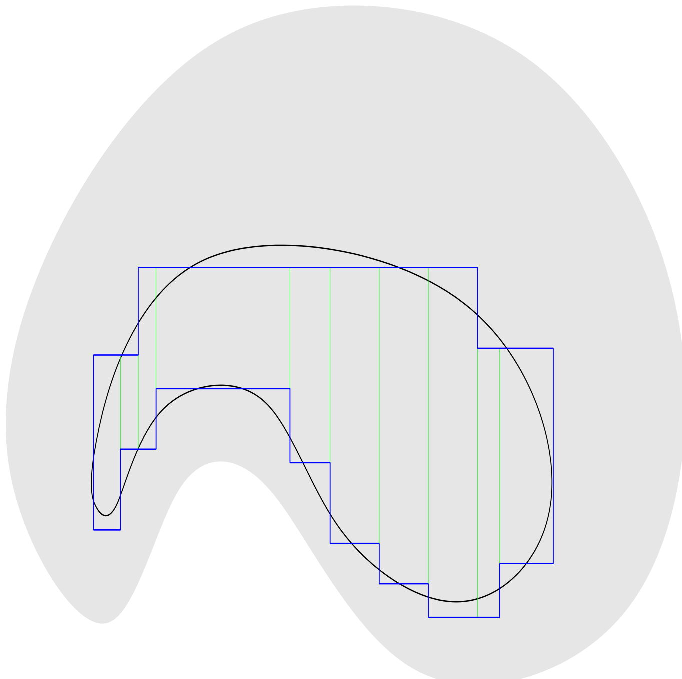

Figure 2.1: An illustration of approximating a loop by a "staricase" curve and the tessellation of it.

1. We can approximate the loop γ by a "staicase" curve σ made only by horizontal and vertical segments, such that σ is "close" to γ uniformly within δ ($\sup_{t\in[0,1]} |\gamma(t)-\sigma(t)| \le \delta$). By a compactness argument and by continuity (hence uniform continuity) of f we can conclude that

$$\left| \oint_\gamma f \mathrm{d}z - \oint_\sigma f \mathrm{d}z \right| \le \epsilon \tag{2.3.27}$$

provided we take σ sufficiently close to γ. (see figure 2.1).

2. The loop integral along such a "staircase" curve is the sum of integrals along boundaries of rectangles, hence equal to zero.

These two statements are rather "intuitive" (especially if you draw some pictures) but not completely simple to prove. Q.E.D.

Definition 3.6 *Given a point $z_0 \in \mathbb{C}$ and a closed contour which does not pass through z_0 we define the* **winding number** *of γ w.r.t. z_0 as*

$$deg(\gamma, z_0) := \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z - z_0} \tag{2.3.28}$$

*This is an integer (***exercise***) which depends only on the homotopy class in $\mathbb{C} \setminus \{z_0\}$.*

4 Cauchy's integral formula

Theorem 2.4.0 *Let $f(z)$ be holomorphic on the domain $\mathcal{D}$ and let $\mathcal{K} \subset \mathcal{D}$ be a simply connected domain such that $\overline{\mathcal{K}}$ is compact and $\partial\mathcal{K}$ is piecewise smooth; then, for any $z \in \mathcal{K}$ we have*

$$\frac{1}{2\pi i}\oint_{\partial\mathcal{K}} \frac{f(w)\mathrm{d}w}{w-z} = f(z) \tag{2.4.1}$$

Proof. Since the integrand is a closed one-form in $\overline{\mathcal{K}} \setminus \{z\}$ and the domain $\mathcal{K}$ is simply connected, we use a homotopy equivalence with a small circle centered at z, $C_\epsilon(z) := \{|w-z| = \epsilon\}$ (oriented counterclockwise).

$$\frac{1}{2\pi i}\oint_{\partial\mathcal{K}} \frac{f(w)\mathrm{d}w}{w-z} = \frac{1}{2i\pi}\oint_{C_\epsilon(z)} \frac{f(z) + f'(z)(w-z) + o(|w-z|)}{w-z}\mathrm{d}w \tag{2.4.2}$$

The first term gives the desired result, the second is identically zero and the third is estimated

$$\left|\frac{1}{2i\pi}\oint_{C_\epsilon(z)} \frac{o(|w-z|)}{w-z}\mathrm{d}w\right| \leq o(1)\epsilon. \tag{2.4.3}$$

Since the result is independent of ϵ, we may send it to zero, thus proving the theorem. Q.E.D.

The theorem can be extended to non simply connected domains and also relaxing the holomorphicity requirement on the boundary to plain continuity. Specifically

Theorem 2.4.3 (Cauchy's theorem for non simply connected domains) *Let $f(z)$ be holomorphic on a bounded domain $\mathcal{D}$ (possibly non simply connected) and continuous on $\overline{\mathcal{D}}$. Suppose that $\partial\mathcal{D}$ is the union of piecewise smooth curves (each one oriented positively). Then*

$$f(z) = \frac{1}{2i\pi}\oint_{\partial\mathcal{D}} \frac{f(w)\mathrm{d}w}{w-z} \tag{2.4.4}$$

where the integral extends to all components of the boundary.

For the "proof" see Fig. 2.2.
We have also the so-called "Cauchy-like" integrals

Theorem 2.4.4 *Let γ a simple, smooth closed curve and $\phi : \gamma \to \mathbb{C}$ a* **continuous** *function. Define the functions*

$$F_m(z) := \oint_\gamma \frac{\Phi(w)\mathrm{d}w}{(w-z)^m} \tag{2.4.5}$$

Then each F_m is differentiable (in the complex sense, i.e. holomorphic) for $z \neq \gamma$ and

$$F'_m(z) = mF_{m+1}(z)\ . \tag{2.4.6}$$

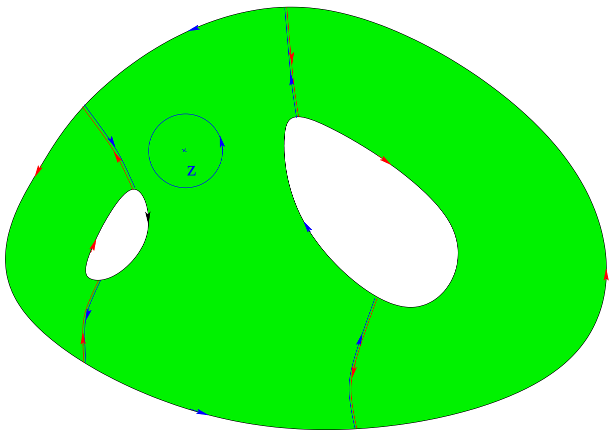

Figure 2.2: An example of Cauchy's integral formula for non simply connected domain. The picture is supposed to be a self-explanation of the proof. The red and blue arcs are drawn separate but they are in fact the same arc traversed in opposite directions.

Proof. By definition of derivative

$$F_m(z) - F_m(a) = \oint_\gamma \phi(w) \left(\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} \right) \mathrm{d}w = \tag{2.4.7}$$

$$= \oint_\gamma \phi(w) \frac{z-a}{(w-z)(w-a)} \sum_{j=0}^{m-1} (w-z)^{-j}(w-a)^{-m+1+j} \tag{2.4.8}$$

Taking the ratio $\frac{F_m(z)-F_m(a)}{z-a}$ and passing to the limit $z \to a$ we obtain the result. Q.E.D.

Corollary 2.4.8 *A holomorphic function $f(z)$ on a domain $\mathcal{D}$ is infinitely differentiable.*

Proof. Note that the definition of holomorphicity *per se* does not require even the condition $\mathcal{C}^1$; this shows that holomorphicity is a very rigid structure. The proof is immediate by noticing that by Cauchy's integral formula any holomorphic function can be written as an integral of the form F_1 in Thm. 2.4.4, with $\phi(w) = f(w)$. Moreover we have obtained an integral formula for all derivatives of $f(z)$. Q.E.D.

4.1 Taylor Series

Definition 4.1 *A function $f : \mathcal{D} \to \mathbb{C}$ ($\mathcal{D}$ a domain) is called* **analytic** *for any $a \in \mathcal{D}$ there is a convergent power series centered at a which represents $f(z)$ in the (nontrivial) disk of convergence.*

We now see that the class of holomorphic functions coincides with the class of analytic functions.

Theorem 2.4.8 *If $f \in \mathcal{H}(\mathcal{D})$ then f is analytic (and viceversa).*

Proof. The implication "analytic⇒ holomorphic" is obvious from Corollary 2.2.10.
Suppose now $a \in \mathcal{D}$ and let $R = \mathrm{dist}(a, \partial\mathcal{D})$. Then for all $z \in D_a(\rho)$, $\rho < R$ we have

$$2i\pi f(z) = \oint_{|w-a|=R} \frac{f(w)\mathrm{d}w}{w-z} = \oint_{|w-a|=R} \frac{f(w)\mathrm{d}w}{w-a-(z-a)} = \oint_{|w-a|=R} \frac{f(w)\mathrm{d}w}{w-a}\frac{1}{1-\frac{z-a}{w-a}} = \tag{2.4.9}$$

$$= \oint_{|w-a|=R} \frac{f(w)\mathrm{d}w}{w-a}\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^n} = \sum_{n=0}^{M}(z-a)^n\frac{2i\pi}{n!}f^{(n)}(a) + \mathcal{R}_M \ , \tag{2.4.10}$$

where

$$R_M := \oint_{|w-a|=R} \frac{f(w)\mathrm{d}w}{w-a}\sum_{n=M+1}^{\infty}\frac{(z-a)^n}{(w-a)^n} \tag{2.4.11}$$

The series under the integral sign is majorized in modulus by

$$|R_M(z)| \le 2\pi\left(\max_{w\in|w-a|=R}|f(w)|\right)\sum_{n=M+1}^{\infty}\left(\frac{\rho}{R}\right)^n \to 0 \tag{2.4.12}$$

as $M \to \infty$. Hence the function is analytic. Q.E.D.

Convergence on the boundary

We only prove a small lemma, but the topic can be a difficult one. Consider a power series: without loss of generality we assume it centered at $z = 0$ and with radius of convergence equal to 1 (we can always reduce to this case with a translation/dilation).

$$f(z) = \sum c_n z^n \ . \tag{2.4.13}$$

Suppose that the following series converges

$$\sum c_n \mathrm{e}^{in\theta} \ , \tag{2.4.14}$$

for some value of θ: namely that $z = \mathrm{e}^{i\theta} \in \partial D_0(1)$ is a point of convergence. Then

Lemma 2.4.14 If $z = \mathrm{e}^{i\theta}$ is a point of convergence on the boundary of the disk of convergence for the series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \ , \qquad \limsup |c_n|^{\frac{1}{n}} = 1 \ , \tag{2.4.15}$$

then the function $f(z)$ tends to $L := \sum c_n \mathrm{e}^{in\theta}$ when approaching the boundary point from the radial direction.

Proof. Again we can assume $\theta = 0$ by re-defining $c_n \to c_n \mathrm{e}^{in\theta}$ (i.e. by $z \mapsto \mathrm{e}^{-i\theta} z$). Thus now we have $L = \sum c_n$ (convergent) and we want to prove

$$\lim_{x\to 1^-} f(x) = L \ . \tag{2.4.16}$$

To this end we should estimate

$$L - f(x) = \sum c_n(1 - x^n) \ . \tag{2.4.17}$$

So we compute

$$\left|\sum c_n(1-x^n)\right| \leq \left|\sum^K c_n(1-x^n)\right| + \left|\sum_{K+1} c_n(1-x^n)\right| \tag{2.4.18}$$

Remark 4.1 If we knew that $\sum c_n$ is **absolutely** convergent the conclusion would be easy: letting $M = \sup |c_n|$ (which is necessarily finite)

$$\left|\sum^K c_n(1-x^n)\right| + \left|\sum_{K+1} c_n(1-x^n)\right| \leq M\sum^K(1-x^n) + \sum_{K+1} |c_n| \leq MK^2(1-x) + \sum_{K+1} |c_n| \tag{2.4.19}$$

and the two terms can be made arbitrarily small by taking K large (for the second one) and $1 - x < \epsilon/(MK^2)$.

Since $\sum c_n$ is not absolutely convergent we must "improve" convergence. This is done by using **Abel's summation formula**

$$\sum_M^\infty a_n b_n = \sum_M (a_n - a_{n+1}) \sum_{j=M}^n b_j \tag{2.4.20}$$

so that

$$\begin{aligned}&\left|\sum^K c_n(1-x^n)\right| + \left|\sum_{K+1} c_n(1-x^n)\right| \leq MK^2(1-x) + \left|\sum_{K+1}(x^n - x^{n+1})\sum_{j=K+1}^n c_j\right| \leq \\ &MK^2(1-x) + \sum_{K+1}(x^n - x^{n+1})\left|\sum_{j=K+1}^n c_j\right| \leq MK^2(1-x) + x^{K+1}\sup_{n\geq K+1}\left|\sum_{j=K+1}^n c_j\right| \leq \\ &\leq MK^2(1-x) + \sup_{n\geq K+1}\left|\sum_{j=K+1}^n c_j\right|\end{aligned} \tag{2.4.21}$$

Since $\sum c_n$ is convergent, the last sup tends to zero as $K \to \infty$, hence we can make it less than $\epsilon/2$. The first term is then less than $\epsilon/2$ for $1 - x < \epsilon(2MK(\epsilon)^2)$. Q.E.D.

4.2 Morera's theorem

Morera's theorem is the "viceversa" of Cauchy's theorem

Theorem 2.4.21 (Morera) *Let $f(z, \overline{z})$ be defined continuous on the domain $\mathcal{D}$ and such that for any closed, simple and $\mathcal{D}$-contractible path $\gamma \subset \mathcal{D}$ we have*

$$\oint_\gamma f(z, \overline{z})\mathrm{d}z = 0 \tag{2.4.22}$$

Then $\partial_{\overline{z}} f \equiv 0$, namely the function is holomorphic.

Proof. Consider a disk $D_a(r) \subset \mathcal{D}$: we prove that $f(z,\overline{z})$ is holomorphic in any such disk. Indeed

$$F_a(z) := \int_a^z f(\xi,\overline{\xi})\mathrm{d}\xi \tag{2.4.23}$$

is defined in the simply connected disk $D_a(r)$ independently of the path of integration. We prove it is has complex derivative.

$$\frac{1}{h}(F(z+h)-F(z)) = \frac{1}{h}\int_z^{z+h} f\mathrm{d}\zeta = \frac{1}{h}\int_z^{z+h}\Big(f(z)+(f(\zeta)-f(z))\Big)\mathrm{d}\zeta = f(z)+R_z(h) \tag{2.4.24}$$

$$R_z(h) := \int_z^{z+h}\frac{f(\zeta)-f(z)}{h}\mathrm{d}\zeta \tag{2.4.25}$$

Since the integration path involved in $R_z(h)$ is irrelevant we take it to be the segment $[z,z+h]$ and then we estimate

$$|R_z(h)| < \frac{1}{h}\sup_{w\in[z,z+h]}|f(w)-f(z)|\mathcal{L}([z,z+h]) = \sup_{w\in[z,z+h]}|f(w)-f(z)| \tag{2.4.26}$$

Since f is continuous on the disk $D_a(r) \subset \mathcal{D}$ it is also uniformly continuous on $\overline{D}_a(r-\epsilon)$ and hence the sup tends to zero as $h \to 0$. This proves that $F_a(z)$ is holomorphic and also that $F_a'(z) = f(z,\overline{z})$. Since we now know that any holomorphic function has also holomorphic derivative, this proves that f is holomorphic as well. Q.E.D.

5 Maximum modulus and mean value

From now on we denote the set of holomorphic functions on a set B by $\mathcal{H}(B)$. We also understand that if the set B is a closed set, we say that $f(z)$ is holomorphic on B if there exists a domain $\mathcal{D}$ containing B where $f(z)$ is holomorphic.

Definition 5.1 *A function $f(z)$ which is holomorphic in $\mathbb{C}$ is called* **entire**

We have the following fundamental

Theorem 2.5.0 (Maximum modulus) *Let B be a relatively compact domain and $f \in \mathcal{H}(\overline{B})$. Then the maximum of $|f(z)|$ on $\overline{B}$ is taken on the boundary ∂B.*

Proof. Let

$$M := \sup_{w\in\partial B}|f(w)| \tag{2.5.1}$$

Note that since ∂B is compact and $|f(w)|$ is continuous, the sup is actually a max. We prove that $\forall z \in B$, $|f(z)| \leq M$. Indeed let $\delta = \mathrm{dist}(z,\partial B)$ and note that from Cauchy integral formula we have, $\forall n \in \mathbb{N}$ that

$$f^n(z) = \frac{1}{2i\pi}\oint_{\partial B}\frac{f^n(w)\mathrm{d}w}{w-z} \tag{2.5.2}$$

Now,

$$|f^n(z)| \leq \frac{1}{2\pi}\left|\oint_{\partial B} ...\right| \leq \frac{\mathcal{L}(\partial B)M^n}{2\pi\delta} \tag{2.5.3}$$

where $\mathcal{L}(\partial\,\mathbf{b})$ denotes the length of a curve[1]. Continuing, we have

$$|f(z)| \leq \left(\frac{\mathcal{L}(\partial B)}{2\pi\delta}\right)^{\frac{1}{n}} M \ , \quad \forall n \in \mathbb{N} \tag{2.5.5}$$

Taking the limit $n \to \infty$ yields the result. Q.E.D.

Corollary 2.5.5 *If the modulus of a holomorphic function has a local maximum in the interior of the domain of holomorphicity then f is identically constant.*

We first need the

Lemma 2.5.5 If $f(z)$ is holomorphic in a neighborhood of a and $|f(z)|$ is constant, then $f(z)$ is constant.

Proof. If $|f(z)| \equiv 0$ then there is nothing to prove. So, let $|f(z)| = R^2 \neq 0$. We have

$$f = u + iv \ , \qquad u^2 + v^2 = R^2 = \text{ const} \ . \tag{2.5.6}$$

Then (by Cauchy-Riemann's $u_x = v_y$, $u_y = -v_x$

$$0 \equiv uu_x + vv_x = uu_x - vu_y \tag{2.5.7}$$

$$0 \equiv uu_y + vv_y = uu_y + vu_x \ , \tag{2.5.8}$$

This implies (since the determinant is $u^2 + v^2 = R^2 \neq 0$) that $u_x \equiv 0 \equiv u_y$. Similarly for v_x, v_y, and hence both u and v are constants. Q.E.D.

Proof of Cor. 2.5.5. Suppose $a \in \mathcal{D}$ is a local max of $|f(z)|$. Then there is a disk $D_a(r) \subset \mathcal{D}$ where a is a global max. Then, for any $\rho < r$

$$|f(a)| = \left|\frac{1}{2i\pi}\oint_{|z-a|=\rho} \frac{f(z)}{z-a}\mathrm{d}z\right| \leq \frac{1}{2\pi}\oint_{|z-a|=\rho} |f(z)|\mathrm{d}s \leq \sup_{|z-a|=\rho} |f(z)| \leq |f(a)| \ . \tag{2.5.9}$$

Note that the second last inequality would be **strict** as soon as there were a z_0 for which $|f(z_0)| < |f(a)|$ (by continuity then also in a neighborhood of z_0). This proves that $|f(z)| = |f(a)|$ in the whole $\mathcal{D}_a(r)$, and hence by the previous lemma is identically constant.

[1] Here and also elsewere we use that, if γ is smooth and s is the arclength parameter (and hence $\dot{x}^2 + \dot{y}^2 \equiv 1$)

$$\left|\int_\gamma f(z)(\mathrm{d}x + i\mathrm{d}y)\right| = \left|\int_0^{\mathcal{L}(\gamma)} f(z)(\dot{x} + i\dot{y})\mathrm{d}s\right| \leq \int_0^{\mathcal{L}(\gamma)} |f(z)|\mathrm{d}s \leq \sup_{z\in\gamma} |f(z)|\mathcal{L}(\gamma) \ , \tag{2.5.4}$$

Let now $M = \max_{z\in\overline{\mathcal{D}}} |f(z)|$ and let a a point where this max is achieved. Suppose that a is in the interior of $\mathcal{D}$ and consider

$$\mathcal{M} := \{z \in \mathcal{D} : \ |f(z)| = M = |f(a)|\} \ . \tag{2.5.10}$$

This set is a level set of the continuous function $|f(z)|$ and hence it is closed (in the relative topology of $\mathcal{D}$). Moreover it is nonempty because $a \in \mathcal{M} \subset \mathcal{D}$ and –most crucially– it is **open**. Indeed if $b \in \mathcal{M}$ then in particular b is a local max. and by the above f is locally constant in a neighborhood of b, meaning that a full neighborhood of b belongs to $\mathcal{M}$. Being both open and closed (and non-empty) in a connected set $\mathcal{D}$ then it must coincide with it. Q.E.D.

Corollary 2.5.10 (Liouville's theorem) *An entire function $f(z)$ which is bounded, $|f(z)| \leq M$ is a constant*

Proof. We prove that $f'(z) \equiv 0$. Indeed using Cauchy's integral formula we have

$$|f'(z)| \leq \frac{1}{2\pi} \oint_{|w|<R} \left| \frac{f(w)}{(w-z)^2} \mathrm{d}w \right| \leq \frac{M}{R} \tag{2.5.11}$$

where R is an arbitrary number $|z| < R$. Since the result is independent of R we take the limit $R \to \infty$ thus concluding that $f'(z) \equiv 0$. Q.E.D.

Theorem 2.5.11 (Fundamental Theorem of Algebra) *Any nonconstant polynomial $P(z)$ has at least one zero.*

Proof. $P(z)$ is clearly an entire function and if it had no zeroes then $1/P(z)$ would be a bounded entire function since actually $|P(z)| \to \infty$ as $|z| \to \infty$ (and hence would be uniformly bounded away from 0), which can only be a constant. Hence a contradiction. Q.E.D.

5.1 Mean value principle

The first coefficient of the Taylor expansion

$$f(a) = c_0 = \frac{1}{2\pi i} \oint_{|w-a|=\epsilon} \frac{f(w)\mathrm{d}w}{w-a} = \frac{1}{2\pi} \int_0^{2\pi} f(a + r e^{i\theta})\mathrm{d}\theta \tag{2.5.12}$$

says that

Theorem 2.5.12 *The value of a holomorphic function $f(z)$ at any point is the average of the values on a circle (of arbitrary radius provided it is contained in the domain of holomorphicity).*

It is useful to introduce the

Definition 5.2 *A function $f(z)$ defined and continous on a domain $\mathcal{D}$ is said to satisfy the property of the mean value if $\forall a \in \mathcal{D}$ and $\forall r > 0$ such that $D_a(r) \subset \mathcal{D}$ we have*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + r\mathrm{e}^{i\theta})\mathrm{d}\theta \tag{2.5.13}$$

The theorem of the maximum modulus applies to all functions which satisfy the mean value property

Theorem 2.5.13 (Maximum modulus principle) *If $f(z)$ defined and continuous on the domain $\mathcal{D}$ satisfies the mean value property and is such that $|f(z)|$ has a local maximum at $z = a$ then $f(z)$ is constant in a neighborhood of a.*

Proof. Exercise. Q.E.D.

We can now prove the following lemma which will be used later on

Theorem 2.5.13 (Schwartz' lemma) *Let $f \in \mathcal{H}(\{|z| < 1\})$ and such that*

$$f(0) = 0,\ |f(z)| < 1 \tag{2.5.14}$$

Then

1. *We have the inequality*
$$|f(z)| \le |z|\ ,\quad \forall z \in D_0(1)\ , \tag{2.5.15}$$

2. *If $\exists z_0 \in D_0(1)$, $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$ then $f(z) \equiv \lambda z$ with $|\lambda| = 1$.*

Proof. Consider the function

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases} \tag{2.5.16}$$

This is holomorphic in $D_0(1)$ since $f(0) = 0$ By the max. modulus theorem $|g(z)|$ takes on the maximum on the boundary $|z| = 1$ or else $g(z)$ is a constant. For any $z \in D_0(1)$ we have $M(|z|) := \sup_{|w|=|z|} |f(w)| < 1$ by the hypothesis. By the max. modulus theorem

$$L(|z|) := \sup_{|w|=|z|} |g(z)| = \frac{M(|z|)}{|z|} \tag{2.5.17}$$

is weakly increasing (actually strictly increasing or constant) and $\sup_{r<1} L(r) \le 1$ since $\sup |f| \le 1$. Therefore $|g| \le 1$ or $|f| \le |z|$. $|f(z)| \le |z|$. The remainder of the proof is left as exercise.Q.E.D

6 Laurent series and isolated singularities

Definition 6.1 *The point $a \in \mathbb{C}$ is an* **isolated singularity** *for the holomorphic function $f(z)$ if there is an open neighborhood I_a of a such that $f \in \mathcal{H}(I_a \setminus \{a\})$.*

Definition 6.2 *An isolated singularities is* **removable** *if $|f(z)|$ is bounded in a punctured neighborhood of a.*

Proposition 6.1 If a is a removable singularity for $f(z)$ then

1. $\exists A := \lim_{z\to a} f(z)$.
2. The function $\tilde{f}(z)$ defined as f on $I_a \setminus \{a\}$ and $\tilde{f}(a) := A$ is holomorphic in the full neighborhood.

Proof By Cauchy we have

$$f(z) = \frac{1}{2\pi i}\oint_{|w-a|=R} \frac{f(w)\mathrm{d}w}{w-z} - \frac{1}{2\pi i}\oint_{|w-a|=\epsilon} \frac{f(w)\mathrm{d}w}{w-z} \tag{2.6.1}$$

where $\epsilon < |z-a| < R$ and both circles are oriented counterclockwise. Since the result is independent of ϵ and $f(w)$ is bounded we can send $\epsilon \to 0$ and the second integral is easily estimated to tend to zero. At this point $f(z)$ is a Cauchy integral and hence holomorphic in the vicinity of a as well, where it admits limit and holomorphic extension. Q.E.D.

In the case where a is not a removable singularity we can use Laurent series.
Laurent series are power series with also negative powers (we consider the case of series centered at $z = 0$ with the understanding that we could replace z by $z - a$ and obtain analogous expressions centered at any other point)

$$f(z) = \sum_{m\in\mathbb{Z}} c_n z^n = \overbrace{\sum_{n=0}^{\infty} c_n z^n}^{=:f_+(z)} + \overbrace{\sum_{n=-\infty}^{-1} c_n z^n}^{=:f_-(z)} \tag{2.6.2}$$

The series f_+ is a usual power series while the series f_- is a series in $\frac{1}{z}$. We call R_+ and $1/R_-$ the respective radii of convergence. Clearly the expression makes sense as a convergent sum iff $R_- < R_+$ and hence the Laurent series is defined in the annulus

$$R_- < |z| < R_+ \tag{2.6.3}$$

Using these remarks we extend the notion of analyticity to the present context

Definition 6.3 *Given a holomorphic function $f(z)$ defined on the annulus $R_- < |z| < R_+$ we say that it admits a Laurent series expansion if there is a convergent Laurent series as above that converges to $f(z)$ in the same annulus.*

We leave as exercises the following statements

Lemma 2.6.3 For a function $f(z)$ as in Def. 6.3 the Laurent series -if it exists- is unique and the coefficients are given by

$$c_n = \frac{1}{2i\pi}\oint_{|w|=\rho} \frac{f(w)}{w^{n+1}} \mathrm{d}w \ , \qquad R_- < \rho < R_+ \ . \tag{2.6.4}$$

Theorem 2.6.4 *Every $f(z) \in \mathcal{H}\big(\{R_- < |z| < R_+\}\big)$ admits expansion in Laurent series.*

Note that if only a finite number of $c_n, n < 0$ is nonzero then f_- is a polynomial in z^{-1} and hence $R_- = 0$. In general we put forward the following

Definition 6.4 *We say that $z = 0$ is a* **pole of order** *if $M = \min\{n : \ c_n \neq 0\}$ is negative for the function $f(z)$. If $M = -\infty$ we say that $z = 0$ is an* **essential singularity**.

Essential singularities have interesting properties;

Theorem 2.6.4 (Sokorskji or Casorati-Weierstrass theorem) *Let $z = a$ be an essential singularity for $f(z) \in \mathcal{H}(\mathcal{D})$; for any punctured disk $D_a(r)^\star$ the values of f on the domain $D_a(r)^\star \cap \mathcal{D}$ is dense in $\mathbb{C}$.*

Proof. Suppose $\exists r > 0$ such that the values of $f(z)$ taken on the domain $I := D_a(r)^\star \cap \mathcal{D}$ is **not** dense. Hence $\exists F_0 \in \mathbb{C}$ and $\epsilon > 0$ such that

$$f(I) \cap D_{F_0}(\epsilon) = \emptyset. \tag{2.6.5}$$

This implies that

$$g(z) := \frac{1}{f(z) - F_0} \tag{2.6.6}$$

is **bounded** in a punctured neighborhood of a and hence can be holomorphically extended there (we use the same symbol for this extension). Thus

$$f(z) = \frac{1}{g(z)} + F_0 \tag{2.6.7}$$

If $g(a) \neq 0$ then $f(z)$ has no singularity at a, in contradiction with the hypothesis. If $g(0) = 0$ and the Taylor series of $g(z)$ starts with a power M then $f(z)$ has a **pole** of order M at a and not an essential singularity (note that $g(z)$ is not identically zero and hence at least one coefficient of the Taylor expansion is nonzero).Q.E.D.

There is a much stronger version of this theorem

Theorem 2.6.7 (Picard's great theorem) *In the same setting of the previous theorem, the set $f(D_a(r) \cap \mathcal{D})$ is $\mathbb{C}$ or the complex plane less a point $\mathbb{C} \setminus \{F_0\}$.*

Example 6.1 *The function $f(z) = \mathrm{e}^{\frac{1}{z}} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$ has an essential singularity at $z = 0$. In any neighborhood of $z = 0$ takes on infinitely many times any complex value c*

$$z_n(c) = \frac{1}{\ln|c| + i\arg(c) + 2i\pi n} \tag{2.6.8}$$

The exceptional value is $c = 0$.

7 Residues

Exercise 7.1 *Prove the following formulæ by explicit parametrization (counterclockwise orientation)*

$$\oint_{|z-a|=R} \frac{\mathrm{d}z}{(z-a)^m} = \begin{cases} 2i\pi & m=-1 \\ 0 & m \in \mathbb{Z}\ , m \neq -1 \end{cases} \tag{2.7.1}$$

Definition 7.1 *Given an isolated singularity a for a holomorphic function $f(z)$ we define the* **residue of** $f(z)$ **at** $z=a$ *the coefficient c_{-1} of the Laurent expansion of f centered at $z=a$. It can be computed by the following integral (***exercise***)*

$$\operatorname*{res}_{z=a} f(z)\mathrm{d}z := \frac{1}{2i\pi}\oint_{|z-a|=\epsilon} f(z)\mathrm{d}z \tag{2.7.2}$$

Theorem 2.7.2 (Residues theorem) *Let $f \in \mathcal{H}(\mathcal{D}\setminus\{a_n\})$ and let $\{a_n\}\subset\mathcal{D}$ be isolated singularities for f. Let γ be a simple and contractible closed curve in $\mathcal{D}$ and $\mathcal{K}\subset\mathcal{D}$ the "interior region" of γ (simply connected). Then*

$$\oint_\gamma f(z)\mathrm{d}z = 2i\pi\sum_{a_j\in\mathcal{K}}\operatorname*{res}_{z=a_j} f(z)\mathrm{d}z \tag{2.7.3}$$

Proof. It is a simple application of Cauchy's theorem by considering the domain (non-simply connected) $\mathcal{K}\setminus\bigcup_{a_j\in\mathcal{K}} D_{a_j}(\epsilon)$, with ϵ small enough. (You should check that there are only a **finite number** of isolated singularities in the compact $\mathcal{K}$).

7.1 Zeroes of holomorphic functions

Let $f\in\mathcal{H}(\mathcal{D})$ and f not identically zero.

Definition 7.2 *If $a\in\mathcal{D}$ is such that $f(a)=0$ we say that f has* **multiplicity of the zero** *equal to M with*

$$M := mult_a(f) := \min\{n|\ f^{(n+1)}(a)\neq 0\} \tag{2.7.4}$$

If $\mathrm{mult}_a(f)=1$ the zero is said to be **simple**, otherwise it is **multiple**. Note that a is a zero of multiplicity m for an analytic function $f(z)$ iff the Taylor series centered at a (which has positive radius of convergence) starts with the power m

$$f(z) = c_m(z-a)^m + \ldots \tag{2.7.5}$$

Exercise 7.2 *Let $f(z)$ be holomorphic in a neighborhood of $z=a$ and suppose that $f(a)=0$ (if $f(a)=A$ we would consider the function $F(z)=f(z)-A$). Suppose that $z=a$ is a* **simple zero** *(i.e. $f'(a)\neq 0$): prove that $f(z)$ admits locally an analytic inverse $g(w)$ such that $f(g(w))=w$.*

Exercise 7.3 *Verify that for polynomials, the notion of multiplicity of a zero coincides with the notion of multiplicity of roots.*

Theorem 2.7.5 (Open mapping theorem) *The image of a domain $\mathcal{D}$ under a holomorphic function $f(z)$ defined over $\mathcal{D}$ is an open set, $G := f(\mathcal{D})$, $G = \overset{\circ}{G}$ (where $\overset{\circ}{G}$ denotes the* **interior** *of G).*

Proof. If $w = f(a)$ and $f'(a) \neq 0$ then $f(z)$ is a local homeomorphism between a neighborhood of a and one of w, which proves that $w \in \overset{\circ}{G}$. If $f(z) - w$ has a zero of multiplicity M at $z = a$ then the equation $f(z) = b$ has M distinct solution in a suitable neighborhood of w (exercise). Hence also in this case $w \in \overset{\circ}{G}$. Since there are no other possibilities we conclude that G coincides with its interior and hence it is open. Q.E.D.

7.2 Meromorphic functions

Definition 7.3 *A function f is said to be* **meromorphic** *on a domain $\mathcal{D}$ if there is an at-most-countable*[2] *set of points $P := \{a_n\}$ without accumulation in $\mathcal{D}$ such that f is holomorphic on $\mathcal{D} \setminus P$ and the points a_n are poles (of finite order).*

Exercise 7.4 *Prove that the set of meromorphic function over a domain $\mathcal{D}$ is a field.*

Exercise 7.5 *Prove that an isolated singularity $z = a$ for a holomorphic function $f(z)$ is a pole (of finite order) iff $|f(z)| \to \infty$ as $z \to a$.*

7.3 Application of the residue theorem to multiplicities of zeroes

Let f be meromorphic on a domain $\mathcal{D}$; we want to count its zeroes and poles. We first have

Proposition 7.1 The function $g(z) = (\ln(f(z))' = \frac{f'(z}{f(z)}$ is meromorphic in $\mathcal{D}$ and the poles are all simple and located precisely at the zeroes and poles of $f(z)$. The residues of $g(z)$ at one such pole a is

$$\operatorname*{res}_{z=a} \frac{f'(z)}{f(z)} \mathrm{d}z = \begin{cases} m \in \mathbb{N} \setminus \{0\} & \text{multiplicity of } a \text{ if } a \text{ is a zero of } f(z) \\ -m \in -\mathbb{N} \setminus \{0\} & \text{order of the pole if } a \text{ is a pole of } f(z) \end{cases} \tag{2.7.6}$$

Proof. Consider the Laurent/Taylor series of $f(z)$ at a

$$f(z) = \sum_{n \geq M} c_n (z-a)^n \tag{2.7.7}$$

$$f'(z) = \sum_{n \geq M} n c_n (z-a)^n \tag{2.7.8}$$

$$\frac{f'(z)}{f(z)} = \frac{M}{z-a} + \mathcal{O}(1) \tag{2.7.9}$$

Q.E.D.

[2] This means that the set is either finite or else infinite but countable.

Corollary 2.7.9 *Let γ be a closed simple curve in $\mathcal{D}$ and $\{a_n\}$ be the zeroes of f with multiplicities m_n and $\{b_j\}$ the poles with orders p_j (there is a finite number of either in the region bounded by γ): then*

$$\frac{1}{2i\pi}\oint_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_n m_n - \sum_j p_j \tag{2.7.10}$$

$$\frac{1}{2i\pi}\oint_\gamma z\frac{f'(z)}{f(z)}\mathrm{d}z = \sum_n m_n a_n - \sum_j p_j b_j \tag{2.7.11}$$

More generally if $g(z) \in \mathcal{H}(\mathcal{D})$ then

$$\frac{1}{2i\pi}\oint_\gamma g(z)\frac{f'(z)}{f(z)}\mathrm{d}z = \sum_n m_n g(a_n) - \sum_j p_j g(b_j) \tag{2.7.12}$$

In other words we can assign a "charge" to each zero (positive) and pole (negative) and then the above integrals compute the total charge and the "center of charge" or (for the last integral) some weighted sum of any holomorphic function at the sites of the charges.

Exercise 7.6 *Prove Corollary 2.7.9.*

Chapter 3

Applications of complex integration

Most of the statement in this chapter are left as **exercise**. We use the notion of analytic continuation which is going to be defined later on, but I hope that you have faint memories of your past undergrad. course.

First of all we have the following alternative expressions for the residues. Let $f(z)$ have a **pole** of order m at $z = a$ then

$$\operatorname*{res}_{z=a} f(z)\mathrm{d}z = \frac{1}{(m-1)!} \lim_{z\to a} \Big((z-a)^m f(z) \Big)^{(m-1)} \tag{3.0.1}$$

In particular for a **simple pole** we have

$$\operatorname*{res}_{z=a} f(z)\mathrm{d}z = \lim_{z\to a} (z-a) f(z) \ . \tag{3.0.2}$$

These formulas follow by direct manipulations of the Laurent series expansion of f around $z = a$. **Exercise, do it!**. One of the most interesting applications of the theorem of residues is in the summation of series.

1 Summation of series

Consider a series of the form

$$S := \sum_{n=-\infty}^{\infty} f(n) \tag{3.1.1}$$

where $f(z)$ is a holomorphic function (or meromorphic) such that

$$|f(z)| = \mathcal{O}(|z|^{-1-\epsilon}) \ . \tag{3.1.2}$$

as $|z| \to \infty$. We find an auxiliary function $g(z)$ which has **simple poles** at the integers with residue 1, typically

$$g(z) = \frac{\pi \cos(\pi z)}{\sin(\pi z)} = \pi \cot(\pi z) \ . \tag{3.1.3}$$

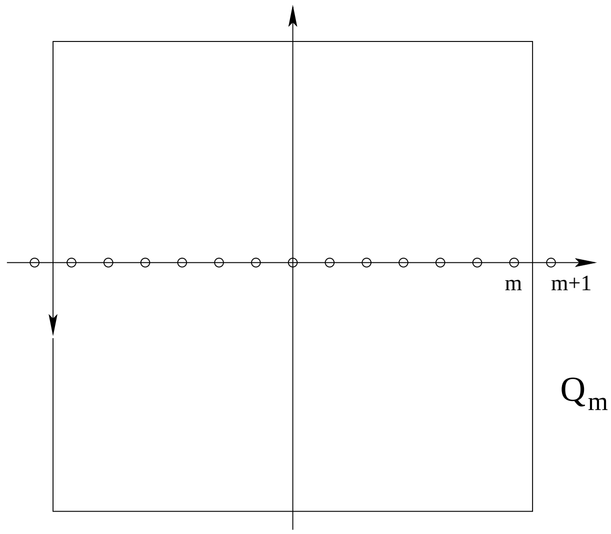

Figure 3.1: The contour for the sum of a series.

Let Q_m be the boundary (counterclockwise) of the square centered at the origin and with sides parallel to the axes, with half length $m + \frac{1}{2}$ (so that the sides intersect the real axis in between two poles). Check that $g(z)$ is uniformly bounded on all four sides of Q_m (uniformly w.r.t. m).

Given the decay condition on $f(z)$ you can check that

$$\lim_{m\to\infty} \frac{1}{2i\pi} \oint_{Q_m} f(z)\pi\cot(\pi z)\mathrm{d}z = 0 \tag{3.1.4}$$

Indeed

$$\left| \oint_{Q_m} f(z)\pi\cot(\pi z)\mathrm{d}z \right| \leq \pi \left(\sup_{z\in Q_m} |\cot(\pi z) \right) (4m+2)\mathcal{O}(m^{-1-\epsilon}) = \mathcal{O}(m^{-\epsilon}) \to 0 \tag{3.1.5}$$

On the other hand this is the sum of **all** the residues of $f(z)\pi\cot(\pi z)$ on the complex plane: the poles of cot give the series, the poles of $f(z)$ give extra contributions. Hence

$$\sum_{n\in\mathbb{Z}} f(n) = -\sum_{z=\text{pole of } f} f(z)\pi\cot(\pi z) \tag{3.1.6}$$

Example 1.1 *The sum* $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$*; consider*

$$I(x) := \sum_{n\in\mathbb{Z}} \frac{1}{n^2 + x^2} \tag{3.1.7}$$

so that $S = \lim_{x\to 0} \frac{1}{2}(I(x) - \frac{1}{x^2})$*. The sum* $I(x)$ *is given by*

$$I(x) = -\operatorname*{res}_{z=ix} \frac{\pi\cot(\pi z)}{z^2+x^2} - \operatorname*{res}_{z=-ix} \frac{\pi\cot(\pi z)}{z^2+x^2} = \frac{\pi\coth(\pi x)}{x} = \frac{1}{x^2} + \overbrace{\frac{\pi^2}{3}}^{2S} + \ldots \tag{3.1.8}$$

For alternating series use instead of cot the function

$$g(z) = \frac{\pi}{\sin(\pi z)} \ , \tag{3.1.9}$$

which has residues at the integers equal to $(-1)^n$. Usually one first needs to "double" the series in question by writing as sum over all $\mathbb{Z}$ instead of just $\mathbb{N}$, as in the example above.

1.1 Fourier series

Consider series of the form

$$F(\omega) := \sum_{n\in\mathbb{Z}} f(n)\mathrm{e}^{i\omega n} \tag{3.1.10}$$

Then we use a similar strategy with

$$g(z) = \frac{2i\pi}{\mathrm{e}^{2i\pi z} - 1} \tag{3.1.11}$$

so that

$$F(\omega) = \sum_{x\in\mathbb{R},\ x=\text{ pole}} \operatorname*{res}_{z=x} f(z)\frac{2i\pi\mathrm{e}^{iz\omega}}{\mathrm{e}^{2i\pi z} - 1} \tag{3.1.12}$$

2 Integrals

2.1 Case 1

Improper integrals of the form

$$I := \int_0^\infty f(x)\sqrt{x}\mathrm{d}x \tag{3.2.1}$$

where $f(z)$ is meromorphic on $\mathbb{C}$. Take the contour $\Gamma(R)$ consisting of the segment $[\frac{1}{R}, R]$ followed by the circle of radius R followed by $[R, \frac{1}{R}]$ and the circle of radius $\frac{1}{R}$. In the limit $R \to \infty$ the two small circles can be discarded if

$$|f(z)| = o(|z|^{-\frac{3}{2}-\epsilon}) \ , \qquad z \to 0|f(z)| = o(|z|^{\frac{3}{2}+\epsilon}) \ , \qquad z \to \infty. \tag{3.2.2}$$

and we have (remember that after analytic continuation $\sqrt{z}$ changes sign.)

$$2I = 2i\pi \sum_{poles\neq 0} \operatorname*{res}_{z} f(z)\sqrt{z}\mathrm{d}z \tag{3.2.3}$$

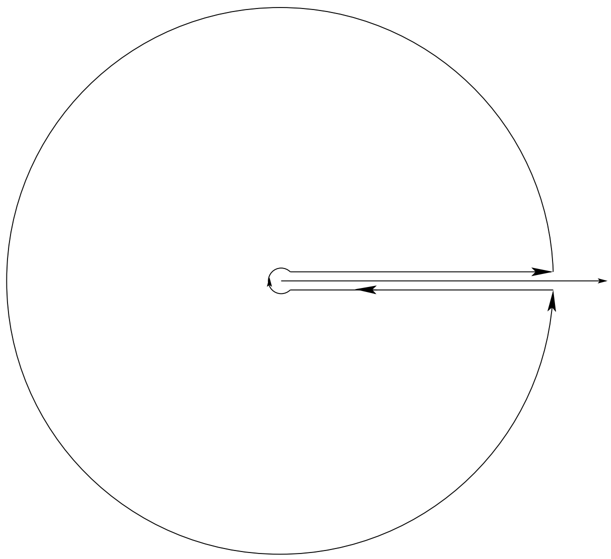

Figure 3.2: The contour for two cases of integrals.

2.2 Case 2

Integrals of the form

$$I := \int_0^\infty f(x)\mathrm{d}x \tag{3.2.4}$$

We take the same path Γ as in the previous case and integrate

$$\lim_{R\to\infty} \oint_\Gamma f(z)\ln(z)\mathrm{d}z = -2i\pi I \tag{3.2.5}$$

So that

$$I = -\sum_{poles\neq 0} \operatorname*{res}_z f(z)\ln(z)\mathrm{d}z \tag{3.2.6}$$

where we understand the principal determination of ln. I'll leave it to you to formulate some condition on the decay of $f(z)$ near $z = 0$ and $z = \infty$ so that everything makes sense.

3 Jordan's lemma and applications

Let $f(z)$ be continuous on $\Im(z) \geq 0$, $|z| \geq R$ and such that $|f(z)| \to 0$ as $|z| \to \infty$, $\Re(z) \geq 0$. We then have

$$\lim_{r\to\infty} \int_{C_r} f(z)\mathrm{e}^{iz}\mathrm{d}z = 0 \ ; \quad C_r := \{|z| = r,\ \Re(z) \geq 0\} \tag{3.3.1}$$

Exercise 3.1 *Prove the lemma using that* $\sin(\theta) \geq \frac{2\theta}{\pi}$ *for* $\theta \in [0, \pi/2]$.

As application, if $f(z)$ is meromorphic in $\Im(z) > 0$ and continuous in $\Im(z) \geq 0$, such that $|f(z)| \to 0$ as $|z| \to \infty$, $\Im(z) \geq 0$, then

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{ix}\mathrm{d}x = 2i\pi \sum_{a_j=\text{pole of } f} \operatorname*{res}_{z=a_j} f(z)\mathrm{e}^{iz} \ . \tag{3.3.2}$$

Exercise 3.2 *Compute*

$$\int_{-\infty}^{\infty} \frac{(x-1)\mathrm{e}^{ix}}{x^2-2x+2}\mathrm{d}x \tag{3.3.3}$$

$$\int_{\mathbb{R}} \frac{\cos(x)}{(x^2+2ix-2)^2}\mathrm{d}x \tag{3.3.4}$$

etc.etc.

Chapter 4

Analytic continuation

1 Analytic continuation

Lemma 4.1.0 Let $f \in \mathcal{H}(\mathcal{D})$ and $a \in \mathcal{D}$ such that $f(a) = 0$; the following statements are mutually exclusive:
(a) There exists a neighborhood U_a of a such that $f(z) \equiv 0$, $\forall z \in U_a$;
(b) There exists a neighborhood U_a of a such that $f(z) \neq 0$, $\forall a \neq z \in U_a$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be the Taylor series of f with radius of convergence $R > 0$. Let M be the **order of the zero**, i.e. $M = \min\{n : \ c_m \neq 0\}$. We assume $M < \infty$, namely the series is not identically zero, for otherwise we are in situation **(a)**. Consider

$$g(z) = \frac{f(z)}{(z-a)^M} \ . \tag{4.1.1}$$

This function is analytic at $z = a$ and $g(a) \neq 0$: hence -by continuity- there is a neighborhood of a where $g(z)$ is never zero. Therefore $f(z) = (z-a)^M g(z)$ is not zero in this punctured neighborhood of $z = a$. Q.E.D.

Corollary 4.1.1 *If the zeroes of $f \in \mathcal{H}(\mathcal{D})$ accumulate in the interior of $\mathcal{D}$ then $f(z)$ is identically zero on $\mathcal{D}$.*

Proof. If this would happen then an accumulation point $z_0 \in \mathcal{D}$ would also be a zero of $f(z)$ (by continuity). Then, by Lemma 4.1.0 $f(z)$ is zero on an open neighborhood of $z = a$. Consider now the set

$$Z(f) := \{z \in \mathcal{D} : \ , \exists \epsilon > 0 : \ f(D_z(\epsilon)) = \{0\}\} \ . \tag{4.1.2}$$

(The set of "fat" zeroes). The definition implies that $Z(f)$ is open and we have just proved that it is not empty since z_0 belongs to it. By continuity of $f(z)$ the set $Z(f)$ is closed since a point z_1 of accumulation for $Z(f)$ has the property contradicting **(a)** of Lemma 4.1.0, and hence belongs to $Z(f)$. Since $Z(f)$ is **open and closed**, then -by elementary topological properties of connected sets- $Z(f) = \mathcal{D}$. Q.E.D.

The above lemma and corollary constitute the **principle of analytic continuation**, stating that

Corollary 4.1.2 *Let $f, g \in \mathcal{H}(\mathcal{D})$. Suppose $\exists a \in \mathcal{D}$ and an infinite set G with a as accumulation point such that $f(z) = g(z)$, $\forall z \in G$: then $f \equiv g$ on $\mathcal{D}$. In particular if $f = g$ on any open disk then they are equal on the whole $\mathcal{D}$.*

Definition 1.1 *The pair $(\mathcal{B}, f)$ where $\mathcal{B}$ is a domain and $f \in \mathcal{H}(\mathcal{B})$ is called an* **analytic element***. We say that the analytic element $(\tilde{\mathcal{B}}, \tilde{f})$ is an* **analytic continuation** *of an analytic element $(\mathcal{B}, f)$ if $\tilde{\mathcal{B}} \supset \mathcal{B}$ and $\tilde{f} \equiv f$ on $\mathcal{B}$.*

It is clear that two analytic continuations of an analytic element must coincide on the intersection of the respective domains (in view of our (4.1.0)).

Corollary 4.1.2 *Let (B, f) be an analytic element and $\tilde{B} \supset B$: let $f_1, f_2 \in \mathcal{H}(\tilde{B})$ two analytic continuations of (B, f). Then $f_1 = f_2$.*

In other words, if an analytic continuation exists, it is unique.

Definition 1.2 *Given $a \in \mathbb{C}$ we define an equivalence relation R_a between all analytic elements (D, g) such that $a \in D$: two analytic elements (B, f) and (D, g) are equivalent if $a \in B \cap D$ and $f \equiv g$ in a neighborhood of a, $U_a \subset B \cap D$. The equivalence classes are called* **germs of analytic functions at** *$z = a$ and denoted by $[f]_a$.*

It is clear that germs of analytic functions at a are in one-to-one correspondence with convergent power series centered at a.

We finally define the analytic continuation of a (germ of) analytic element along a path.

Definition 1.3 *Let (B, f) be an analytic element and $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) \in B$. The* **analytic continuation of** f **along** γ *is a family (B_t, f_t) of analytic elements such that*

1. $[f]_{\gamma(0)} = [f_0]_{\gamma(0)}$
2. $\gamma(t) \in B_t$
3. $\forall t \in [0, 1]$ $\exists \delta > 0$ *such that* $|s - t| < \delta$ *implies* $\gamma(s) \in B_t$
4. $[f_t]_{\gamma(s)} = [f_s]_{\gamma(s)}$.

We say also that (B_1, f_1) is the **analytic continuation** *of (B_0, f_0) along the given path.*

The following Lemma holds (without the nondifficult proof)

Lemma 4.1.2 Given a curve $\gamma : [0,1] \to \mathbb{C}$ connecting the points $\gamma(0) = a$ to $\gamma(1) = b$ and two analytic continuations of analytic elements (B_t, f_t) and (D_t, g_t) such that the initial germs coincide $[f_0]_a = [g_0]_a$ then so do all germs

$$[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)} \ . \tag{4.1.3}$$

This lemma implies that what we actually continue are **germs** of analytic functions.

Definition 1.4 *The* **complete analytic function** *(in Weierstrass' sense) of an analytic element (B, f) is the collection of all germs $[g]_b$ for which there is a point $a \in B$ and a path γ connecting a to b such that $[g]_b$ is the continuation of the germ $[f]_a$ along γ.*

Example 1.1 *Consider the function $\ln(z) : \{\Re(z) > 0\} \to \mathbb{C}$, defined as*

$$\ln z := \int_1^z \frac{\mathrm{d}\xi}{\xi}. \tag{4.1.4}$$

Such function has the properties of the usual logarithm since

$$\mathrm{e}^{\ln(z)} = z \tag{4.1.5}$$

which can be verified by taking the derivative of both sides and comparing the values at $z = 1$. Explicitly we know that

$$\ln(z) = \ln|z| + i\arg(z) \ , \tag{4.1.6}$$

where in the RHS ln *means the usual logarithm of real numbers and here the argument is chosen in $(-\pi/2, \pi/2)$. Here it is "simple" to see what is the analytic continuation of $\ln(z)$ along the counterclockwise circle centered at $z = 0$ and radius 1; the result will be the germ of an analytic function $\tilde{\ln}(z)$ at $z = 1$ such that*

$$\tilde{\ln}(z) = \ln(z) + 2i\pi \tag{4.1.7}$$

in a suitable neighborhood of $z = 1$. Viceversa, analytic continuation in the clockwise orientation will produce a **different** *germ*

$$\hat{\ln}(z) = \ln(z) - 2i\pi \ . \tag{4.1.8}$$

Exercise 1.1 *Consider the function*

$$f(z) = |z|^{1/2}\mathrm{e}^{i\arg(z)/2} \ , \Re(z) > 0. \tag{4.1.9}$$

Describe the complete analytic function of this analytic element, which we will call $\sqrt{z}$.

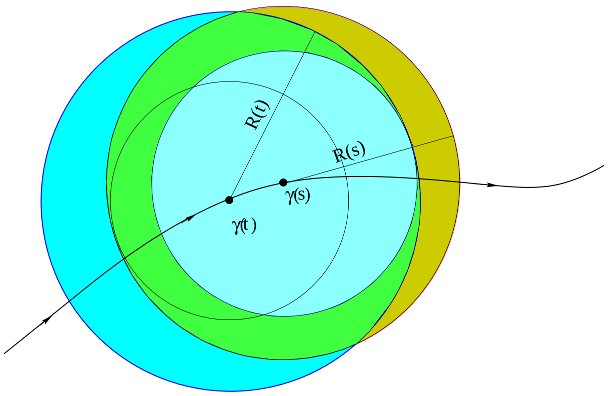

Figure 4.1: The disks used in the proof of Lemma 4.1.9.

1.1 Monodromy theorem

Let $a, b \in \mathbb{C}$ and $\gamma, \tilde{\gamma}$ be two paths connecting them. Let (B_t, f_t) and (D_t, g_t) two analytic continuation of the same initial germ $[f_0]_a = [g_0]_a$ along the two different paths. In general the final germs need not be equal as the example of the logarithm shows.

So we ask the question as to under what conditions the analytic continuation is independent of the path. As we will see, a homotopy equivalence will be involved.

We start with the

Lemma 4.1.9 Let $(B_t, f_t)_\gamma$ be an analytic continuation along a contour $\gamma : [0,1] \to \mathbb{C}$. Let $R(t)$ be the radius of convergence of the power series representation of f_t centered at $\gamma(t)$. Then either $R(t) \equiv \infty$ or $R(t) : [0,1] \to (0,\infty)$ is continuous.

Proof. If there is t such that $R(t) = \infty$ then f_t is an entire function and so must be all f_s, and hence $R(s) \equiv \infty$. Suppose now that $R(t) < \infty$ for at least one t (and hence for all t's). Let $|t - s| < \delta$ in the definition of analytic continuation. Then (see figure)

$$R(s) \geq \mathrm{dist}(\gamma(s), \{|z - \gamma(t)| = R(t)\}) \geq R(t) - |\gamma(s) - \gamma(t)| \tag{4.1.10}$$

so that $R(t) - R(s) \leq |\gamma(s) - \gamma(t)|$. Interchanging the rôles of $\gamma(s)$ and $\gamma(t)$ we obtain also $R(s) - R(t) \leq |\gamma(t) - \gamma(s)|$ and hence

$$|R(t) - R(s)| \leq |\gamma(t) - \gamma(s)| \tag{4.1.11}$$

Whence the proof since $\gamma : [0,1] \to \mathbb{C}$ is continuous. Q.E.D.

Lemma 4.1.11 Let $(B_t, f_t)_\gamma$ be the analytic continuation of the analytic element (B_0, f_0) from $a = \gamma(0)$ to $b = \gamma(1)$. Let $(D_t, g_t)_\rho$ be another analytic continuation of the same initial germ with ρ connecting a to b.Then there exists $\epsilon > 0$ such that if the parametrized paths γ and ρ are ϵ close

$$|\gamma(t) - \rho(t)| < \epsilon \tag{4.1.12}$$

the final germs coincide $[f_1]_b = [g_1]_b$. (Invariance of the analytic continuation under small perturbations of the path).

Proof. We define $R(t)$ as in Lemma 4.1.9. If $R(t) \equiv \infty$ any ϵ will suffice and the proof is left as exercise. Suppose $R(t) < \infty$. Since it is continuous on a compact, it has a **positive** minimum. We take ϵ as half of this minimum. This guarantees that $\rho(t)$ is always within the disk of convergence of the power series representation of f_t. Without loss of generality we assume that B_t and D_t are the respective disks of convergence of the p.s. reps. of f_t and g_t. Consider the set

$$T := \{t \in [0,1] \text{ s.t. } \ f_t(z) \equiv g_t(z) \ , \forall z \in B_t \cap D_t\} \tag{4.1.13}$$

Such set is nonempty since by assumption $t = 0 \in T$. We prove that it is open and closed in $[0,1]$ and connectivity of the interval will conclude that $T = [0,1]$. In particular f_1 and g_1 coincide in a neighborhood of $z = b$ and hence define the same germ. To prove that it is open we take a neighborhoot U of $t \in T$ such that $\forall t' \in U$ we have

$$|\gamma(t) - \gamma(t')| < \frac{\epsilon}{2} \ \text{ and } \ |\rho(t) - \rho(t')| < \frac{\epsilon}{2} \ . \tag{4.1.14}$$

We now show that $\gamma(t')$ belongs to $D_{t'}$ and symmetrically that $\rho(t') \in B_{t'}$. Indeed

$$|\gamma(t') - \rho(t')| \le |\gamma(t') - \gamma(t)| + |\gamma(t) - \rho(t)| + |\rho(t) - \rho(t')| \le 2\epsilon \le R(t). \tag{4.1.15}$$

Therefore $\gamma(t')$ and $\rho(t')$ belong to $D_t \cap D_{t'} \cap B_t \cap B_{t'}$ and hence all power series reps of $f_t, f_{t'}, g_t, g_{t'}$ have a nonempty common intersection of the respective disks. Thus we have the equality of $f_{t'}(z) \equiv g_{t'}(z)$, $z \in D_t \cap D_{t'} \cap B_t \cap B_{t'}$ and hence T is open.

We prove now that T is closed hence concluding the proof. Let t_0 be an accumulation point of T: as above

$$\exists t \text{ s.t. } \ |\gamma(t) - \gamma(t_0)| < \frac{\epsilon}{2} \text{ and } |\rho(t) - \rho(t_0)| < \frac{\epsilon}{2} \ . \tag{4.1.16}$$

As before one has that the fourtuple intersection of the disks of convergence is nonempty (and open) so t_0 belongs to T as well and T is closed. Q.E.D.

Definition 1.5 *An analytic element (B, f) admits* **unrestricted analytic continuation** *to the domain $\mathcal{D} \supset B$ if for any path $\gamma \subset \mathcal{D}$ with initial point in B there exists an anlytic continuation of (B, f) along γ.*

Theorem 4.1.16 *Let $\mathcal{D}$ be a domain where the analytic element (B, f) can be unrestrictedly analytically continued. Let $\gamma_1, \gamma_2 \subset \mathcal{D}$ be curves $\mathcal{D}$-homotopic at fixed end-points. Then the analytic continuations of (B, f) along γ_1, γ_2 yield the same germ of analytic function at $\gamma_1(1) = \gamma_2(1)$.*

Corollary 4.1.16 *If $\mathcal{D}$ above is simply connected then there is an analytic function $F : \mathcal{D} \to \mathbb{C}$ such that $F_{|_B} \equiv f$.*

Example 1.2 *The domain $\mathcal{D}$ where $(\{\Re(z) > 0\}, \ln(z))$ admits unrestricted analytic continuation is $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. However there is no analytic function $F(z)$* **defined on** *$C^\times$ which restricts to $\ln(z)$ on $\Re(z) > 0$. If this where the case the analytic continuation of $\ln(z)$ around the origin counterclockwise or clockwise would yield the same result, which is not the case.*

Exercise 1.2 *Prove Corollary 4.1.16.*

2 Schwartz reflection principle

This is a particularly useful application of analytic continuation

Theorem 4.2.0 *Let $\mathcal{D}_+$ a domain in the upper half plane $\Im(z) > 0$ such that $I := \overline{\mathcal{D}_+} \cap \mathbb{R} \neq \emptyset$ is a union of intervals. Suppose $f(z)$ is holomorphic on D_+ and continuous on $\overline{D_+}$ and also $f(x) \in \mathbb{R}$, $\forall x \in I \subset \mathbb{R}$. Then there exists a holomorphic function g defined on $\overline{D}_+ \cup \overline{D}_+^*$ such that $g_{|_{\mathcal{D}_+}} = f$.*

Proof. We define

$$g(z) = \begin{cases} f(z) & z \in \mathcal{D}_+ \\ \overline{f(\overline{z})} & z \in \mathcal{D}_- := {\mathcal{D}_+}^* \end{cases} \tag{4.2.1}$$

Such function is also continuous on I. Since it can be represented as a Cauchy integral on a contour embracing subintervals of I it is also holomorphic at points of I. Q.E.D.

Chapter 5

One-dimensional complex manifolds

We begin with some general facts about topological spaces and differential geometry.

1 Definition

Definition 1.1 *A (real/complex) manifold of dimension n is a set M with a collection of pairs* $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ *where* $U_\alpha \subset M$ *and* $\phi_\alpha : U_\alpha \to (\mathbb{R}/\mathbb{C})^n$ *on their respective images and such that*

1. $\phi_\alpha(U_\alpha)$ *is open in* $[\mathbb{R}/\mathbb{C}]^n$ *and* $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ *is one-to-one.*
2. *The sets* U_α *are a covering of M*
$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = M \tag{5.1.1}$$
3. *If* $U_{\alpha,\beta} := U_\alpha \cap U_\beta \neq \emptyset$ *then both* $\phi_\alpha(U_{\alpha,\beta})$ *and* $\phi_\beta(U_{\alpha,\beta})$ *are open and*
$$G_{\alpha,\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha,\beta}) \to \phi_\alpha(U_{\alpha,\beta}) \tag{5.1.2}$$
are ($\mathcal{C}^k$*/analytic) functions of all the respective variables.*

The maps ϕ_α *are called* **local coordinates***, the sets* U_α *are called* **local charts***. The functions* $G_{\alpha,\beta}$ *are called* **transition functions***.*

Given two collections of local coordinate-charts $\{\phi_\alpha, U_\alpha\}_\alpha$ and $\{\psi_\beta, V_\beta\}_\beta$, we say that they are **equivalent** if their union still defines a (real/complex) manifold structure. The equivalence classes of local coordinate-charts $[\{(U_\alpha, \phi_\alpha)\}_\alpha]$ are called **atlases** (or **conformal structure** in the complex case).

Note that –interchanging $\alpha \leftrightarrow \beta$ in the last point of the definition– we have that $G_{\alpha,\beta}$ are invertible and the inverse is in the same class ($\mathcal{C}^k$ or analytic), $G^{-1}_{\alpha,\beta} = G_{\beta,\alpha}$.

A complex n-dimensional manifold is also a real $\mathcal{C}^\infty$ manifold of dimension $2n$. We will be concerned with manifold of complex dimension 1 and hence the local charts $z_\alpha = \phi_\alpha(p)$ will be complex valued functions providing local identification of M with a domain in $\mathbb{C}$. The set M becomes immediately a topological space with the topology inherited via ϕ_α^{-1} from $[\mathbb{R}/\mathbb{C}]^n$; an open set U in M is a set such that $\phi_\alpha(U)$ is open $\forall\alpha$.

From now on we restrict the formulation to complex one–dimensional manifolds, but many definitions and statements are obvious specializations of more general ones where either we have more dimensions or we change the " category" of functions from "analytic" (holomorphic) to $\mathcal{C}^k$ or else.

Definition 1.2 *Let M be a complex one-dimensional manifold with atlas $\{(U_\alpha, z_\alpha)\}$. A function $f : M \to \mathbb{C}$ is said to be* **holomorphic** *(***meromorphic***) if for each local chart we have*

$$\begin{array}{rcl} f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) & \to & \mathbb{C} \\ z_\alpha & \mapsto & f_\alpha(z_\alpha) := f(\phi_\alpha^{-1}(z_\alpha)) \end{array} \tag{5.1.3}$$

is holomorphic/meromorphic on the open set $\phi_\alpha(U_\alpha)$).

Note that on the intersection of charts $U_{\alpha,\beta}$ the notion of holomorphicity/meromorphicity in the different coordinates is the same since the transition functions are biholomorphic.

Theorem 5.1.3 *Let M be connected and compact in the topology of the atlas. Then the only holomorphic functions are constants.*

Proof. Since $|f|$ is continuous on the compact M then it takes on a maximum at $p \in M$. Let $p \in U_\alpha$, then f_α has a maximum modulus in the interior of $\phi_\alpha(U_\alpha)$ and hence it is constant on U_α. Let $q \in M$ and since M is connected it is also arcwise connected (**exercise**). Let γ be a continuous path from p to q: by compactness of γ it can be covered by a finite number of charts U_{α_j}, with $U_{\alpha_0} = U_\alpha$. By induction you can show that $f_{\alpha_k} \equiv C \Rightarrow f_{\alpha_{k+1}} \equiv C$ and hence $f_{\alpha_N} = C = f_{\alpha_0}$. Q.E.D.

Definition 1.3 *Let M and N be two complex one-dimensional manifolds with atlases respectively (U_α, ϕ_α) and (V_β, ψ_β). We say that a map*

$$\varphi : M \to N \tag{5.1.4}$$

is **holomorphic** *if at any point $p \in M$, $p \in U_\alpha$, $\varphi(p) \in V_\beta$ then $w_\beta = \psi_\beta(f(\phi_\alpha^{-1}(z_\alpha)$ is holomorphic in a small disk arount $\phi_\alpha(p)$.*

Remark 1.1 It is customary to abuse the notation and identify a point $p \in U_\alpha$ with its *coordinate* $z_\alpha = z_\alpha(p) := \phi_\alpha(p)$. The above function then would be written as $w_\beta = f(z_\alpha)$.

Definition 1.4 *Two complex manifolds M, N are* **biholomorphic** *(or* **biholomorphically equivalent***) if there exist two holomorphic bijections $\varphi : M \to N$ and $\psi : N \to M$ such that $\varphi \circ \psi = Id_N$ and $\psi \circ \varphi = Id_M$. This defines an equivalence relation (***exercise***).*

When considering complex manifolds we do not distinguish between manifolds which are biholomorphically equivalent and hence we re-define a **complex manifold** to be the equivalence class of complex manifolds (as in the former definition).

Definition 1.5 *A holomorphic map $\varphi : M \to M$ which admits holomorphic inverse is called auto-biholomorphism or* **automorphism** *(for short). The set of automorphisms of a (complex) manifold M will be denoted by $Aut(M)$ and it is a group with respect to the composition of maps.*

2 One-forms and integration

We always assume that $\left(M, \{(U_\alpha, \phi_\alpha)\}_\alpha\right)$ is a complex one-dimensional variety with atlas of charts $z_\alpha = z_\alpha(p) = \phi_\alpha(p)$. Instead of introducing the abstract notion of tangent and cotangent space (which would be the "comme-il-faut" way) we give a "hands-on" definition of forms.

Definition 2.1 *A [holomorphic/meromorphic] one form ω is a collection of functions $f_\alpha : U_\alpha \to \mathbb{C}$ which are [holomorphic/meromorphic] in the local chart z_α and such that on the intersection $U_\alpha \cap U_\beta$ we have*

$$f_\alpha = f_\beta \frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha} \tag{5.2.1}$$

Here the notation $\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}$ means the Jacobian of the function $z_\beta(z_\alpha) = \phi_\beta(\phi_\alpha^{-1}(z_\alpha)$. and we tacitly understand that we will think of the f_α's as functions either of the abstract point $p \in M$ or as "concrete" functions of the local coordinate z_α as most convenient for the context and without explicit mention.

If we treat the symbols $\mathrm{d}z_\alpha$ formally then the mnemonics behind the definition is

$$f_\alpha \mathrm{d}z_\alpha = f_\beta \mathrm{d}z_\beta \ , \tag{5.2.2}$$

on the intersection, and hence the reference to the chart becomes irrelevant.

A particular class of one-forms are differentials: given a [holomorphic/meromorphic] function $f : M \to \mathbb{C}$ we define its differential $\mathrm{d}f$ as the one form given in local coordinates by

$$\mathrm{d}f = f'(z_\alpha)\mathrm{d}z_\alpha \ . \tag{5.2.3}$$

As a consequence of the chain-rule this definition does provide a differential in the sense specified above.

The use of one-forms is -as always- that of being integrated alont one-dimensiona **real** manifolds, namely curves. A curve is simply a map $\gamma : [0, 1] \to M$ which is continuous. The notion of smoothness $\mathcal{C}^k$ is easy to formulate and it is left as **exercise**.

If the whole support of a curve γ lies within a single coordinate chart then integration of a one–form ω along the curve is defined as in the complex plane using the local coordinate. Some care should be paid (but "only the first time you do it") to define it consistently when the support of γ extends to more than one chart.

Integration of one-forms along curves Let ω be a [holomorphic/meromorphic] one–form and $\gamma : [0,1] \to M$ be a continuous curve. We define the integral of ω along γ

$$\int_\gamma \omega \tag{5.2.4}$$

as follows. Let U_j be a covering of γ by charts. By compactness we can take it finite and we can order it so that $U_j \cap U_{j+1} \neq \emptyset$. We take points t_j such that $\gamma(t_j) \in U_j \cap U_{j+1}$. By construction the arc $\gamma([t_j, t_{j+1}])$ lies within a single chart U_{j+1} and then we define

$$\int_\gamma \omega = \sum_j \int_{t_j}^{t_{j+1}} f_{j+1}(z_{j+1}(t))\dot{z}_{j+1}(t)\mathrm{d}t\ . \tag{5.2.5}$$

One should then prove that the definition is independent of the choice of the splitting points. This follows from the following exercise.

Exercise 2.1 *Let $\gamma := [a,b] \to U_\alpha \cap U_\beta$ and let ω be a one form: prove that*

$$\int_a^b f_\alpha(z_\alpha(t))\dot{z}_\alpha(t)\mathrm{d}t = \int_a^b f_\beta(z_\beta(t))\dot{z}_\beta(t)\mathrm{d}t \tag{5.2.6}$$

(The notation $z_\alpha(t)$ is a short-hand for $z_\alpha(\gamma(t)) = \phi_\alpha(\gamma(t))$).

We also need the generalization of Stokes' theorem. To this end we must define two–forms

Definition 2.2 *A two–form η is a collection of smooth-functions $g_\alpha(x_\alpha, y_\alpha)$ of the* **real variables** *$x_\alpha = \Re(z_\alpha)$, $y_\alpha = \Im(z_\alpha)$ such that*

$$g_\alpha = g_\beta \det\begin{pmatrix} \frac{\partial x_\beta}{\partial x_\alpha} & \frac{\partial x_\beta}{\partial y_\alpha} \\ \frac{\partial y_\beta}{\partial x_\alpha} & \frac{\partial y_\beta}{\partial y_\alpha}\end{pmatrix} = g_\beta \left|\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}\right|^2 \tag{5.2.7}$$

As before the mnemonics is

$$-2ig_\alpha \mathrm{d}x_\alpha \wedge \mathrm{d}y_\alpha = g_\alpha \mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z_\alpha} = 2ig_\beta \mathrm{d}x_\beta \wedge \mathrm{d}y_\beta = g_\beta \mathrm{d}z_\beta \wedge \mathrm{d}\overline{z_\beta} \tag{5.2.8}$$

The reason of the definition is that now we can integrate η over a domain (open connected set) $\mathcal{D} \subset M$ which may extend over several charts. Suppose now we have a non-holomorphic one form, i.e.

$$\omega = f_\alpha \mathrm{d}z_\alpha + \widetilde{f_\alpha}\mathrm{d}\overline{z}_\alpha \tag{5.2.9}$$

where the functions $f_\alpha, \widetilde{f_\alpha}$ are defined such that

$$f_\alpha = f_\beta \frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}\ , \qquad \widetilde{f_\alpha} = \widetilde{f_\beta}\overline{\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}} \tag{5.2.10}$$

but in general we admit that $f, \tilde{f}$ are not holomorphic. The total differential is defined as earlier in the course to be the two-form given in local coordinates

$$\mathrm{d}\omega = \left(\partial_{z_\alpha}\widetilde{f}_\alpha - \partial_{\overline{z}_\alpha} f_\alpha\right)\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z_\alpha} \tag{5.2.11}$$

We then have

Theorem 5.2.11 *Let $\mathcal{D}$ a domain in M and ω a one form defined and smooth over $\mathcal{D}$. Then*

$$\int_{\partial\mathcal{D}} \omega = \iint_{\mathcal{D}} \mathrm{d}\omega \ . \tag{5.2.12}$$

In particular if ω is a meromorphic one form and $\mathcal{D}$ does not include any of the poles then $\int_{\partial\mathcal{D}} \omega = 0$ since $\mathrm{d}\omega$ is smooth and identically zero on $\mathcal{D}$.

2.1 Zeroes and poles: residues

In general it makes no sense to talk about the value of a one-form at a point because the function f_α depends on the local coordinate: if we re-define the local coordinate $w_\alpha = W_\alpha(z_\alpha)$ by a biholomporphic function W_α then we have

$$\tilde{f}_\alpha = f_\alpha \frac{\mathrm{d}W_\alpha}{\mathrm{d}z_\alpha} \tag{5.2.13}$$

and the value will change. Ditto for points in the intersection of different charts. However it $p \in M$ is such that $f_\alpha(p) = 0$ then the same holds in any local coordinate. Similarly if p is a pole and hence f_α has a pole at $z_\alpha(p)$ then so it must be for all coordinates whose charts cover the point p and for all other local coordinate we can choose on the given chart. Even more, if p is a zero/pole of ω, its multiplicity/order is independent of the choice of coordinate. This motivates the following

Definition 2.3 *Let ω be a [holomorphic/meromorphic] one form on M. We say that $p \in M$ is a zero/pole of ω of multiplicity/order m if $f_\alpha(z_\alpha)$ has a zero/pole of the same multiplicity/order at $z_\alpha(p)$.*

Exercise 2.2 *Prove that if we change local coordinate $w = W(z_\alpha)$ in a neighborhood of $z_\alpha(p)$ the notion multiplicity/order of the zero/pole does not change. Prove the same also if p lies in the intersection of different charts.*

Residues

Let ω be a meromorphic one–form and p a pole. Let $z = z_\alpha$ be a local chart covering p and $a = z_\alpha(p)$. Then (denoting $f(z) = f_\alpha(z_\alpha)$)

$$\omega = f(z)\mathrm{d}z = \sum_{n \geq -M} c_n (z-a)^n \mathrm{d}z \tag{5.2.14}$$

is the local Laurent series expansion.

Definition 2.4 *The residue of ω at p is*

$$\operatorname*{res}_p \omega = c_{-1} \tag{5.2.15}$$

where c_{-1} is defined by the local expansion (5.2.14).

Note that the coefficients c_n **depend** on the choice of coordinate and that c_{-1} is the only one which does not. In order to see this we can note that

$$c_{-1} = \frac{1}{2i\pi} \oint_{|z_\alpha - a| = \epsilon} f_\alpha(z_\alpha) \mathrm{d}z_\alpha = \frac{1}{2i\pi} \oint_{\gamma_\epsilon} \omega \tag{5.2.16}$$

where γ_ϵ is a small loop around p in M. Since the integral of the one–form is independent of the coordinate, then so is c_{-1}.

Exercise 2.3 *Verify directly on the Laurent expansion that c_{-1} is independent of the local coordinate.*

One-forms on a complex one–dimensional manifold are customarily classified in three kinds.

Definition 2.5 *A one form ω on M is called of the* **first kind** *if it is holomorphic, of the* **second kind** *if it is meromorphic but the residues at all poles are zero, and of the* **third kind** *if at least one residue is non-zero.*

We have however the following

Theorem 5.2.16 *Let M be a* **compact complex curve** *(=one–dimensional manifold≡ two-dimensional real surface). Let ω be a meromorphic one–form. Then the sum of all residues is zero*

$$\sum_{p = \text{pole of } \omega} \operatorname*{res}_p \omega = 0 \tag{5.2.17}$$

Proof. We take the domain $\mathcal{D} = M \setminus \bigcup_{p = \text{ poles of } \omega} D_p(\epsilon)$ where $D_p(\epsilon)$ stands for small disks centered at p (in a local coordinate). Then the sum of residues is precisely (minus) the integral of ω on the boundary $\partial\mathcal{D}$

$$\sum_{p = \text{pole of } \omega} \operatorname*{res}_p \omega = -\int_{\partial \mathcal{D}} \omega = -\iint_{\mathcal{D}} \mathrm{d}\omega = 0 \tag{5.2.18}$$

by Stokes' theorem. Q.E.D.

Exercise 2.4 *Let $f : M \to \mathbb{C}$ be a meromorphic function with only one simple pole. Prove that this establishes a biholomorphic equivalence with the Riemann–Sphere.*
[Hint; apply the residue theorem to $\frac{\mathrm{d}f}{f-a}$, $a \in \mathbb{C}$]

3 Coverings and Riemann surfaces

Definition 3.1 *Given a one-dimensional complex manifold N, a* **Riemann surface** *over N is a pair (M, f) where M is another one-dimensional complex manifold and $f : M \to N$ is a holomorphic map onto.*

Definition 3.2 *The map $f : M \to N$ is called* **univalent** *if $f(a) \neq f(b)$ for $a \neq b$.*

The local description of univalent maps is easy. Let $a \in M$ and $A = f(a)$. We take local coordinates z and w in a neighborhood of a and A and we can assume that $z(a) = 0$, $w(A) = 0$ (we may have to shift the coordinates supplied with the atlas and redefine them). Then we can think of f restricted to the chart of z as a holomorphic function of z

$$f \in \mathcal{H}(D_0(\epsilon)) \ , \qquad w = f(z) \ , \ f(0) = 0 \ . \tag{5.3.1}$$

The condition of local univalency translates in $f'(0) \neq 0$.

More generally, for holomorphic maps $f : M \to N$ we introduce the following

Definition 3.3 *A point $a \in M$ is called a* **ramification point** *if, in local coordinates a and w centered at a and $A = f(a)$ respectively, we have $f'(z)|_{z=z(a)} = 0$. The point $A \in N$ is called* **branchpoint**. *The multiplicity of the zero of $f'(z)$ at $z = z(a)$ is called the* **index of ramification** *(and it is independent of the choiche of local coordinates). A map $f : M \to N$ for which there are no ramification points is called* **unramified**.

Example 3.1 *The map* $\exp : \mathbb{C} \to \mathbb{C}^\times$, $\exp(z) := \mathrm{e}^{2i\pi z}$ *is unramified but not univalent.*

We have the extension of the open-mapping theorem, whose proof is virtually identical and left as **exercise**.

Lemma 5.3.1 Let $\mathcal{D} \subset M$ be a domain (=open and connected set). Then the image $f(\mathcal{D}) \subset N$ under a non-constant holomorphic map is open as well (and connected, hence a domain).

Proposition 3.1 A holomorphic onto map $f : M \to N$ is a biholomorphic equivalence iff it is univalent.

Proof. Only one direction is nontrivial. Let f be univalent. Then we can always define uniquely a set-theoretical inverse function. Since we have proved that the map is unramified, this global inverse is also a local holomorphic function f^{-1} and hence it is a global holomorphic function. Q.E.D.

Then

Theorem 5.3.1 *Let (M, f) a compact Riemann surface over N and $B \subset N$ the collection of branchpoints (it is a discrete set, namely for each $b \in B$ there is a neighborhood of b that does not contain any other branchpoint). For each $y \in \dot{N}$ the number of preimages is independent of y*

$$\sharp f^{-1}(y) = \sharp f^{-1}(y'), \quad \forall y, y' \in \dot{N}. \tag{5.3.2}$$

Proof. First of from a compactness argument (plus analyticity) one finds that the number of preimages of **any** point $y \in N$ is finite. This is left as exercise. Consider the function

$$Num : N \to \mathbb{R} \ , \quad Num(y) = \sharp f^{-1}(y) \ . \tag{5.3.3}$$

We claim that this function is continuous and hence (since it takes only integer values) constant. Indeed let $n_0 = Num(y_0)$ for some $y_0 \in \dot{N}$. For each $x_j \in f^{-1}(y_0)$ there is a neighborhood $U_j \subset M$ of x_j and a neighborhood $V_j \subset N$ of y_0 where the restriction of f is one-to-one. Take the intersection of all $\tilde{V} = \cap V_j$ (which is still a neighborhood of y because it is a **finite** intersection) and $\tilde{U}_j = f^{-1}(\tilde{V}) \cap U_j$. Note that $\tilde{U}_j \cap \tilde{U}_k = \emptyset$ for $j \neq k$. We claim first that $f^{-1}(\tilde{V}) = \bigcup_{j=1}^n \tilde{U}_j$ (i.e. there are no other pieces of the preimage). Indeed suppose that $\hat{U} = f^{-1}(\tilde{V}) \setminus \bigcup_{j=1}^n \tilde{U}_j \neq \emptyset$ (this set is open). Then $f(\hat{U})$ must be a open set in $\dot{N}$ with accumulation point at y_0. Take a sequence $y_j \to y_0$ and a corresponding preimage subsequece $x_j \in \hat{U}$. By compactness x_j has some accumulation point x_0 and by continuity $f(x_0) = y_0$. Hence $x_0 \in \overline{\hat{U}} \cap f^{-1}(y_0)$; this would imply that in any neighborhood of x_0 $f(x)$ takes on the same value more than once and hence can't be invertible at x_0. Therefore y_0 should be a branchpoint (which is not the case).

Then all this implies that Num is constant in a neighborhood of any $y_0 \in \dot{N}$ and hence continuous at any point. Note that $\dot{N}$ is connected. Since Num is continuous then $Num^{-1}((n_0 - \epsilon, n_0 + \epsilon)) = Num^{-1}(\{n_0\})$ is both open and closed, hence coincides with $\dot{N}$. Q.E.D.

Definition 3.4 *For (M, f) a Riemann surface over N such that $\sharp f^{-1}(y) \leq \infty$ for each non-branchpoint $y \in N$, we call this integer the* **number of sheets** *of the surface.*

Example 3.2 *A polynomial $P(z) = z^n + \ldots : \overline{C} \to \overline{C}$ is a n-sheeted Riemann surface over $N = \overline{\mathbb{C}}$. The branchpoints are the* **critical values** *namely the values of P at the zeroes of the derivatives. In other words they are the values of λ such that $P(z) - \lambda$ has at least on multiple roots.*

Exercise 3.1 *Prove that if M and N are connected and compact and $f : M \to N$ is holomorphic and univalent then it is a biholomorphic equivalence.*

4 Fundamental group and universal covering

Let be given a (complex or real) connected manifold M.

Definition 4.1 *A manifold M is said to be arc-connected if $\forall x, y \in M$ there is a continuous curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = x, \gamma(1) = y$.*

For general topological spaces the two notions of connectedness are not equivalent, arc-connectedness being stronger than connectedness alone. However

Proposition 4.1 [**Exercise**] A manifold M is connected iff it is arc-connected.

Let $x \in M$ be chosen arbitrarily and then fixed (the "basepoint"). We consider the collection of all closed curves starting and ending at x

$$\mathcal{L}(x, M) := \{\gamma : [0,1] \to M\ ,\ \gamma \in \mathcal{C}([0,1], M),\ \gamma(0) = \gamma(1) = x\} \tag{5.4.1}$$

We take the set-theoretical quotient of this set by the relation of homotopy equivalence at fixed end-points $\sim$

$$\pi_1(x, M) := \mathcal{L}(x, M)/\sim \tag{5.4.2}$$

This new set is called the **fundamental group of** M (or **first homotopy group**). The name is unjustified so far, since we did not define a group structure. We do it now: for any $[\gamma]$, $[\eta] \in \pi_1(x, M)$ with representatives $\gamma, \eta \in \mathcal{L}(x)$ we define the loop

$$\gamma \odot \eta(t) = \begin{cases} \eta(2t) & t \in \left[0, \frac{1}{2}\right) \\ \gamma(2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{5.4.3}$$

This defines a new loop that in the first half time runs along η and then along γ. The symbol $\odot$ here stands for the **concatenation** of contours and can be more generally defined for any two curves γ, η such that the endpoint of η is the starting point of γ. Then we define the product in the fundamental group

$$[\gamma] \cdot [\eta] = [\gamma \odot \eta] \tag{5.4.4}$$

The unit of the multiplication is the class of the constant loop $[0,1] \to \{x\}$; the inverse of a loop γ is the class of the same loop run in the opposite sense.

Exercise 4.1 *Prove that this definition is well-posed (independent of the choice of representatives).*

Exercise 4.2 *Let M be an arc-connected manifold. Prove that $\pi_1(x, M)$ and $\pi_1(x', M)$ are naturally isomorphic and specify this isomorphism.*

This exercise implies that the fundamental group π_1 is "the same" no matter what basepoint is used in the definition and hence we can refer just to the manifold and omit the basepoint $\pi_1(x, M) \equiv \pi_1(M)$.

Note that saying that $\pi_1(M) = \{\mathbf{1}\}$ (the trivial group) is a rephrasing of saying that the arc-connected M is simply connected (and viceversa).

Exercise 4.3 *Let $M = \{z :\ |z| = 1\}$ with the standard topology. Prove that $\pi_1(M) \simeq \mathbb{Z}$ (the additive group of integers).*

4.1 Intersection number

The notion of intersection number is more general than the one given here as it applies to any two submanifolds of complementary dimensions. In our case of complex one-dimensional manifold (i.e. real surface) two submanifolds of complementary dimension must have both dimension 1 (i.e. they must be curves) or 0 and 2 (points and domains). The latter case is rather degenerate (although not meaningless) and we focus only on the first case. Moreover the following discussion is rather qualitative and we omit several details.

In this context one uses the definition of

Definition 4.2 *A cycle $\gamma \subset M$ is a closed piecewise smooth curve.*

Given a cycle it is not difficult to imagine (but less easy to prove) that in the same homology class there is a representative which is **smooth**: in view of this fact we always tacitly assume (since we will be interested only in the homology class of the cycle) that the cycle is smooth.

Given two cycles γ and η we consider their intersection: again, possibly by a small deformation of one or both contours we can reduce to the situation that
(a) the intersection is finite and
(b) all intersections occur **transversally**, i.e. the tangents to γ and η at the point of intersection are not parallel.

Given $p \in \gamma \cap \eta$ one such point of intersection, we associate a number $\nu(p) \in \{+1, -1\}$ as follows. Let z be a local coordinate at p: the two (arcs) of γ and η now are arcs in a neighborhood of $z(p) = 0$ crossing each other transversally. We denote by $\dot{\gamma}_0$ and $\dot{\eta}_0$ the two tangent vectors at $z(p) = 0$; if the determinant of their compoments is positive we set $\nu(p) = 1$, if it is negative we set $\nu(p) = -1$. In other words the number $\nu(p)$ indicates the orientation of the axis spanned by $\dot{\gamma}_0$ and $\dot{\eta}_0$ (in this order!) relative to the orientation of the standard $\Re(z)$, $\Im(z)$ axes.

Definition 4.3 *The intersection number between γ and η is then defined by*

$$\gamma \sharp \eta := \sum_{p \in \gamma \cap \eta} \nu(p) \ . \tag{5.4.5}$$

It follows immediately from the definition that $\gamma \sharp \eta = -\eta \sharp \gamma$ and the intersection number is an integer. One can also prove that;

Proposition 4.2 The intersection number is invariant under smooth homotopy deformations of γ and η.

Therefore the intersection number depends only on the *homotopy classes* of γ and η, which we then denote by $[\gamma]\sharp[\eta]$.

In particular it makes sense to compute the **self-intersection** of a cycle

$$[\gamma]\sharp[\gamma] = 0 \ . \tag{5.4.6}$$

This makes sense because in the actual computation one chooses two different representatives in the same class of γ which intersect transversally: the fact that the result is zero then follows from the antisymmetry.

Note also that the intersection number depends on the orientation of the contours: if we reverse one contour the int. number changes sign

$$[\gamma]\sharp[\eta] = -[\gamma]^{-1}\sharp[\eta] \ . \tag{5.4.7}$$

You should convince yourself that the intersection number is independent of the class by drawing examples.

5 Universal covering

Definition 5.1 *Let M be a (real/complex) connected manifold. A (the) universal covering space is a pair (Π, M') $\Pi : M' \to M$ continuous and onto, M' connected and simply connected and such that the preimage of any open simply connected set $U \subset M$ is the disjoint union of open, connected and simply connected sets $\{U_g\}$, such that $\Pi : U_g \to U$ is a homeomorphism (one-to-one).*

The nice fact is that –no matter what is $\pi_1(M)$– there always is a universal cover. The proof of existence is in fact constructive.

Theorem 5.5.0 *Let M be a (real/complex) connected manifold with fundamental group $\pi_1(M) \neq \{\mathbf{1}\}$. Then there exists a universal cover (Π, M').*

Remark 5.1 In fact one can prove that all universal coverings are homeomorphic.

Sketch of proof. We sketch the proof for one-dimensional complex manifolds (the case relevant to us) but the idea is in no way specific and it is an **exercise** to extend it to arbitrary dimensional manifolds. Fix $x_0 \in M$ and $\pi_1(x_0, M)$. Let $\mathcal{P}(x_0, x)$ be the quotient of the paths from x_0 to x modulo homotopy at fixed end-points. Define

$$M' := \{(x, [\gamma]) \ , \quad x \in M, \ [\gamma] \in \mathcal{P}(x_0, x)\} \tag{5.5.1}$$

The topology is defined by displaying a basis of neighborhoods of any point $(x, [\gamma]) \in M'$; such a basis is constructed as follows. Let z be a local coordinate centered arount x ($z(x) = 0$). A basis of neighborhoods of x is given in M by the coordinate disks $D_0(r)$, with r sufficiently small so that the disk is contained in the local chart. For $y \in z^{-1}(D_0(r))$ we can connect with a coordinate segment x to y, calling this curve $\vec{xy}$. Then we deifine a basis of neighborhoods for $(x, [\gamma]) \in M'$ as

$$\tilde{D}_r := \{(y, [\vec{xy} \odot \gamma]) \ , \quad y \in D_0(r)\}. \tag{5.5.2}$$

The coordinate on this open set is defined to be the same

$$\tilde{z} : \tilde{D}_r \to D_0(r) \ , \qquad \tilde{z}\Big((y, [\vec{xy} \odot \gamma]\Big) = z(y) \ . \tag{5.5.3}$$

This way we have defined a manifold. We claim that this yields a universal covering. Indeed a closed loop starting and ending at $(x, [\gamma])$ is –by definition–

$$\tilde{\eta} : [0,1] \to M' \tag{5.5.4}$$

$$\tilde{\eta}(t) = (\eta(t), [\gamma](t)) \tag{5.5.5}$$

where the (admittedly confusing) notation is that $\eta(t)$ is a smooth closed curve in M and $[\gamma](t)$ is the class of loops from x_0 to $\eta(t)$ along γ first ant then along the arc of η. Since by definition of closed loop we must have $[\gamma](0) = [\gamma](1)$, this means that $\gamma \odot \eta$ is homotopic to γ. This can be iff η is a closed contractible loop. Q.E.D.

Example 5.1 *Consider $M = \mathbb{C}^\times$; the universal cover is equivalent to $\mathbb{C}$ and the covering map is $e^z : \mathbb{C} \to \mathbb{C}^\times$.*

Exercise 5.1 *Prove that the number of preimages of a point $x \in M$ in its universal covering is labelled by the elements of the fundamental group $\pi_1(x, M)$.*

Deck transformations

Let M be a complex manifold and (Π, M') its universal covering. Let ρ be a closed loop and x_0 be a base-point used in the construction of the universal covering. The deck-transformation is defined by the map (biholomorphism)

$$\begin{array}{ccc} \phi_\rho : M' & \longrightarrow & M' \\ (x, [\gamma]) & \mapsto & (x, [\gamma \odot \rho]). \end{array} \tag{5.5.6}$$

Deck transformations are a group homomorphism $\pi_1(M) \to Aut(M')$ where $Aut(M')$ denotes the group of biholomorphisms from M' to itself.

6 Riemann's sphere

We consider the standard sphere $S^2 := \{X^2+Y^2+Z^2 = 1\}$ in $\mathbb{R}^3$. This is a two-dimensional real compact manifold. The north/south poles N, S are the points $N = (0, 0, 1)$ and $S = (0, 0, -1)$. We define the two

stereographic projections as follows: the projection $\phi_N : U_N := S^2 \setminus \{N\} \to \mathbb{R}^2$ is the intersection of the line passing through the point N and $P \in U_N$ with the plane $Z = 0$. The projection ϕ_S is defined analogously. Explicitly one obtains, for the North projection

$$x = \frac{X}{1-Z} \ , \qquad y = \frac{Y}{1-Z} \tag{5.6.1}$$

Note that $Z \neq 1$ on U_N. The complex coordinate $z_N = x + iy$ gives the first map

$$z_N = \frac{X+iY}{1-Z} \ , \qquad Z \neq 1 \tag{5.6.2}$$

Similarly one obtains the South projection

$$z_S = \frac{X-iY}{1+Z} \ , \qquad Z \neq -1 \tag{5.6.3}$$

In the intersection $U_N \cap U_S$ (where $Z \neq 1, -1$) we have

$$z_N z_S = \frac{X^2+Y^2}{1-Z^2} \equiv 1 \tag{5.6.4}$$

Therefore the transition function $g_{NS} = \frac{1}{z_S}$ is holomorphic (since $z_S \neq 0$ in $U_N \cap U_S$).

The equivalence class of this complex structure is called **Riemann's sphere** and it can be viewed as the Alexandrov's compactification (i.e. one-point compactification) of the complex plane $\mathbb{C}$, $\overline{\mathbb{C}}$, where the complex plane is thought of as the plane of the coordinate $z = z_N$ (or z_S) and the point at infinity is the North pole.

A basis of open neighborhoods of the North pole is thus $|z| > R$ or, which is the same $|z_S| < r = \frac{1}{R}$. A germ of analytic function at N is nothing but a convergent power series in z_S centered at $z_S = 0$ or equivalently a Laurent series in $z = z_N$ without regular part

$$\sum_{n=0}^{\infty} c_n (z_S)^n = \sum_{n=0}^{\infty} c_n \frac{1}{z^n} \ . \tag{5.6.5}$$

We also have

Proposition 6.1 The meromorphic function on $\overline{C}$ are rational functions of $z = z_N$.

Proof. The poles of a nonzero function $f : \overline{C} \to \mathbb{C}$ are discrete and hence (due to compactness) in finite number. So are the zeroes. Let $\{a_j\}_{j=1\ldots,M}$ and $\{b_\ell\}_{\ell=1\ldots N}$ the $z = z_N$ coordinates of the poles/zeroes in U_N (i.e. with finite z-coordinate). Let p_j, m_ℓ the multiplicities of the poles/zeroes and consider

$$g(z) = f(z) \frac{\prod_{j=1}^{M} (z-b_j)_j^p}{\prod_{\ell=1}^{N} (z-a_\ell)^{m_\ell}} \tag{5.6.6}$$

This function has no zeroes and no poles for finite z: if g has a zero at ∞ then it is identically zero by Liouville's thm. If it has a pole then $\frac{1}{g}$ has a zero and is also bounded so it is zero, which would yield

$f(z) \equiv \infty$ and this is absurd. Therefore the only possibility is that $g(z)$ tends to a nonzero constant C at infinity and hence it is that constant

$$g \equiv C \ \Rightarrow \ \ f(z) = C\frac{\prod_{\ell=1}^{N}(z-a_\ell)^{m_\ell}}{\prod_{j=1}^{M}(z-b_j)^p_j} \tag{5.6.7}$$

This concludes the proof. Q.E.D.

Note that a rational function can be viewed as a holomorphic function $\overline{\mathbb{C}} \to \mathbb{C}$ or as a holomorphic map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, namely

Corollary 5.6.7 *The holomorphic maps $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are in one-to one correspondence with rational functions.*

Exercise 6.1 **(a)** *Prove that if $f(z)$ is entire and has a pole at ∞ then it is a polynomial.*
(b) *Prove that if $f(z)$ is entire and there exist $M > 0, \alpha > 0$ and $R > 0$ such that $|f(z)| < M|z|^\alpha$ for $|z| > R$ then $f(z)$ is a polynomial.*

6.1 One forms

We Riemann sphere $\overline{\mathbb{C}}$ is covered by two charts: one is $z \in \mathbb{C} \simeq \overline{\mathbb{C}} \setminus \{\infty\}$ with $z(0) = 0$ and the other is ζ in $\overline{\mathbb{C}} \setminus \{0\}$, with $\zeta(\infty) = 0$. The transition function is $\zeta(z) = \frac{1}{z}$ in the intersection $\mathbb{C}^\times$. We now show that **there are no (nontrivial) holomorphic one forms**: indeed a holomorphic one form would be the collection of two functions $f_1(z)$ and $f_2(\zeta)$ with

$$f_1(z)\mathrm{d}z = f_2(\zeta)\mathrm{d}\zeta \qquad \in \mathbb{C}^\times \tag{5.6.8}$$

Both f_1 and f_2 should be holomorphic in their respective variables and hence entire; moreover

$$f_1(z) = f_2(z^{-1})\frac{-1}{z^2} \tag{5.6.9}$$

Since $f_2(\infty)$ has a well defined value, we see that $f_1(z)$ is not only bounded but also vanishing at $z = \infty$, hence it is identically 0 and so is f_2.

Exercise 6.2 *Prove that all meromorphic one forms are of the form $f_1(z)\mathrm{d}z$ with $f(z)$ rational.*

6.2 Automorphisms

An automorphism $\varphi : \overline{C} \to \overline{\mathbb{C}}$ is a rational function

$$\varphi(z) = \frac{P(z)}{Q(z)}\ , \tag{5.6.10}$$

with P, Q some polynomials with no common factors. The equation

$$\varphi(z) = w \quad P(z) = wQ(z) \tag{5.6.11}$$

has as many solutions in z as the maximum of the degrees of P and Q; since φ must have only one inverse value, it follows that $\deg(P) \leq 1 \geq \deg(Q)$. In other words

$$\varphi(z) = \frac{az+b}{cz+d} \ , \quad a,b,c,d \in \mathbb{C} \tag{5.6.12}$$

The invertibility of $\varphi(z)$ requires $\Delta := ad - bc \neq 0$. By multiplying $a' = \lambda a, \ldots, d' = \lambda d$ for $\lambda^2 = \frac{1}{\Delta}$ the transformation does not change, so that we can always write it as

$$\varphi(z) = \frac{a'z+b'}{c'z+d} \ , \quad a',b',c',d' \in \mathbb{C} \ , \quad a'd' - b'c' = 1 \tag{5.6.13}$$

In other words the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, characterizes the automorphisms of $\overline{\mathbb{C}}$. More precisely the group that acts faithfully is $PSL_2(\mathbb{C}) := SL_2(\mathbb{C})/\{\pm\mathbf{1}\}$ since the matrix $g \in SL_2(\mathbb{C})$ and the matrix $-g$ give the same transformation. The transformations of this form are called **fractional linear** transformations or **homographies** or **Möbius** transformations: they have the following interesting property

Proposition 6.2 If $\Gamma \subset \mathbb{C}$ is a circle or a straight line then its image under a homography is also a circle or a straight line.

Proof. A homography is the composition of transformations of the type (**exercise**)

$$w_1 = \lambda z + \rho \ , \ \lambda \neq 0 \ , \qquad w_2 = \frac{1}{z} \ , \tag{5.6.14}$$

Since transformations of the first type are roto-translations, they clearly have the required property. It remains to check it only for the inversion $z \mapsto \frac{1}{z}$.

The line. We have $z = at + b$, $a \neq 0$, $t \in \mathbb{R}$ and without loss of generality $|a| = 1$. Since $\frac{1}{z} = \lambda \frac{1}{\lambda z}$ trivially, this means that we can roto-dilate the line so as to have in a "canonical" position. We do so and hence assume that either is the imaginary axis (if $\Im(a/b) = 0$) or the line $z' = it + 1$. This line is easily seen to be mapped by $z \mapsto \frac{1}{z}$ to the circle of radius $\frac{1}{2}$ and center $\frac{1}{2}$ (hence passing through the origin). In general one can find that the line $z = at + b$ with $\Im(a/b) \neq 0$ (i.e. not passing through the origin) is mapped to the circle of center C and radius R

$$C = \frac{\overline{a}}{2i\Im(b\overline{a})}, \qquad R = |C|. \tag{5.6.15}$$

The circle. If we have a circle and by the same argument used above we can always assume it is either centered at the origin with radius 1 (trivially mapped to itself) or centered at $z = 1$ and radius r, $z = 1 + re^{\theta}$. The center and radius of the new circle is

$$C = \frac{1}{1 - r^2} \ , \qquad R = \frac{r}{|1 - r^2|} \ , \tag{5.6.16}$$

indeed

$$\left|\frac{1}{1+r\mathrm{e}^{i\theta}} - \frac{1}{1-r^2}\right| = \left|\frac{-r^2 - r\mathrm{e}^{i\theta}}{(1-r^2)(1+r\mathrm{e}^{i\theta})}\right| = R \tag{5.6.17}$$

Note that if $r = 1$ the circle passes through the origin and hence it is the image of a line (and since the map $z \to 1/z$ is its own inverse, also its image is the same line).

Exercise 6.3 *Prove that a circle in $\mathbb{C}$ is given by the following equation:*

$$A|z|^2 + \overline{B}z + B\overline{z} + C = 0 \ , \qquad , A, C \in \mathbb{R},\ B \in \mathbb{C},\ AC < |B|^2 \ . \tag{5.6.18}$$

Find center and radius.

Exercise 6.4 *Prove that the equation of the circle through the three points z_1, z_2, z_3 (or the line if they are aligned) is given by*

$$\det\begin{pmatrix} |z|^2 & z & \overline{z} & 1 \\ |z_1|^2 & z_1 & \overline{z}_1 & 1 \\ |z_2|^2 & z_2 & \overline{z}_2 & 1 \\ |z_3|^2 & z_3 & \overline{z}_3 & 1 \end{pmatrix} = 0 \tag{5.6.19}$$

Exercise 6.5 *Prove that the following equation determines a circle (or a line if $K = 1$) and find center and radius.*

$$\left|\frac{z-z_1}{z-z_2}\right| = K \ , \quad z_1 \neq z_2 \ , \ K > 0 \ . \tag{5.6.20}$$

7 The unit disk

The unit disk $\mathbf{D} := \{|z| < 1\}$ is naturally a simply connected complex manifold. It is biholomorphically equivalent to the upper half plane $\mathcal{H}_+ := \{w : \ \Im(w) > 0\}$ and the equivalence is given by the map

$$z = \frac{w-i}{w+i} \tag{5.7.1}$$

$$w = \frac{i+iz}{1-z} \tag{5.7.2}$$

Indeed the boundary of $\mathbf{D} \setminus \{1\}$ is mapped on the real axis for we have ($z = x + iy$)

$$w = \frac{i(1-|z|^2 + z - \overline{z})}{|1-z|^2} \tag{5.7.3}$$

$$\Im(w) = \frac{1-|z|^2}{|1-z|^2} \geq 0 \ . \tag{5.7.4}$$

7.1 Automorphisms

Theorem 5.7.4 *Any automorphism of $\mathcal{H}_+$ is of the form*

$$w' = \frac{aw+b}{cw+d} \ , \ a,b,c,d \in \mathbb{R}, ad-bc = 1 \ . \tag{5.7.5}$$

Proof. The fact that the above transformations are automorphisms is left as exercise (you need to check that $\Im(w') > 0$ and that the inverse has the same properties). Also, they form a group (the composition of two such transformations is again a transformation of the same form). Note that in particular we have transformations of the form

$$w' = \lambda w + k \ , \quad \lambda > 0, \ k \in \mathbb{R} \tag{5.7.6}$$

(here $\lambda = a^2, \ k = ba$ since $d = 1/a$).

Let $\varphi : \mathcal{H}_+ \to \mathcal{H}_+$ be an automorphism: let $w_0 = \varphi^{-1}(i) = i\lambda + k$. Then

$$\psi(w) := \varphi(\lambda w + k) \tag{5.7.7}$$

is another automorphism which fixes the point $w = i$. Using the biholomorphic equivalence with the unit disk (5.7.2) (which maps $w = i$ to $z = 0$) we think of it as a map $f : \mathbf{D} \to \mathbf{D}$ which fixes the origin

$$f(0) = 0 \ . \tag{5.7.8}$$

By Schwartz' lemma 2.5.13 we have $|f(z)| \leq |z|$. Since the inverse $f^{-1} : \mathbf{D} \to \mathbf{D}$ also fixes the origin we must also have $|f^{-1}(z')| \leq |z'|$. Taking $z' = f(z)$ we find

$$|z| \leq |f(z)| \leq |z| \ , \Rightarrow |f(z)| \equiv |z| \ , \tag{5.7.9}$$

and hence by the same Schwartz' lemma, $f(z) = \mathrm{e}^{i\theta} z = \alpha z$. Going back to ψ we have then proved that

$$\frac{\psi - i}{\psi + i} = \alpha \frac{w-i}{w+i} \tag{5.7.10}$$

$$\psi(w) = \frac{i(\alpha+1)w + \alpha - 1}{(1-\alpha)w + i(\alpha+1)} \tag{5.7.11}$$

In order to recognize that this is a trasformation of the form (5.7.5) we introduce $\omega := \mathrm{e}^{i\theta/2}$ so that $\omega^2 = \lambda$ and $\overline{\omega} = \frac{1}{\omega}$. Factoring $2i\omega$ from numerator and denominator of $\psi(w)$ we have

$$\psi(w) = \frac{\cos(\theta/2)w + \sin(\theta/2)}{-\sin(\theta/2)w + \cos(\theta/2)} \tag{5.7.12}$$

and hence the matrix in $SL_2(\mathbb{R})$ representing the initial transformation φ is

$$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \tag{5.7.13}$$

[Iwasawa decomposition of $SL_2(\mathbb{R})$] Q.E.D.

8 Complex tori and elliptic functions

Another way of obtaining complex one-dimensional manifold is by quotienting

Definition 8.1 *Let* Γ *be a at-most countable* **group** *and* (X, τ) *a topological space. We say that* Γ *acts on* (X, τ) *if there for each* $g \in \Gamma$ *there is a continous map (which we denote by the same symbol)* $g : X \to X$ *with the properties*

1. $\mathbf{1}(x) = x \ \forall x \in X$

2. $h(g(x)) = hg(x)$

The action is called **proper discontinous** *if* $\forall x$ *there is a neighborhood* U_x *such that* $g(U_x) \cap U_x = \emptyset$, $\forall g \neq \mathbf{1}$.

If a group Γ acts properly discontinuously on X we can define a new topological space denoted by X/Γ. As a set it is the quotient of X by the equivalence relation

$$x \sim x' \text{ iff } \exists g \in \Gamma \text{ s.t. } x' = g(x) \tag{5.8.1}$$

A basis of neighborhoods in X/Γ is defined as the equivalence classes of the neighborhoods which are acted upon properly discontinously. The definition can be extended easily to manifolds.

There are many other ways to define a topological structure on the quotient space of X/G even if the action is not proper discontinuous but this would lead us astray.

One important extension of the above definition (which we will need for the modular group) is

Definition 8.2 (Proper discontinuos action/extended definition) *Let the group* Γ *be acting on a topological space* (X, τ)*: the action is called* **proper discontinuous** *if* $\forall x \in X$ *the* **stabilizer** *of* x

$$\Gamma_x := \{g \in \Gamma \ s.t. \ gx = x\} \ , \tag{5.8.2}$$

is a finite *subgroup of* Γ *and if there is a neighborhood* U_x *such that*

$$g(U_x) \cap U_x = \emptyset, \quad \forall g \notin \Gamma_x \ . \tag{5.8.3}$$

Once more the topology can be easily constructed on the quotient space using **precisely invariant** neighborhoods of points of X as basis; a precisely invariant neighborhood is a neighborhood V_x of $x \in X$ such that

$$h(V_x) = V_x \ , \forall h \in \Gamma_x \text{ and } g(V_x) \cap V_x = \emptyset \ , \quad \forall g \notin \Gamma_x \ . \tag{5.8.4}$$

In the case of manifolds, however, the ensuing topological space X/Γ not always can be endowed with the structure of a smooth manifold (sometimes it is an **orbifold**).

In this section we consider two examples where the stabilizer of each point is trivial (only the identity transformation) and hence we don't need the refined version. Our first example is $\Gamma = \mathbb{Z}$ and $X = \mathbb{C}$ with action

$$n(z) := z + n. \tag{5.8.5}$$

The quotient manifold is equivalent to an infinite cylinder; it can be covered with two (or more) charts, for example $U_0 = [\{\Re(z) \in (0, 1)\}]$ and $U_1 = [\{\Re(z) \in (-1/2, 1/2)\}]$. The coordinate there can be taken as the value of the *representative* of the class whose real part falls in the given intervals. The first chart covers all classes except those of the line $[\{\Re(z) = 0\}]$ and the second covers all except $[\{\Re(z) = \frac{1}{2}\}]$. Another point of view is to cover the manifold $\mathbb{C}/\Gamma$ with infinitely many charts

$$U_{2j+1} = [\{\Re(z) \in (j - \frac{1}{2}, j + \frac{1}{2})\}], \qquad U_{2j} = [\{\Re(z) \in (j, j + 1)\}]\ , \tag{5.8.6}$$

and use the tautological coordinates z_j, which are all related by addition of a suitable integer.

It should be then clear that [holomorphic/meromorphic] functions are functions periodic with period 1

$$f(z) = f(z + 1) \tag{5.8.7}$$

i.e. such that the value does not depend on the representative of the point.

Exercise 8.1 *The manifold $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^\times$ are biholomorphically equivalent and the map is*

$$\begin{array}{rcl} \varphi : \mathbb{C}/\mathbb{Z} & \longrightarrow & \mathbb{C}^\times \\ {[z]} & \mapsto & \mathrm{e}^{2i\pi z} \end{array} \tag{5.8.8}$$

The next example is the one that motivates the title of the section; we take $\Gamma = \mathbb{Z} \times \mathbb{Z}$ (i.e. pairs of integers with obvious group structure) acting on $\mathbb{C}$ as a **lattice**

$$(n, m)(z) := z + n\omega_1 + m\omega_2\ , \tag{5.8.9}$$

where the complex numbers ω_i are called the **periods**.

Exercise 8.2 *Prove that the action is proper-discontinuous iff $\omega_1/\omega_2 \notin \mathbb{R} \setminus \mathbb{Q}$, namely iff the two periods are not collinear.*

It is customary to order the periods so that $\tau := \frac{\omega_2}{\omega_1}$ has **positive** imaginary part. The quotient $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is equivalent to the topological manifold consisting of the parallelogram of vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$ where opposite sides are identified pointwise, namely a torus (doughnut), which is **compact**; in this topology a curve exiting on the right re-enters from the left at the corresponding point of the opposite side (the topology of many old-style videogames). We denote the lattice by

$$\Lambda := \{n\omega_1 + m\omega_2,\ \ n, m \in \mathbb{Z}\} \subset \mathbb{C} \tag{5.8.10}$$

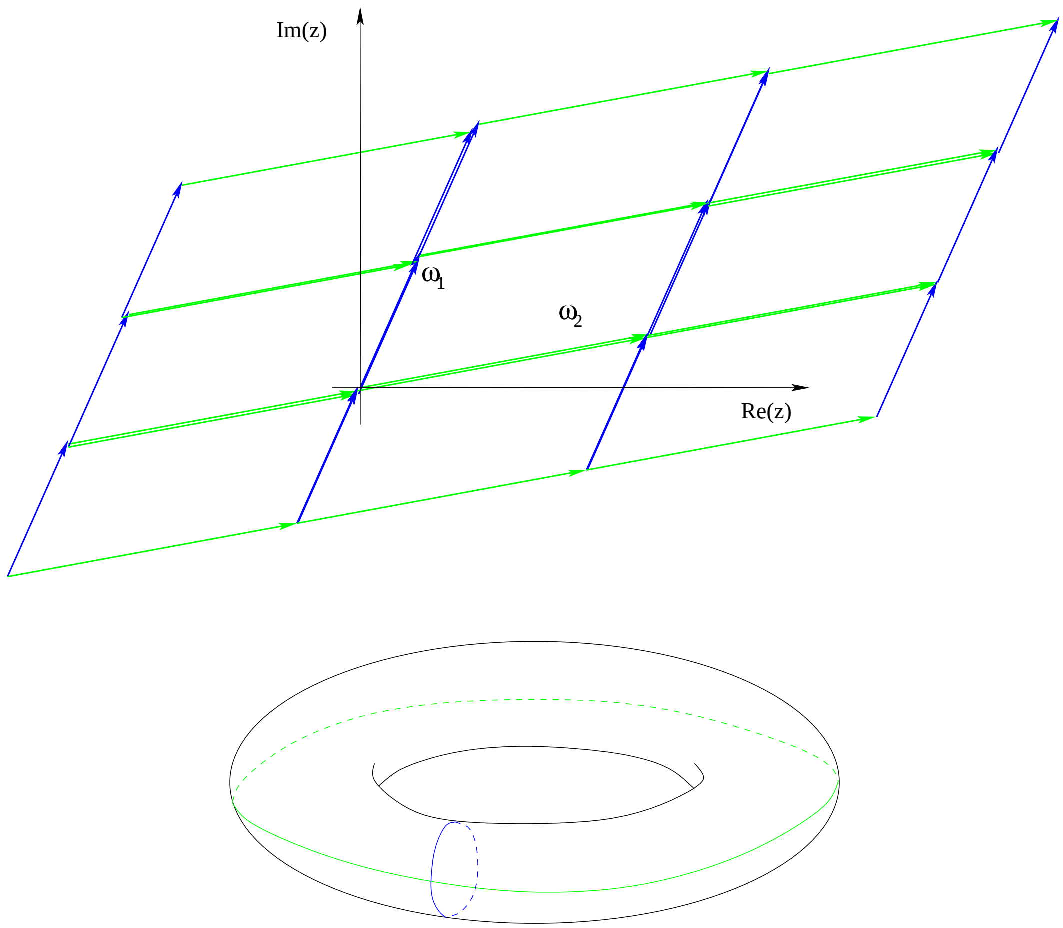

Figure 5.1: The tessellation of the plane by the fundamental domains Π and the topological model of the torus.

and by

$$\Pi := \{z = \omega_1 s_1 + \omega_2 s_2, \quad s_1, s_2 \in [0,1)\} \tag{5.8.11}$$

the **fundamental parallelogram** (any point $[z] \in \mathbb{C}/\Lambda$ admits a unique representative in the fundamental parallelogram).

We will denote by $E(\omega_1, \omega_2)$ the **complex torus** obtained in this way by quotienting $\mathbb{C}/\Lambda(\omega_1, \omega_2)$.

A holomorphic/meromorphic function on $\mathbb{C}/\Lambda$ is –similarly to the previous example– a **doubly periodic** function

$$f(z) = f(z + \omega_1) = f(z + \omega_2)\ . \tag{5.8.12}$$

It is easy to see directly (i.e. without invoking Thm. 5.1.3) that a doubly holomorphic function is constant, for it is bounded on the parallelogram and –by periodicity– also on the whole complex plane and hence by Liouville's theorem is constant. Therefore the only "interesting" functions are the meromorphic ones, which are called **elliptic functions** (by definition *doubly periodic* **meromorphic** *functions*[1]).

8.1 One-forms

We start by observing that the one form $C\mathrm{d}z$, C =constant, is well defined since different local coordinates differ by addition of constants and hence the Jacobian is 1. The one-form $\mathrm{d}z$ is clearly then holomorphic because the local charts are bounded parallelograms in the complex $z-$plane (this is in contrast with the case of the Riemann–sphere since $\mathrm{d}z$ is **not** holomorphic, it has a double pole at ∞ (**exercise**)). Hence the vector space of holomorphic one-forms is at least one–dimensional: in fact there are no other holomorphic one-forms, since then it should be of the form

$$\omega = f(z)\mathrm{d}z\ , \tag{5.8.13}$$

with $f(z)$ doubly periodic and hence constant.

8.2 Elliptic functions

The goal of the following paragraph is to give a precise describition of the field of elliptic functions.

Theorem 5.8.13 *The following* **Weierstrass' $\wp$ function** *is an elliptic function with double poles at the points of* Λ

$$\wp(z) = \wp(z; \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda'} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \tag{5.8.14}$$

$$\Lambda' := \Lambda \setminus \{0\}. \tag{5.8.15}$$

[1]Note that this excludes doubly periodic functions with essential singularities, which are simply referred to as "doubly periodic functions".

Proof. We outline the main steps of the proof. We first prove that the series converges to a meromorphic function with poles at Λ. For any compact $K \subset \mathbb{C} \setminus \Lambda$ we prove uniform convergence. The only concern arises from the infinite sum over the points of Λ: let R be such that the compact K is contained in the disk $D_0(R)$; since only a finite number of λ's fall within $D_0(R)$ only the part of the sum coming from the external ones is of concern, since the rest can be easily estimated uniformly (using the $\delta = \mathrm{dist}(K, \Lambda) > 0$)

$$\left|\sum_{|\lambda|>R} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)\right| \leq \sum_{|\lambda|>R} \left|\frac{2\lambda z - z^2}{\lambda^2 (z-\lambda)^2}\right| \leq \sum_{|\lambda|>R} \left|\frac{2\lambda z - z^2}{\lambda^2 (z-\lambda)^2}\right| \leq \sum_{|\lambda|>R} \frac{2|\lambda| R + R^2}{|\lambda|^2 (|\lambda| - R)^2} \tag{5.8.16}$$

This last double series is estimated by an integral of C/r^3 over the plane and is therefore convergent.

We also note that $\wp(z)$ is **even**, $\wp(z) = \wp(-z)$. We now prove that $\wp$ is doubly periodic; first of all we note that

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} \tag{5.8.17}$$

and that it is easy to see that this series as well is uniformly convergent on any compact set in $\mathbb{C} \setminus \Lambda$. It is also easy to see that $\wp'(z+\lambda) = \wp'(z)$, $\forall \lambda \in \Lambda$ by shifting the indices of the sum (**exercise**). Since the derivative is doubly periodic, the function $\wp$ must be doubly periodic as well up to additive constant

$$\wp(z+\omega_1) - \wp(z) = C \ , \qquad \wp(z+\omega_2) - \wp(z) = D \tag{5.8.18}$$

Recalling that $\wp$ is even we evaluate at $z = \omega_i/2$ (the half periods)

$$\wp(\omega_1/2) - \wp(-\omega_1/2) = 0 \qquad \wp(\omega_2/2) - \wp(-\omega_2/2) = 0 \ , \tag{5.8.19}$$

and hence $D = C = 0$ and $\wp$ is doubly periodic. Q.E.D.

We need to write also the Laurent series of $\wp$ at $z = 0$

Lemma 5.8.19 The Laurent series expansion of $\wp(z)$ around $z = 0$ is

$$\wp = \frac{1}{z^2} + 3z^2 G_2 + 5z^4 G_3 + \ldots + (2k+1) z^{2k} G_{k+1} + \ldots \tag{5.8.20}$$

$$G_j = \sum_{\lambda \in \Lambda'} \frac{1}{\lambda^{2j}} \tag{5.8.21}$$

Proof. We have

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda} \frac{1}{\lambda^2} \left(\frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} - 1\right) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda'} \sum_{k=1}^{\infty} (k+1) \frac{z^k}{\lambda^{k+2}} \tag{5.8.22}$$

The last triple series is uniformly absolutely convergent for $|z| < \min(|\omega_1|, |\omega_2|) - \epsilon$ and hence we can interchange the order of the sums obtaining (note that the odd terms cancel because of the symmetry $\Lambda = -\Lambda$)

$$\wp = \frac{1}{z^2} + 3z^2 G_4 + 5z^4 G_6 + \ldots + (2k+1) z^{2k} G_{2k+2} + \ldots \tag{5.8.23}$$

$$G_{2j} = \sum_{\lambda \in \Lambda'} \frac{1}{\lambda^{2j}} = \sum_{\mathbb{Z}^2 \ni (m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^{2j}} = \frac{1}{{\omega_1}^{2j}} \sum_{\mathbb{Z}^2 \ni (m,n) \neq (0,0)} \frac{1}{(m + n\tau)^{2j}} \tag{5.8.24}$$

The functions G_{2j} are called **Eisenstein's series** and it is easily seen that they are absolutely convergent (and uniformly in $\tau = \omega_2/\omega_1$ for $\Im(\tau) > \epsilon > 0$). Q.E.D.

Theorem 5.8.24 *Weierstrass' function $\wp$ satisfies the differential equation*

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \ , \qquad g_2 := 60G_4 \ , \ g_3 = 140G_6 \ . \tag{5.8.25}$$

Proof. We take the Laurent expansion of $\wp'$ and square it

$$\wp' = \frac{-2}{z^3} + 6zG_4 + 20z^3G_6 + \dots \tag{5.8.26}$$

$$(\wp')^2 = \frac{4}{z^6} - \frac{24}{z^2}G_4 - 80G_6 + \mathcal{O}(z) \ , \tag{5.8.27}$$

Therefore we subtract a suitable polynomial in $\wp$ so as to eliminate the singular part:

$$(\wp')^2 - 4\wp^3 = -\frac{60}{z^2}G_4 - 140G_6 - \mathcal{O}(z) \tag{5.8.28}$$

$$(\wp')^2 - (4\wp^4 - g_2\wp - g_3) = \mathcal{O}(z). \tag{5.8.29}$$

Since the LHS is obviously an elliptic function so is the RHS. Moreover the LHS may have poles a priori only on Λ, but the RHS is telling that it has instead a zero at $z = 0 \in \Lambda$ and hence is holomorphic and vanishes at one point so it is identically zero. Q.E.D.

The content of Thm. 5.8.24 is that if we set $X = \wp(z)$, $Y = \wp(z)$ we obtain a parametrization of the following **algebraic curve**

$$Y^2 = 4X^3 - g_2X - g_3 \ . \tag{5.8.30}$$

According to the terminology of algebraic curves, this is an instance of hyperelliptic curve of genus 1. In order to prove that $\wp$ and $\wp'$ actually parametrize completely the curve (5.8.30) we prove the following result

Proposition 8.1 Let $f(z)$ be an elliptic function (meromorphic). Let $Z := \{z_n\}$ and $P := \{p_\ell\}$ be the sets of zeroes/poles (both are finite) chosen in the fundamental parallelogram Π and let m_n, k_ℓ be the multiplicities/orders. Then

1. The number of zeroes (conted with multiplicities) minus the number of poles (counted with their order) is zero

$$\sum m_n = \sum k_\ell \tag{5.8.31}$$

2. The sum of the positions of the zeroes (with multiplicities) minus the sum of the positions of the poles (with the orders) is a point of Λ

$$\sum_n m_n z_n - \sum_\ell k_\ell p_\ell = \lambda \in \Lambda \tag{5.8.32}$$

Proof. We will use Corollary 2.7.9. For the path γ we choose the boundary $\partial\Pi$ and obtain

$$\oint_{\partial\Pi} = \int_0^{\omega_1} + \int_{\omega_1}^{\omega_1+\omega_2} + \int_{\omega_1+\omega_2}^{\omega_2} + \int_{\omega_2}^{0} \tag{5.8.33}$$

where the integrals are supposed to be straight segments (which is mostly irrelevant due to Cauchy's theorem). We now integrate the differential $\frac{f'(z)}{f(z)}\mathrm{d}z$ (which is elliptic!) on this path and note that -due to ellipticity- the first integral is minus the third and the second is minus the fourth, e.g. (actually for any elliptic function $g(z)$)

$$\int_0^{\omega_1} g(z)\mathrm{d}z = \int_{\omega_2}^{\omega_1+\omega_2} g(z-\omega_2)\mathrm{d}z \tag{5.8.34}$$

[If by any chance one of the poles of $\frac{f'(z)}{f(z)}$ falls exactly on the boundary, we translate the fundamental parallelogram by ϵ to avoid this.]
Therefore we have

$$\oint_{\partial\Pi} \frac{f'(z)}{f(z)}\mathrm{d}z = 0 \tag{5.8.35}$$

and the first assertion follows. For the second we integrate $z\frac{f'(z)}{f(z)}\mathrm{d}z$ which is **not** elliptic, so some care must be paid

$$\int_0^{\omega_1} z\frac{f'(z)}{f(z)}\mathrm{d}z - \int_{\omega_2}^{\omega_1+\omega_2} z\frac{f'(z)}{f(z)}\mathrm{d}z = \int_0^{\omega_1}\left(z\frac{f'(z)}{f(z)} - (z+\omega_2)\frac{f'(z+\omega_2)}{f(z+\omega_2)}\right)\mathrm{d}z = \omega_2\int_0^{\omega_1}\mathrm{d}(\ln(f(z)))$$
$$\int_{\omega_1}^{\omega_1+\omega_2} z\frac{f'(z)}{f(z)}\mathrm{d}z - \int_0^{\omega_2} z\frac{f'(z)}{f(z)}\mathrm{d}z = \int_0^{\omega_1}\left((z+\omega_1)\frac{f'(z+\omega_1)}{f(z+\omega_1)} - z\frac{f'(z)}{f(z)}\right)\mathrm{d}z = \omega_1\int_0^{\omega_2}\mathrm{d}(\ln(f(z)))$$

The integrals in the RHS's are integer multiples of the periods since -e.g.- the segment $[0,\omega_1]$ is mapped to a loop by the elliptic map $f(z)$ and hence the integral just counts the winding number of the loop w.r.t. zero. Q.E.D.

We thus have

Proposition 8.2 The funcion $\wp(z)$ takes on every value $c \in \mathbb{C}$ exactly twice (counting with multiplicity) in each fundamental domain.

Proof. Since $\wp$ has only one double pole, it must have two zeroes by Prop. 8.1. The same applies to the elliptic function $\wp(z) - c$. Q.E.D.

Note that since $\wp$ is even the two values are taken on at z_c and $-z_c \equiv \omega_1 + \omega_2 - z_c$, i.e. at points symmetrically placed w.r.t. the center of the fundamental parallelogram Π: at those points $\wp'$ takes on opposite values and hence we have proved

Corollary 5.8.35 *The functions $X = \wp(z)$ and $Y = \wp'(z)$ give a complete parametrization of the locus (5.8.30).*

We finally characterize all meromorphic elliptic functions.

Theorem 5.8.35 *Any meromorphic elliptic function $f(z)$ can be uniquely written as*

$$f = R_1(X) + YR_2(X) \ , \qquad X = \wp(z) \ , \qquad Y = \wp'(z) \tag{5.8.36}$$

where R_1, R_2 are rational functions.

Proof. We split f in its even and odd parts (which are still elliptic),

$$f_e(z) = \frac{1}{2}(f(z) + f(-z)) \qquad f_o(z) = \frac{1}{2}(f(z) - f(-z)) \ . \tag{5.8.37}$$

Since $f_o/\wp'(z)$ is even we need only to prove that any even elliptic function can be (uniquely) written as a rational function of $\wp$. This is left as **exercise**. Q.E.D.

Definition 8.3 *The algebraic curve (5.8.30) is called a* **elliptic curve** *of modulus $\tau = \frac{\omega_2}{\omega_1}$.*

Coordinate charts and compactification of the algebraic curve

Consider now an arbitrary algebraic curve

$$\dot{\mathcal{L}} := \left\{ (X, Y) : \quad Y^2 = 4X^3 - g_2 X - g_3 \right\} \tag{5.8.38}$$

where we assume that the roots of $P(X) = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$ are distinct (hence ${g_2}^3 - 27{g_3}^2 \neq 0$). This subset of $\mathbb{C}^2$ is clearly not compact: we perform a one-point compactification

$$\mathcal{L} = \dot{\mathcal{L}} \cup \{\infty\} \tag{5.8.39}$$

by adding a formal point at infinity. We need to define the local charts and local coordinates.

1. For a neighborhood of a point $(X_0, Y_0) \in \dot{\mathcal{L}}$ with $X_0 \neq e_j$ we declare that the local coordinate is X itself (which locally and univocally parametrizes a neighborhood of $\dot{\mathcal{L}}$ near the given point.

2. In a neighborhood of $(e_j, 0)$ we declare that the local coordinate is Y (which locally and univocally parametrizes a neighborhood of $\dot{\mathcal{L}}$ near the given point).

3. A neighborhood of ∞ is in the usual sense: local coordinate is $\xi = \frac{X}{Y}$. Introducing $\eta = \frac{1}{Y}$, the functions ξ, η satisfy (from the same algebraic equation)

$$\eta = 4\xi^3 - g_2\xi\eta^2 - g_3\eta^3 \tag{5.8.40}$$

   and the point that we are adding has $\xi = 0 = \eta$.

Remark 8.1 This definition of coordinate at the ∞ point comes from a "natural" embedding of $\dot{\mathcal{L}}$ as a subset of a 2-dimensional complex compact manifold called $\mathbb{CP}^2$. It is defined as

$$\mathbb{C}P^2 := \mathbb{C}^3 \setminus \{0\}/\mathbb{C}^\times \ , \tag{5.8.41}$$

where the quotient is with respect to the group action of the (Abelian) group $\mathbb{C}^\times$ of nonzero complex numbers, acting on $\mathbb{C}^3 \setminus \{0\}$ by

$$\lambda \cdot (z_1, z_2, z_3) := (\lambda z_1, \lambda z_2, \lambda z_3) \tag{5.8.42}$$

We do not enter in any further detail here.

Exercise 8.3 *Prove that the unique (up to scalar multiplication) differential of the first kind on the elliptic curve of modulus τ can be written as*

$$\omega = \frac{\mathrm{d}X}{Y} \ . \tag{5.8.43}$$

In particular you should prove (check) that the given differential is holomorphic at all points (in particular at the branchpoints e_j and at ∞).

Hence conclude that the coordinate z is given by the **elliptic integral**

$$z = \int \frac{\mathrm{d}X}{\sqrt{4X^3 - g_2 X - g_3}} \tag{5.8.44}$$

Exercise 8.4 *Prove that the Eisenstein series G_k are all polynomials in G_2, G_3 with rational coefficients by following the given steps.*

1. *From the ODE for $\wp$ derive the second order ODE*

$$\wp''(z) = 6\wp^2 - g_2 \tag{5.8.45}$$

2. *Match the Laurent series expansions of the two sides (the LHS is linear in the G_k's, the RHS is quadratic) and derive a recurrence relation for the G_k's.*

3. *Draw the conclusion by induction on the recurrence relations found in the previous step.*

There is still a problem that we did not address: indeed so far we have proved that **if** g_2, g_3 are given by the expressions (5.8.25) then $\wp$ and $\wp'$ give a parametrization (a.k.a. **uniformization**) of the locus of $\mathbb{C}^2$

$$Y^2 = 4X^3 - g_2 X - g_3 \ , \tag{5.8.46}$$

in terms of the complex torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$. Suppose that we instead assign two complex numbers g_2, g_3 and seek an uniformization for this locus: the natural question is wether there is an elliptic curve with some modulus τ and hence some periods ω_1, ω_2 for which the Weierstrass functions $\wp, \wp'$ yield the desired uniformization.

Before addressing this in full we make the following remark on an elliptic curve:

Proposition 8.3 Let g_2, g_3 be given by (5.8.25). Then

$$\Delta := {g_2}^3 - 27{g_3}^2 \neq 0 \tag{5.8.47}$$

Proof. The expression for Δ is nothing but the **discriminant** of the polynomial

$$P(X) = 4X^3 - g_2 X + g_3 \tag{5.8.48}$$

defined as the product of the squares of the differences of the roots. We call the roots of $P(X)$ e_1, e_2, e_3

$$P(X) = 4(X - e_1)(X - e_2)(X - e_3) \ . \tag{5.8.49}$$

We claim that (up to permutation)

$$e_1 = \wp\left(\frac{\omega_1}{2}\right) \ , \qquad e_2 = \wp\left(\frac{\omega_2}{2}\right) \ , \qquad e_3 = \wp\left(\frac{\omega_3}{2}\right) \qquad \omega_3 := \omega_1 + \omega_2 \tag{5.8.50}$$

Indeed at these points (called the **half periods**) the function $\wp'$ is zero by antisymmetry and hence so is $P(X)$. Since the polynomial is of the third degree, there are no other roots. We have thus

$$g_2 = 4(e_1e_2 + e_2e_3 + e_3e_1) \ , \qquad g_3 = 4e_1e_2e_3. \tag{5.8.51}$$

It is an exercise to see that

$$(e_1 - e_2)^2(e_2 - e_3)^2(e_1 - e_3)^2 = \Delta \ . \tag{5.8.52}$$

Therefore we must prove that $e_i \neq e_j$, $i \neq j$. Suppose that $e_1 = e_2$ (the other cases will be ruled out similarly). Since $\wp - e_1$ has precisely two zeroes (including multiplicities) and they are symmetric around the center of Π, this would imply that $\omega_2/2 = -\omega_1/2 \bmod \Lambda$, which is not the case. (Note that this argument also proves that $\wp - e_j$ has a double zero since $\omega_j/2 \equiv -\omega_j/2$). Q.E.D.

We should now prove the converse, namely

Theorem 5.8.52 *For any two complex numbers g_2, g_3 such that*

$${g_2}^3 - 27{g_3}^2 \neq 0 \tag{5.8.53}$$

there is an elliptic curves with periods ω_1, ω_2 and modular parameter $\tau = \frac{\omega_2}{\omega_1} \in \{\Im(t) > 0\}$ uniformizing the algebraic curve

$$Y^2 = 4X^3 - g_2 X - g_3 \ . \tag{5.8.54}$$

Proof. The requirement on the discriminant says that the curve is a non-singular hyperelliptic curve of genus 1 (see Section 11.3). Let us call e_1, e_2, e_3 the distinct roots of $P(X) = 4X^3 - g_2 X - g_3$. Let $\mathcal{D}$ the domain obtained by cutting $\mathbb{C}$ along a curve ℓ_1 from e_1 to e_2 and one ℓ_2 from e_3 to ∞. On this domain

$Y = \sqrt{P}$ is single valued (choosing arbitrarily one determination of the square root and analytically continuing to $\mathcal{D}$) as discussed in Sec. 11.3. We define

$$\Omega_1 := \oint_{\gamma_1} \frac{\mathrm{d}X}{\sqrt{P(X)}} , \qquad \Omega_2 := \oint_{\gamma_2} \frac{\mathrm{d}X}{\sqrt{P(X)}} , \tag{5.8.55}$$

where γ_1 encircles e_1, e_2 and γ_2 encircles e_2, e_3. They are the periods of the unique Abelian integral of the first kind

$$u(X) = \int_{X_0}^{X} \frac{\mathrm{d}x}{\sqrt{P(x)}} \tag{5.8.56}$$

where the path of integration is a curve avoiding the branchpoints. We claim that $u(X)$ defines a local coordinate in the neighborhood of the compactified elliptic curve; indeed near e_j the integrand has an integrable singularity (inverse square root) and hence $u(e_j)$ are well defined. According to the general theory of hyperelliptic curves (Sec. 11.3) near e_j the local coordinate is Y and $u(Y)$ is holomorphic there (check!). The only remaining point is ∞: since the integrand is $\mathcal{O}(X^{-3/2})$ also $X = \infty$ is an integrable singularity the check of the holomorphicity is left as exercise (note that acccording to the homotopy class of the contour chosen to approach $X = \infty$ the resulting values of $u(\infty)$ will differ by an integral linear combination of the periods).

Since $u(X)$ depends on the contour of integration from X_0 to X only by addition of $m\Omega_1 + n\Omega_2$, $m, n \in \mathbb{Z}$ (see Sec. 11.3), we have (almost) completed the proof if we can show that $\frac{\Omega_2}{\Omega_1}$ is **not** a real number, so that the map

$$u : \mathcal{L} \to \mathbb{C}/\mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2 \tag{5.8.57}$$

defines a biholomorphism between the algebraic curve and the elliptic curve (in the sense of quotient of $\mathbb{C}$ by the lattice of periods). The fact that $\Im\left(\frac{\Omega_2}{\Omega_1}\right) \neq 0$ is a particular case of a theorem going under the name of Riemann theorem (one of the many under the same name!). At this point one should even prove that Ω_i are nonzero[2]. Consider the simply-connected domain $\mathcal{L}_0 \subset \mathcal{L}$ obtained as follows: it is described as a two–sheeted covering of the domain $\mathcal{D}_0$ obtained by cutting $\mathbb{C}$ from e_1 to e_2 and from e_2 to e_3 and from e_3 to ∞; the two sheets $\mathcal{D}_{0,\pm}$ will be glued **only along one segment**, for example the one from e_3 to ∞ (see Figure 5.2).

We invoke your intuition that this domain is **simply connected** and hence the function $u(X)$ admits analytic continuation to a well-defined **holomorphic** function (the domain so constructed is the so–called **canonical dissection** of the compact curve $\mathcal{L}$). If your intuition fails then we can actually prove that the value of $u(X)$ is the same whether we use for the integration $\int_{X_0}^{X} \mathrm{d}u$ the "direct" (homotopy class of

[2]Consider the function $F(X) = \Im(P(X))$; it splits the X plane into two domains $F(X) < 0$ and $F(X) > 0$. The boundary $F(X) = 0$ contains the branchpoints e_j's (since $P(e_j) = 0$) and it is a piecewise smooth curve. On an arc joining e_1 to e_2 the real part has a definite sign (positive or negative) since otherwise at some other point also $\Re(P(X)) = 0$ and we would have another root of $P(X)$. The abelian integrals on these segments cannot vanish.

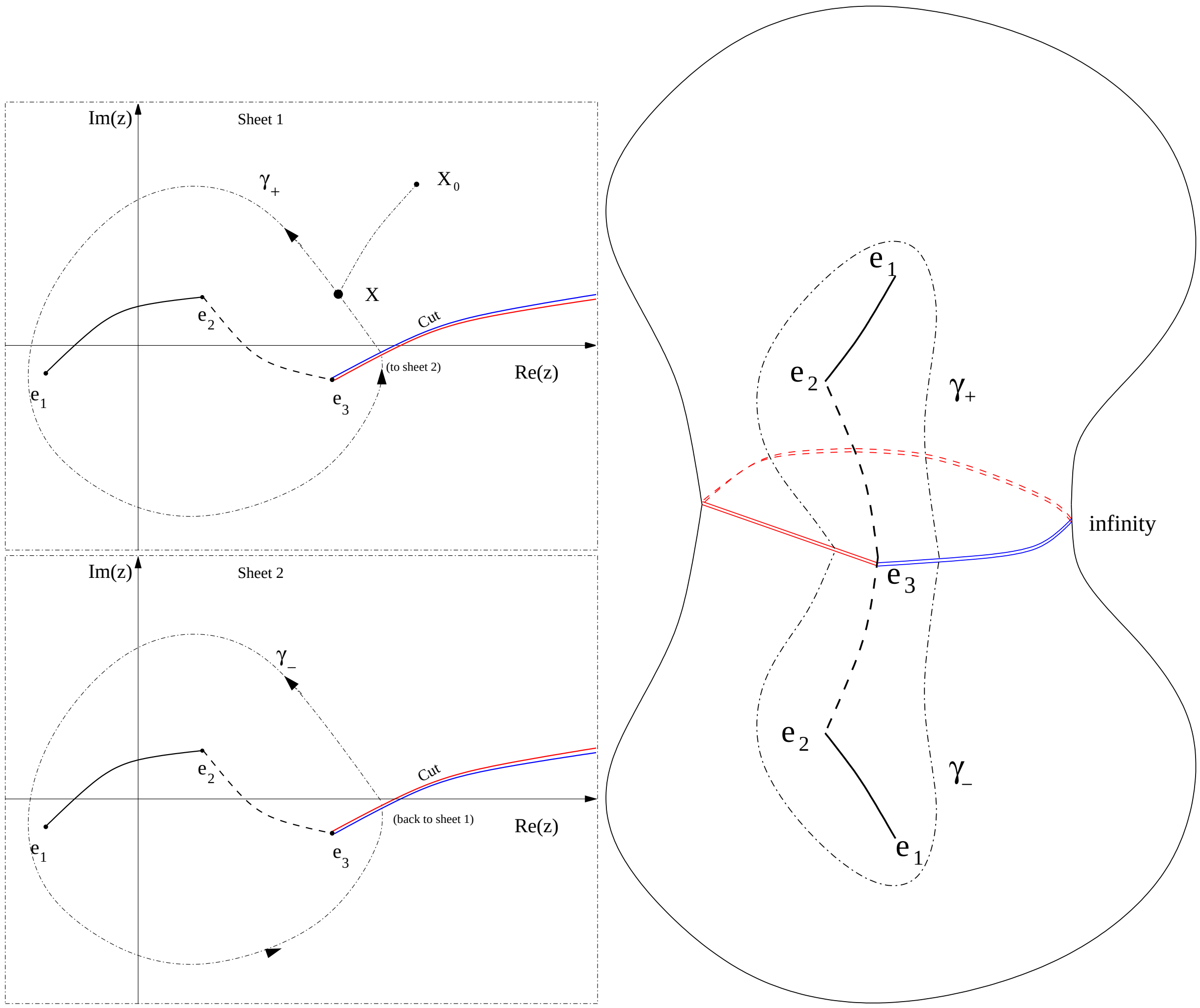

Figure 5.2: Example of the simply connected domain $\mathcal{L}_0$ and the topological model of the simply connected Riemann region (on the right).

the) segment in Figure or the (class of) the contour obtained by concatenation of the segment and the contour $\gamma_+ \cup \gamma_-$: indeed it suffices to prove that the integral of $\mathrm{d}u$ along the closed (on the Riemann surface!) loop $\gamma_+ \cup \gamma_-$ is zero.

For this we denote the differentials

$$\omega_\pm := \frac{\mathrm{d}X}{\sqrt{4X^3 - g_2 X - g_3}} \tag{5.8.58}$$

which are defined on the two copies domain $\mathcal{D}_0$ by choosing the two opposite determinations of the square-root. Then

$$\oint_{\gamma_+ \cup \gamma_-} \mathrm{d}u = \int_{\gamma_+} \omega_+ + \int_{\gamma_-} \omega_- \tag{5.8.59}$$

Since $\gamma_\pm$ have (by our choice) the same image on the X-plane, and since $\omega_- = -\omega_+$, the two last integrals are exactly one the opposite of the other, and hence the total is zero as asserted.

Remark 8.2 Note that the fact that we have chosen γ_+ and γ_- to have the same image in the X-plane is irrelevant, as well as the fact that the points of entry/exit on each sheets are the same (it is only important that the exit point from sheet 1 is the same as the entry point on sheet 2 and viceversa). I urge you to think carefully of why these details are irrelevant for the value of the integral.

Consider now the integral

$$\frac{1}{2i} \iint_{\mathcal{L}_0} \mathrm{d}u \wedge \mathrm{d}\overline{u} = 2\mathcal{R} \tag{5.8.60}$$

where

$$\mathcal{R} := \frac{1}{2i} \iint_{\mathcal{D}_0} \frac{\mathrm{d}X \wedge \mathrm{d}\overline{X}}{|P(X)|} = \frac{1}{2i} \iint_{\mathcal{D}_0} \mathrm{d}u \wedge \mathrm{d}\overline{u} = \frac{1}{2i} \iint_{\mathcal{D}_0} \mathrm{d}\left(\overline{u}\mathrm{d}u\right) > 0 \tag{5.8.61}$$

Here the above integral has to be understood with a grain of salt: the form $\mathrm{d}X \wedge \mathrm{d}\overline{X}$ is proportional to the Lebesgue measure on the X plane, and the denominator has singularities at e_j of type $|X - e_j|^{-1}$, which are integrable in the Lebesgue sense. As $X \to \infty$ the integral is also convergent because of the decay of $|P(X)|^{-1} \sim |X|^{-3}$. Note also that –as a two–dimensional integral– we have

$$\mathcal{R} := \frac{1}{2i} \iint_{\mathcal{D}_0} \frac{\mathrm{d}X \wedge \mathrm{d}\overline{X}}{|P(X)|} = \iint_{\mathbb{C}} \frac{\mathrm{d}^2 X}{|P(X)|} \tag{5.8.62}$$

since $\mathcal{D}_0$ differs from $\mathbb{C}$ by a set of zero measure (the cuts).

By Stokes theorem the integral over $\mathcal{L}_0$ is

$$\frac{1}{2i} \int_{\partial \mathcal{L}_0} \overline{u} \mathrm{d}u. \tag{5.8.63}$$

The boundary of $\mathcal{L}_0$ (oriented positively) consists of the union of the following segments:

1. the left edge of the segment from e_1 to e_2 on sheet 1 (green in figure (5.3)):
2. the left edge of the segment from e_2 to e_3 on sheet 1(purple):

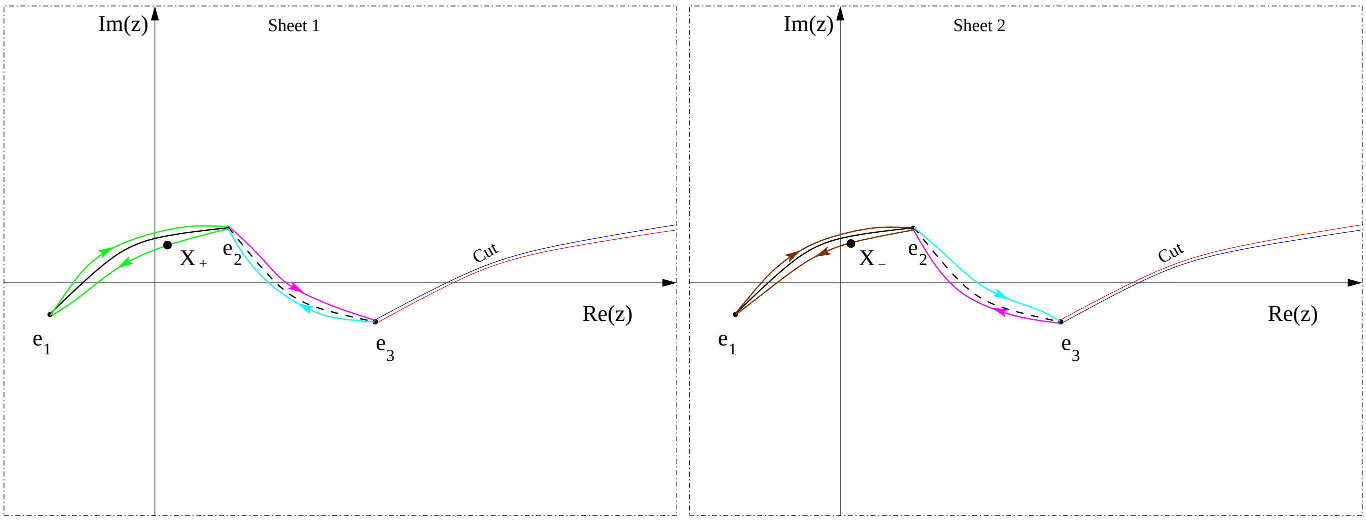

Figure 5.3: The boundary of the simply connected Riemann region $\mathcal{L}_0$ (the two edges of each segment are drawn separate only for didactical purposes).

3. the left edge of the segment from e_3 to e_2 on sheet 2 (purple):
4. the left edge of the segment from e_2 to e_1 on sheet 2 (brown):
5. the left edge of the segment from e_1 to e_2 on sheet 2 (brown):
6. the left edge of the segment from e_2 to e_3 on sheet 2(cyan):
7. the left edge of the segment from e_3 to e_2 on sheet 1 (cyan):
8. the left edge of the segment from e_2 to e_1 on sheet 1 (green).

Note that two segments of the same color close to a loop on our original Riemann surface $\mathcal{L}$; we call the green one the a-cycle and the puple one the b-cycle. For reasons that should become clear later on we call the brown one $a^{(-)}$ and the cyan one $b^{(-)}$.

Next we note that the value of $u(X)$ on corresponding points of the green and brown contours differ by the period on the cyan (b) cycle: indeed if X_+ is a point on the "lower" edge of the green contour and X_- is the same point on the lower edge of the brown contour, then the value of u are related by (refer to Fig. 5.3).

$$u(X_-) = u(X_+) - \oint_\gamma \mathrm{d}u \ , \tag{5.8.64}$$

where γ is a loop winding around e_2 and e_3 clockwise, hence homotopic to the opposite of the b-cycle. Therefore we have

$$u(X_+) - u(X_-) = \Omega_2 =: \oint_b \frac{\mathrm{d}X}{Y} \tag{5.8.65}$$

Similarly for points $Y_\pm$ on the purple and cyan contours

$$u(Y_+) - u(Y_-) = -\Omega_1 =: -\oint_a \frac{\mathrm{d}X}{Y} \ . \tag{5.8.66}$$

Note also that $\oint_a \mathrm{d}u = -\oint_{a^{(-)}} \mathrm{d}u$ (and similarly for the b-cycles) due to the fact that the analytic continuation of the square root changes sign. More explicitly

$$\oint_a \mathrm{d}u = \int_{e_2}^{e_1} \omega_+ + \int_{e_1}^{e_2} \omega_+ = 2\int_{e_1}^{e_2} \omega_+ \tag{5.8.67}$$

$$\oint_{a^{(-)}} \mathrm{d}u = \int_{e_2}^{e_1} \omega_- + \int_{e_1}^{e_2} \omega_- = 2\int_{e_1}^{e_2} \omega_- = -\oint_a \mathrm{d}u \ . \tag{5.8.68}$$

This explains the notation: a and $a^{(-)}$ are in fact the same contour (homologically) run in opposite direction (ditto for b and $b^{(-)}$).

We can now proceed with the computation of the integral (5.8.60):

$$\begin{aligned}
\int_{\partial\mathcal{L}_0} \overline{u}\mathrm{d}u &= \int_a \overline{u}\mathrm{d}u + \int_b \overline{u}\mathrm{d}u + \int_{a^{(-)}} \overline{u}\mathrm{d}u + \int_{b^{(-)}} \overline{u}\mathrm{d}u = &(5.8.69)\\
&= \int_a \overline{(u(X_+) - u(X_-))}\mathrm{d}u + \int_b \overline{(u(Y_+) - u(Y_-))}\mathrm{d}u = &(5.8.70)\\
&= \overline{\Omega_2}\oint_a \mathrm{d}u + \overline{\Omega_1}\oint_b \mathrm{d}u = &(5.8.71)\\
&= \overline{\Omega_2}\Omega_1 - \Omega_2\overline{\Omega_1} &(5.8.72)
\end{aligned}$$

so that we obtain

$$0 < \frac{1}{2i}\iint_{\mathcal{L}_0} \mathrm{d}u \wedge \mathrm{d}\overline{u} = \frac{1}{2i}\left(\overline{\Omega_2}\Omega_1 - \overline{\Omega_1}\Omega_2\right) = \Im(\overline{\Omega_2}\Omega_1) = \Im\left(|\Omega_2|^2 \frac{\Omega_1}{\Omega_2}\right) \tag{5.8.73}$$

which proves that the imaginary part of the ratio $\tau = \Omega_1/\Omega_2$ must be nonzero and actually positive (the sign is due to the definition of Ω_i and the relative orientations of the a and b cycles).

We have therefore proved that our curve $\mathcal{L}$ is biholomorphically equivalent to the complex torus $E(\Omega_1, \Omega_2)$. If we consider the Abelian integral $u(X)$ and invert it, we obtain an elliptic function which we identify with the $\wp$-function (up to a shift) since $\wp$ and $\wp'$ satisfy the same algebraic equation as X, Y. Q.E.D.

We conclude this section with a few remarks on the choices of the cycles a and b made in the proof of Thm. 5.8.52. One can in fact choose different ways of "canonically" dissecting the elliptic curve but the dissection must be such that the resulting domain is connected and simply-connected. The different choices depend on where we *decide* to put the various cuts and glueing; for example we could have decided that the cuts be done from e_1 to e_3 and e_3 to e_2 and e_2 to ∞ instead of our choice. The resulting new $\widetilde{a}$

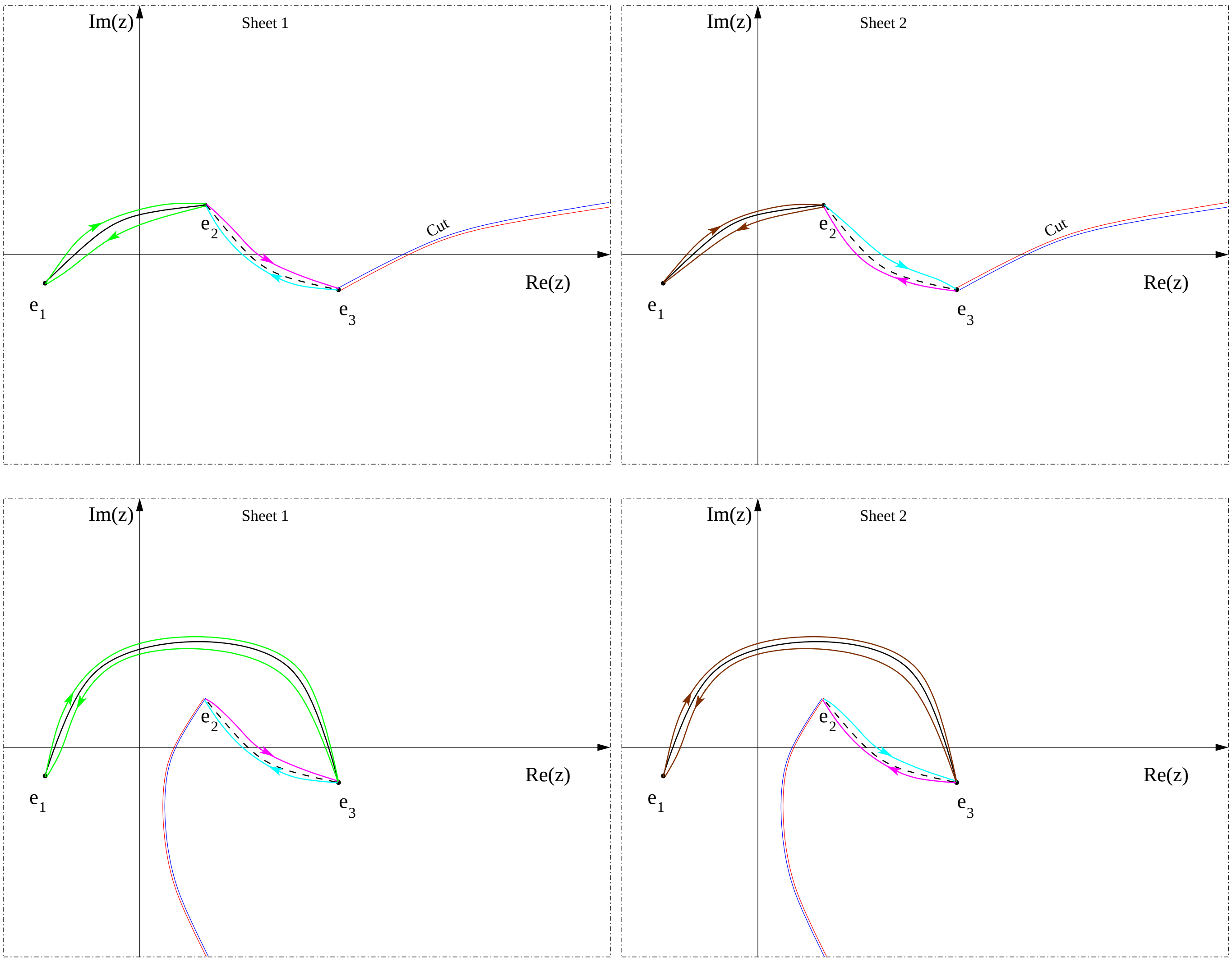

Figure 5.4: Two equivalent but distinct canonical dissections of the same elliptic curve.

and $\widetilde{b}$ cycles would then differ from the previous ones. Looking at Fig. 5.4 we see that the new $\widetilde{a}$-cycle (green) is homotopic to the concatenation of the previous green and purple. It follows that

$$\widetilde{\Omega_1} = \oint_{\widetilde{a}} \mathrm{d}u = \oint_a \mathrm{d}u + \oint_b \mathrm{d}u = \Omega_1 + \Omega_2 \ . \tag{5.8.74}$$

Once a dissection has been made the identification of a and b cycles follows in such a way that the **intersection number** between a and b cycle is $+1$.

8.3 Automorphisms and equivalence of complex tori

We now study the equivalence classes of elliptic curves (modulo the relation of biholomorphic equivalence): since we have proved that any elliptic curve is equivalent to a complex torus, the study is tantamount classifying complex tori. In particular we would like to "parametrize" the collection of all classes of inequivalent complex tori.

By the very definition any complex torus $E(\omega_1, \omega_2)$ is uniquely defined by its periods ω_1 and ω_2: however we cannot exclude that there is another complex torus $E(\eta_1, \eta_2)$ with different periods but biholomorphically equivalent to the first.

Since all the information is encoded in the lattices of the periods, the question to ask is; given two lattices $\Lambda, \tilde{\Lambda}$ of periods, find under what conditions there is a holomorphic and invertible map $f : \mathbb{C} \to \mathbb{C}$ such that $f(\Lambda) = \tilde{\Lambda}$ and

$$\begin{aligned}
&f(z+\omega_1) = f(z) + a\eta_1 + b\eta_2\\
&f(z+\omega_2) = f(z) + c\eta_2 + d\eta_2 \ , \qquad a,b,c,d \in \mathbb{Z}\\
&f^{-1}(w+\eta_1) = f^{-1}(w) + a'\omega_1 + b'\omega_2\\
&f^{-1}(w+\eta_2) = f^{-1}(w) + c'\omega_2 + d'\omega_2 \ , \qquad a',b',c',d' \in \mathbb{Z} \ .
\end{aligned} \tag{5.8.75}$$

We immediately observe that the only candidates for such function f are linear (or affine maps)

$$f(z) = Az + B \ , \tag{5.8.76}$$

for, from the requirements, $f'(z)$ is elliptic (w.r.t. the first lattice) and bounded and hence constant. The constant B can be reabsorbed in a suitable shift of z and hence we are left only with $f = Az$ (and $A \neq 0$ for otherwise the map is not invertible). The next observation is that we can certainly substitute a complex torus $E(\omega_1, \omega_2$ a biholomorphically equivalent one $E(1,\tau)$, with $\tau = \frac{\omega_2}{\omega_1}$ by using $A = \frac{1}{\omega_1}$. Therefore without loss of generality the conditions (5.8.75) can be posed for $\omega_1 = 1, \ \omega_2 = \tau$ and $\eta_1 = 1 \ , \ \ \eta_2 = \tau'$ with both $\tau, \tau' \in \{\Im(w) > 0\}$. Then

Lemma 5.8.76 The matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ which enter (5.8.75) must be inverse of each other and hence have determinant $= \pm 1$. If in addition the periods are ordered so that the modular parameters have both positive imaginary parts then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ . \tag{5.8.77}$$

Exercise 8.5 *Prove the Lemma.*

Proposition 8.4 Two complex tori $E(1,\tau)$ and $E(1,\tau')$, with $\tau, \tau' \in \{\Im(w) > 0\}$ are biholomorphically equivalent iff there is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\tau' = g \cdot \tau := \frac{a\tau + b}{c\tau + d} \tag{5.8.78}$$

Exercise 8.6 *Prove that the group* $\Gamma = SL_2(\mathbb{Z})$ *acts on* $\{\Im(\tau) > 0\}$*, namely that*

$$\tau' = g \cdot \tau := \frac{a\tau + b}{c\tau + d} \tag{5.8.79}$$

has $\Im(\tau') > 0$ *and that this is a group action.*

Exercise 8.7 *Prove the Proposition and find also the constant* A *appearing in the isomorphism.*

9 Modular group and modular forms

The group $SL_2(\mathbb{Z})$ is called the **(full) modular group**. The action of Γ on $\mathcal{H}_+ := \{\Im(\tau) > 0\}$ has a kernel $\{\pm \mathbf{1}\}$. The quotient group is called $PSL_2(\mathbb{Z})$ and the action is then faithful; it is however more practical to study directly the full modular group.

It can be proved that any $g \in \Gamma$ can be obtained by composing the following two generators

$$T: \qquad T \cdot \tau = \tau + 1 \ , \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.9.1}$$

$$S: \qquad S \cdot \tau = -\frac{1}{\tau} \ , \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{5.9.2}$$

The **fundamental domain** for the action namely a set such that any $\tau \in \mathcal{H}_+ := \{\Im(\tau) > 0\}$ has a **unique** $\tau' = g \cdot \tau$ in such set, can be chosen as

$$D_0 := \left\{ \tau : \ |\tau| > 1, \ \Re(\tau) \in [-\frac{1}{2}, \frac{1}{2}), \ \Im(\tau) > 0 \right\} \cup \left\{ |\tau| = 1, \ \Im(\tau) > 0, \Re(\tau) \in [-\frac{1}{2}, 0] \right\} \tag{5.9.3}$$

Exercise 9.1 *Prove that* $\forall \tau \in \mathcal{H}_+$ *there is an element* $g \in SL_2(\mathbb{Z})$ *such that* $g\tau \in \mathcal{D}_0$*. (Use an appropriate sequence of* T *and* S *transformations to place* τ *in the fundamental domain. Draw pictures!).*

The action of Γ is proper-discontinuous only in the extended definition Def. 8.2 and hence we have no guarantee that the quotient topological space is a smooth complex manifold (however we will see that it is).

In particular we have

Proposition 9.1 For τ in the closure of the fundamental domain D_0 we have the following stabilizer subgroups

1. $\Gamma_\tau = \pm\{\mathbf{1}, S\}$ for $\tau = i$;
2. $\Gamma_\tau = \pm\{\mathbf{1}, TS, (TS)^2\}$ if $\tau = \omega := \frac{1}{2} + i\frac{\sqrt{3}}{2}$;
3. $\Gamma_\tau = \pm\{\mathbf{1}, ST, (ST)^2\}$ if $\tau = -\overline{\omega} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$;

whereas for all other points the stabilizer is $\pm\mathbf{1}$.

The proof is just a straightforward computation. Note that –apart from the trivial ±– the stabilizer subgroups of $\omega, -\overline{\omega}$ are the cyclic groups $\mathbb{Z}_3$, whereas for i it is the cyclic group $\mathbb{Z}_2$.

Since $\mathcal{H}_+$ is not complete in the natural topology there is no hope that its quotient is; in order to have a compact topological space after quotienting $\mathcal{H}_+/\Gamma$ we extend $\mathcal{H}_+$ and its topology as follows. First define

$$\overline{\mathcal{H}}_+ := \mathcal{H} \cup \{\infty\} \cup \mathbb{Q} \ . \tag{5.9.4}$$

The points ∞ and $\mathbb{Q}$ are called **cusps**; the modular group Γ permutes the cusps amongst themselves (**exercise**) and hence they constitute a single point in the quotient space $\overline{\mathcal{H}}_+/\Gamma$.

The topology is extended by declaring that a basis of neighborhoods of ∞ is given by the sets $U_\infty := \{\Im(\tau) > C,\ C \in \mathbb{R}_+\} \cup \{\infty\}$. Neighborhoods of the other cusps are defined by transporting these by an appropriate element $g \in \Gamma$ (which maps ∞ to $p/q \in \mathbb{Q}$): the result are small open disks tangent to the real axis at $p/q \in \mathbb{Q}$ (together with the point of tangency). Note that this topology induces a nonstandard topology on $\mathbb{Q}$, for which any rational number does not contain any other rational number in its neighborhoods.

Note that if we map $\mathcal{H}_+$ to the unit disk by

$$\tau \mapsto q := \mathrm{e}^{2i\pi\tau} \tag{5.9.5}$$

and we stipulate that ∞ is mapped to $q = 0$, then a neighborhood of ∞ is just a disk in the q-plane of radius $\mathrm{e}^{-2\pi C}$.

The situation is a bit dangerous at this point since the cusps have infinite stabilizers; for example ∞ is stabilized by all powers of the translation $T : \tau \mapsto \tau + 1$.

However the intersection of any neighborhood of ∞ with the fundamental domain has (obviously) trivial stabilizer (see Figure 5.5). We therefore consider the extended fundamental domain $\overline{D}_0 = D_0 \cup \{\infty\}$

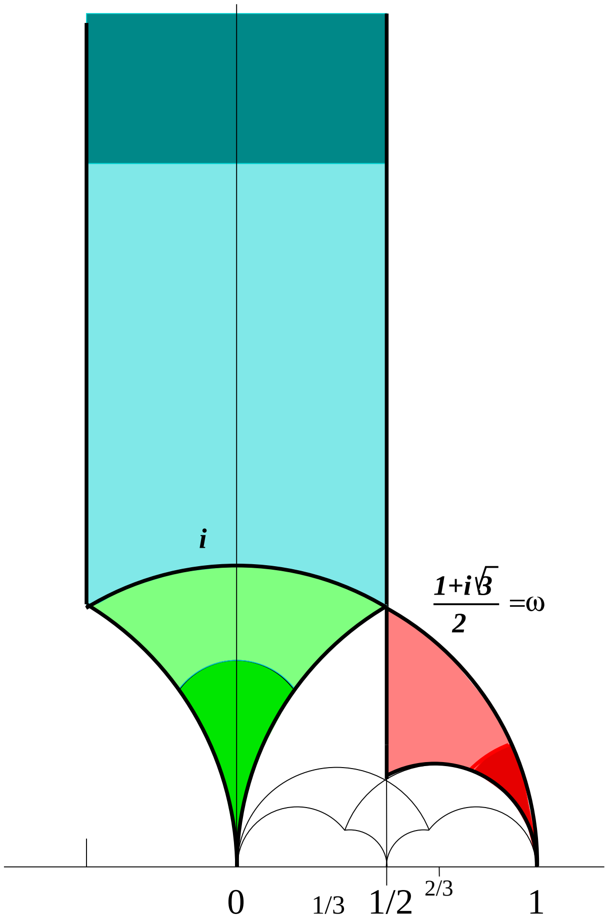

Figure 5.5: The fundamental domain (cyan) with a neighborhood of ∞; also marked are other two fundamental domains with corresponding neighborhoods of the cusp.

where ∞ has been added to the standard fundamental domain and a basis of neighborhoods is the intersection of the previously defined ones with this domain.

The topology defined on $\overline{D}_0$ near ∞ is precisely such that the map $\tau \mapsto q = e^{2i\pi\tau}$ is continuous at the cusp $i\infty$; more than that, it has defined a local coordinate which we use in order to define the analytic structure. We will say that a holomorphic/meromorphic **periodic** function $f(\tau) = f(T\tau) = f(\tau+1)$ has a pole/zero of order/multiplicity m if

$$\tilde{f}(e^{2i\pi\tau}) := f(\tau) \tag{5.9.6}$$

has a Laurent-series expansion of the form

$$\tilde{f}(q) = \sum_{n=\pm m}^{\infty} c_n q^m \ . \tag{5.9.7}$$

Exercise 9.2 *Show that topologically $\mathcal{H}_+/\Gamma$ is homeomorphic to the set $\overline{D_0}/\sim$ where the relation $\sim$ identifies*

$$\tau = -\frac{1}{2} + ir \sim \frac{1}{2} + ir, \tag{5.9.8}$$

$$\tau = e^{i\theta} \sim -e^{-i\theta} \quad \textit{for} \quad \theta \in [\pi/3, \pi/2) \tag{5.9.9}$$

So far we have proved that the equivalence classes of elliptic curves/complex tori are in one-to-one correspondence with the **orbits** of $\mathcal{H}_+$ under the action of $\Gamma = PSL_2(\mathbb{Z})$[3]. In order to define a complex structure on this set we seek a function $J(\tau)$ which has the following properties

1. $J : \mathcal{H}_+ \to \mathbb{C}$ is holomorphic in $\mathcal{H}_+$
2. $J(\tau)$ takes on the same values on points of the same orbit, namely

   $$J\left(\frac{a\tau+b}{c\tau+d}\right) = J(\tau)\ \forall\tau \in \mathcal{H}_+\ ,\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{5.9.10}$$

   In such a way we may think of $\tilde{J} : \mathcal{H}_+/\Gamma \to \mathbb{C}$ so that $\tilde{J}([\tau]) = J(\tau)$.
3. For any two disjoint orbits the values of J are different i.e. J **separates the points** of $\mathcal{H}_+/\Gamma$.

If such a function exists (and in fact it does) then we **define** the complex structure on $\mathcal{H}_+/\Gamma$ by decreeing that $J([\tau])$ is a complex coordinate: note that this implies that we can cover the whole space with only one chart. Before constructing $J(\tau)$ we make an important digression in the following section.

[3]The orbits of a group acting on a manifold are the equivalence classes of points which are connected by a trasformation of the said group.

9.1 Modular functions and forms

Definition 9.1 *A meromorphic function $f(\tau)$ defined on the upper half-plane $\mathcal{H}_+$ is called a* **meromophic modular function of weight** *k (k an integer) if it satisfies the relation*

$$f(g\tau) = (c\tau + d)^k f(\tau) \ , \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ , \tag{5.9.11}$$

and it is meromorphic at infinity in the sense that its Fourier series representation has at most a finite number of negative frequencies

$$f(\tau) = \sum_{n>>-\infty} c_n q^n \ , \quad q = \mathrm{e}^{2i\pi\tau} \ . \tag{5.9.12}$$

If $f(\tau)$ is holomorphic at all points including infinity, we then call it a **modular form** *of weight k and the set of modular forms of weight k will be denoted by $M_k = M_k(\Gamma)$. A modular form is called a* **cusp form** *if it vanishes at the cusp $i\infty$ and their set is denoted by $E_k = E_k(\Gamma)$.*

In the definition we have used the full modular group $\Gamma = SL_2(\mathbb{Z})$ but one could give a similar definition for subgroups Γ' of Γ; in this case one should specify "modular form for Γ'".

In the definition of modular function we have in particular for T and S that

$$f(\tau+1) = f(\tau) \ , \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.9.13}$$

$$f(-1/\tau) = (-\tau)^k f(\tau) \ . \ , \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.9.14}$$

Remark 9.1 Since T, S generate $SL_2(\mathbb{Z})$ in order to verify that a function is modular of weight k it is sufficient (and usually much easier) to verify the two relations (5.9.14): the invariance under T in particular implies that modular functions are periodic functions of period 1 and hence can be expanded in Fourier series of $q = \mathrm{e}^{2i\pi\tau}$. It is also easy to see (using $-\mathbf{1} \in SL_2(\mathbb{Z})$ in eqs. (5.9.14)) that there are no (nontrivial) modular functions of odd weight, and hence we assume from now on that the weight is an even number.

Remark 9.2 Since

$$\frac{\mathrm{d}g\tau}{\mathrm{d}\tau} = (c\tau + d)^{-2} \tag{5.9.15}$$

we can rephrase Def. 9.1 by saying that the tensor $\omega := f(\tau)(\mathrm{d}\tau)^{\frac{k}{2}}$ is invariant under the modular group.

It is obvious that the set of modular functions/forms/cusp forms of a fixed weight is a vector space. Moreover if we multply/divide two modular functions f, h of weights k, j we obtain another modular function of weight $k+j$ or $k-j$, hence the set of modular functions of all weights is a field. In particular we have the subfield of modular functions of weight zero.

We have previously introduced the Eisenstein series in the course of the proof of Lemma 5.8.19 as functions on the space of lattices. Since they are homogeneous in the periods we can always set one of the periods to 1 and the other to $\tau \in \mathcal{H}_+$. We now reintroduce the definition of the Eisenstein series which is more appropriate in the context of modular functions as follows

$$G_{2k}(\tau) := \sum_{n,m}{}^{\prime} \frac{1}{(m\tau+n)^{2k}} , \quad k \geq 2 , \tag{5.9.16}$$

where the prime indicates that the term $m = 0 = n$ is omitted. We have already remarked that it is easy to show that the double series are absolutely convergent and uniformly over any compact set of $\mathcal{H}_+$.

By elementary manipulation of series it follows that

$$G_{2k}(\tau+1) = G_{2k}(\tau) \tag{5.9.17}$$

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau) , \tag{5.9.18}$$

and hence they are modular functions. Moreover they have a well-defined value at $\tau \to i\infty$

$$\lim_{\tau\to i\infty} G_{2k}(\tau) = \lim_{\tau\to i\infty} \sum_{n,m}{}^{\prime} \frac{1}{(m\tau+n)^{2k}} = \sum_{n\neq 0} \frac{1}{n^{2k}} = 2\zeta(2k) \tag{5.9.19}$$

In the literature it is often found a **normalized Eisenstein series**, where the normalization is such that they take value 1 at ∞

$$E_{2k}(\tau) := \frac{1}{2\zeta(2k)} G_{2k}(\tau) . \tag{5.9.20}$$

They have the property (useful in the applications to number theory) that the coefficients in the q expansion are rational

$$E_{2k}(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{5.9.21}$$

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1} \tag{5.9.22}$$

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} , \quad B_k = \text{ Bernoulli numbers.} \tag{5.9.23}$$

Recall that in Weierstrass uniformization we have the functions g_2, g_3 which are multiples of G_4, G_6; in particular a computation shows

$$Y^2 = 4X^3 - g_2 X - g_3 \tag{5.9.24}$$

$$g_2 = 60 G_4 = \frac{4}{3}\pi^4 E_4 , \tag{5.9.25}$$

$$g_3 = 140 G_6 = \frac{8}{27}\pi^6 E_6 . \tag{5.9.26}$$

The discriminant of the elliptic curve becomes

$$\Delta(\tau) = {g_2}^3 - 27{g_3}^2 = \frac{(2\pi)^{12}}{1728}\left({E_4}^3 - {E_6}^2\right) \tag{5.9.27}$$

which shows, –since E_4, E_6 are normalized at $\tau = i\infty$– that Δ is a **cusp form** of weight 12. Summarizing we have shown that

Proposition 9.2 The (normalized) Eisenstein series are modular forms $G_{2k} \in M_{2k}$ and the discriminant Δ is a cusp form $\Delta \in E_{12}(\Gamma)$.

Poles and residues of modular functions

We first prove

Lemma 5.9.27 Let $f(\tau)$ a modular function of weight k and γ be a contour in $\mathcal{H}_+$ avoiding its poles and zeroes: let $g \in SL_2(\mathbb{Z})$, then

$$\int_\gamma \frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} - \int_{g\gamma} \frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} = -k \int_\gamma \frac{\mathrm{d}\tau}{\tau + d/c} \tag{5.9.28}$$

Proof. Differentiating the identity

$$f(g\tau) = (c\tau + d)^k f(\tau) \tag{5.9.29}$$

we obtain

$$f'(g\tau)(c\tau + d)^{-2} = kc(c\tau + d)^{k-1} f(\tau) + (c\tau + d)^k f'(\tau) \tag{5.9.30}$$

$$\frac{f'(g\tau)}{f(g\tau)}\mathrm{d}g\tau = \frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} + k\frac{1}{\tau + d/c} \tag{5.9.31}$$

so that the LHS of the lemma gives

$$\int_\gamma \frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} - \frac{f'(g\tau)}{f(g\tau)}\mathrm{d}g\tau = -k \int_\gamma \frac{\mathrm{d}\tau}{\tau + d/c} \tag{5.9.32}$$

The lemma is proved. Q.E.D.

Proposition 9.3 Let $f(\tau)$ be a nontrivial modular function of weight k for $SL_2(\mathbb{Z})$. let $v_p(f)$ denote the order of the zero (or minus the order of the pole) for $p \in \mathcal{H}_+$. Let $v_\infty(f)$ denote the index of the first nonvanishing term in the q-expansion of f at ∞ (i.e. (minus) the order of the zero/pole at infinity), then

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{i,\omega \neq p \in \mathcal{H}_+/\Gamma} v_p(f) = \frac{k}{12} \tag{5.9.33}$$

Proof. We first apply the residue theorem to count poles/zeroes inside the fundamental domain; it could happen that some (e.g. P, Q in Fig. 5.6) fall exactly on the boundary. In this case some care must be paid so as to avoid counting the other images of the same points that fall also on the boundary (this is the reason of the small bumps in the contour).

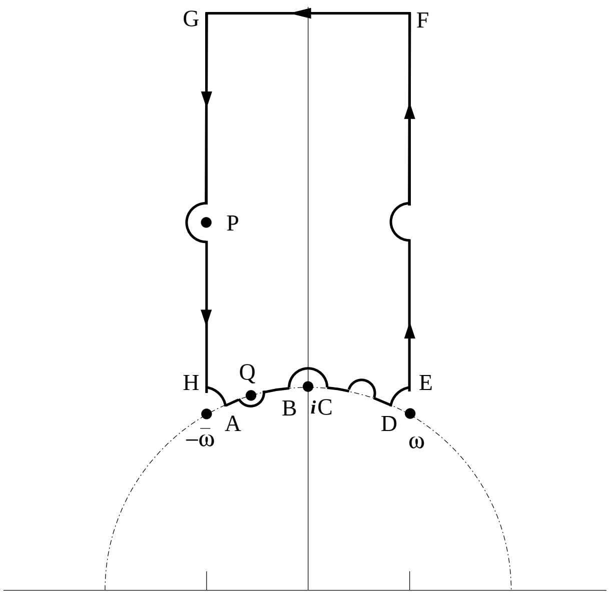

Figure 5.6: The contour used in the proof of Prop. 9.3.

Let γ denote the contour in the figure, where the "ceiling" is high enough so as to include all finite zeroes and poles of $f(\tau)$ that fall in the fundamental domain. Then, by Corollary 2.7.9 we have

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{f'(\tau)\mathrm{d}\tau}{f}(\tau) = \sum_{i,\omega\neq p\in\mathcal{H}_+/\Gamma} v_p(f)\ . \tag{5.9.34}$$

On the other hand we can compute the same integral in a different way. First of all the contribution of the small circular arcs at $\omega, -\overline{\omega}$ and i : since they are $1/6$ and $1/2$ (in the limit of small radii) of a complete circle clockwise around the points, it is easy to see that they compute the corresponding fractions of residues[4] as in the following example

$$\lim_{\epsilon\to 0}\frac{1}{2i\pi}\int_{\substack{|z|=\epsilon\\ \arg(z)\in(\theta_1,\theta_2)}}\frac{f'(z)\mathrm{d}z}{f(z)} = \lim_{\epsilon\to 0}\frac{1}{2i\pi}\int_{\substack{|z|=\epsilon\\ \arg(z)\in(\theta_1,\theta_2)}}\left(\frac{\nu_0(f)}{z}+\mathcal{O}(z)\right)\mathrm{d}z = \tag{5.9.35}$$

$$= \frac{\theta_2-\theta_1}{2\pi} \tag{5.9.36}$$

Hence the three arcs contribute $-1/2$ of $\nu_i(f)$ and $-1/6$ of $\nu(\omega) = \nu(-\overline{\omega})$ where the equality follows from the fact that they are representative of the same orbit and the sign is due to the fact that the integrals are clockwise. Finally the "ceiling" of the contour can be written as

$$\frac{1}{2i\pi}\int_{\frac{1}{2}+iR}^{-\frac{1}{2}+iR}\frac{f'\mathrm{d}\tau}{f} = \frac{1}{2i\pi}\oint_{|q|=\mathrm{e}^{-2\pi R}}\frac{\mathrm{d}f}{\mathrm{d}q}\frac{\mathrm{d}q}{f} = -\nu_\infty(f)\ . \tag{5.9.37}$$

[4]This is true only if the integrand has at most a simple pole at the point, which is the case for us.

The two arcs on the unit circle AB and CD are interchanged by S, whereas GH and FE by T; however the two integrals on GH and FE cancel each other because of the opposite orientation and the fact that the differential $f'\mathrm{d}\tau/f$ is invariant under T. Therefore it only remains to compute the integrals along AB and CD. Noting that $S\cdot(AB) = DC$ (i.e. $S : \tau \mapsto -\frac{1}{\tau}$ inverts the orientation) we have

$$\frac{1}{2i\pi}\int_{AB}\frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} + \frac{1}{2i\pi}\int_{CD}\frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} = \frac{1}{2i\pi}\int_{AB}\frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} - \frac{1}{2i\pi}\int_{S\cdot(AB)}\frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} = \tag{5.9.38}$$

$$= -k\frac{1}{2i\pi}\int_{AB}\frac{\mathrm{d}\tau}{\tau} \to \frac{k}{12} \tag{5.9.39}$$

in the limit of small radii of the arcs HA, BC and DE. Therefore we have proved

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{f'(\tau)\mathrm{d}\tau}{f(\tau)} = -v_\infty(f) - \frac{1}{2}v_i(f) - \frac{1}{3}v_\omega(f) + \frac{k}{12} \tag{5.9.40}$$

and the proof of the proposition follows by rearranging the terms. Q.E.D.

We also have

Proposition 9.4 The modular forms of all weights form a ring with a gradation induced by the weight

$$M(\Gamma) = \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma)\ . \tag{5.9.41}$$

The subspaces of modular forms of fixed weight have the following structure:

1. $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$ (i.e. the only modular forms of weight zero are constants).
2. $M_k(SL_2(\mathbb{Z})) = \{0\}$ if k is negative or odd or $k = 2$.
3. $M_k(SL_2(\mathbb{Z})) = \mathbb{C}\{E_k\}$ (i.e. one-dimensional and generated by an Eisenstein series) for $k = 4, 6, 8, 10, 14$.
4. $S_{12}(SL_2(\mathbb{Z})) = \mathbb{C}\{\Delta\}$.
5. $S_k(SL_2(\mathbb{Z})) = \{0\}$ for $k < 12$ and $k = 14$, and for $k \geq 16$ we have $S_k(SL_2(\mathbb{Z})) = \Delta M_{k-12}(SL_2(\mathbb{Z}))$.
6. $M_k(SL_2(\mathbb{Z})) = S_k(SL_2(\mathbb{Z})) \oplus \mathbb{C}\{E_k\}$ for $k > 12$.

Proof. The fact that modular forms form a graded ring is obvious. For the rest the idea is to use Prop. 9.3 and the fact that a modular form (by definition) has no poles and hence the RHS of (5.9.33) is nonnegative.

(1,2,3) If $k = 0$ in (5.9.33) then there are no zeroes; if $f(\tau)$ is in M_0 then so is $f(\tau) - c$ which has at least one zero if f is nonconstant. We readily get a contradiction.
If $k = 2$ in (5.9.33) there is no way of getting $\frac{1}{6}$ also nin the RHS so there are no such modular forms.
If $k = 4$ we must have $v_\omega(f) = 1$; since E_4 is one such modular form, then E_4 has only a simple zero at

ω. Then f/E_4 is a modular form of weight 0 and hence constant, so any modular form of weight 4 is proportional to E_4. Similarly for E_6 (with a simple zero at i), E_8 (double zero at ω), E_10 (simple zero at i and ω), E_14 (simple zero at i and double zero at ω).

(4) For the case $k = 12$ we have Δ (cusp form) and E_{12}, so the dimension of M_{12} is at least 2; note also that Δ has **only** one zero at infinity. Let $f \in M_{12}$; if $f(\infty) = 0$ then f is proportional to Δ (because f/Δ is a modular form of weight 0): if $f(\infty) = c$ then $f - cE_{12}$ is proportional to Δ. Hence M_{12} is exactly 2-dimensional.

(5) By definition of cusp forms, they must have $v_\infty > 0$ and hence there are none for $k < 12$. For $k = 12$ we have already seen that they are spanned by Δ. If $f \in S_k$ then it has a (at least simple) zero at ∞ and hence f/Δ is a modular form in M_{k-12}.

(6) If $f \in M_k$ then either it is a cusp form or $f(\tau) - f(\infty)E_k(\tau)$ is a cusp form. Q.E.D¿

Theorem 5.9.41 (Ring of modular forms) *The ring of modular forms is a free graded polynomial ring in* E_4, E_6

$$M(\Gamma) = \bigoplus_{\substack{k\in\mathbb{N}\\ k\neq 2}} M_k(\Gamma) = \mathbb{C}[E_4, E_6] \tag{5.9.42}$$

Proof. We need to prove that any modular form can be written as a polynomial in E_4, E_6. This is done by induction on k. There is nothing to prove for $k = 0, 4, 6, 12$ (recall that $\Delta = \frac{(2\pi)^{12}}{1728}\left(E_4{}^3 - E_6{}^2\right)$). Moreover since $M_k = \Delta M_{k-12} \oplus \mathbb{C}E_k$ we need to express E_k as polynomials of E_4, E_6.

For $k = 8$ clearly $E_8 - E_4{}^2$ is a cusp form of weight 8 and hence vanishes (by the previous proposition).

For $k = 10$ we use $E_{10} - E_4E_6$, for $k = 14$ we use $E_{14} - E_4{}^2E_6$.

For $k > 14$, in view of the decomposition $M_k = \Delta M_{k-12} \oplus \mathbb{C}E_k$ and proceeding by induction, we need only to express E_k as a polynomial in E_4, E_6; to this end note that the diophantine equation

$$4a + 6b = k \tag{5.9.43}$$

admits solution for all even $k \geq 10$; if $k \ mod\ 6 = 0$we just take $b = k/6$; if $k \ mod\ 6 = 2$ we take $b = (k-8)/6$ and $a = 2$; if $k \ mod\ 6 = 4$ we take $b = (k-4)/2$ and $a = 1$. Hence $E_k - E_4{}^aE_6{}^b$ is a cusp form of weight k and hence also a polynomial in E_4, E_6. Q.E.D.

Theorem 5.9.43 *The function*

$$J(\tau) = \frac{g_2{}^3(\omega_1, \omega_2)}{g_2{}^3 - 27g_3{}^2} = \frac{E_4{}^3}{E_4{}^3 - E_6{}^2}\ . \tag{5.9.44}$$

is $SL_2(\mathbb{Z})$*–invariant and takes on each complex value* $c \in \mathbb{C}$ *exactly once in the fundamental domain* D_0 *and hence gives a bijection of* $\overline{\mathcal{H}}_+/\Gamma$ *with the Riemann sphere* $\overline{\mathbb{C}}$.

Remark 9.3 For applications to number theory the J function is normalized differently as $j = 1728J$.

Proof Clearly $J - c$ for $c \in \mathbb{C}$ is a modular function of weight zero, hence invariant under $SL_2(\mathbb{Z})$. Since it has a simple pole at ∞ it must have either (depending on c) a double zero at i, a triple zero at ω or a simple zero at some other point of D_0. This proves that it takes any value exactly once and hence gives a bijection with $\mathbb{C}$ (and with $\overline{\mathbb{C}}$ if we throw in $\tau = i\infty$). Q.E.D.

Remark 9.4 More precisely, since $E_4(\omega) = 0$ (see proof. of Prop. 9.4) J has a triple zero at ω. Moreover

$$\frac{{E_4}^3}{{E_4}^3 - {E_6}^2} = 1 + \frac{{E_6}^2}{{E_4}^3 - {E_6}^2} \tag{5.9.45}$$

and since we have discussed that $E_6(i) = 0$ then $J(\tau) - 1$ has a double zero at $\tau = i$. In other words $J(\omega) = 0$, $J(i) = 1$ (and clearly $J(\infty) = \infty$).

We can finally put together the theory of the moduli space of complex tori=elliptic curves with the just developed theory of modular functions by saying that the moduli space of elliptic curves can be given a structure of smooth compact complex manifold equivalent to the Riemann sphere.

The theory of moduli for other Riemann surfaces is much more complicated and invariably leads to objects that are not smooth manifolds but have some sort of singularities.

10 Classification of complex one-dimensional manifolds

Exercise 10.1 *Prove that $\mathbb{C}$ and the unit disk $D_0(1)$ are not biholomorphically equivalent. [Hint: use Liouville's theorem]*

More generally we have

Theorem 5.10.0 (Classification of simply connected curves) *Let M be a one-dimensional complex manifold, connected and simply connected. Then it is biholomorphically equivalent to one of the following*

1. *The Riemann–Sphere $\overline{C}$ (see later)*
2. *The complex plane $\mathbb{C}$*
3. *The unit disk $D_0(1) := \{z : \ |z| < 1\}$*

The classification of non simply–connected one dimensional complex manifolds is reduced to the study of proper-discontinuous group actions on simply connected manifolds. Indeed we have seen that for any non simply connected manifold M its universal covering $\tilde{M}$ is acted upon properly-discontinously by the deck transformations. It can be proved for instance that if $\pi_1(M) = \mathbb{Z} \times \mathbb{Z}$ then it is an elliptic curve of the kind seen before (for some τ).

11 Algebraic functions and algebraic curves

Definition 11.1 *A function $f(z)$ defined on a domain $\mathcal{D}$ is called* **algebraic** *if there exists a polynomial function $P(w, z)$ such that*

$$P(f(z), z) \equiv 0, z \in \mathcal{D}. \tag{5.11.1}$$

The locus

$$\mathcal{C} := \{(w, z) \in \mathbb{C}^2 : \ P(w, z) = 0\} \tag{5.11.2}$$

is called an **algebraic curve***.*

Sometimes it is useful to consider a rational function $R(X, Y)$ instead of a polynomial and the definition requires a certain specification so as to "avoid" the zeroes of the denominator.

The second remark is that if $P(f(z), z) \equiv 0$ in $\mathcal{D}$ then so must be for **any analytic continuation** of f along any path: indeed if $\tilde{f}$ is the analytic continuation of f then the analytic continuation of $P(f(z), z)$ is $P(\tilde{f}(z), z)$ and since it is the continuation of the zero function it must be identically zero.

We now prove that a polynomial equation $P(w, z) = 0$ of degree n in w defines locally n (germs) of analytic functions. More precisely

Proposition 11.1 Given the algebraic equation $P(w, z) = 0$ with

$$P(w, z) = A_n(z)w^n + \ldots + A_0(z) \ , A_n(z) \not\equiv 0 \ , \tag{5.11.3}$$

and a point $(w_0, z_0) \in \mathbb{C}^2$ such that $\partial_w P\Big|_{(w_0,z_0)} \neq 0$ then there is a germ of analytic function $f(z) = w_0 + \sum_{n\geq 1} c_n(z - z_0)^n$ which satisfies the functional equation.

Sketch of proof. We regard the function $P(w, z) : \mathbb{C}^2 \to \mathbb{C}$ as a $\mathcal{C}^\infty$ function $\tilde{P} : \mathbb{R}^4 \to \mathbb{R}^2$. Then the condition $P_w \neq 0$ at (w_0, z_0) guarantees that the rank of the Jacobian of the function $\tilde{P} : \mathbb{R}^4 \to \mathbb{R}^2$ is maximal and can be solved locally for $\Re(w), \Im(w)$ yielding differentiable (actually infinitely differentiable) functions of $\Re(z), \Im(z)$. Then one has to check that these functions satisfy also Cauchy–Riemann equations. Q.E.D.

From now on we assume (to avoid trivial occurrences) that $P(w, z)$ is **irreducible** namely (def) cannot be written as the product of two non-constant polynomials. For simplicity in the discussions below we may also require that the **discriminant** of $P(w, z)$ (viewed as a polynomial in w with parameter z) is not the identically zero function of z. We recall the definition of the discriminant

Definition 11.2 *Given a (monic) polynomal $P(X) = a_N X^N + \ldots$ of degree N in the indeterminate X, and $X_1, \ldots, X_N$ its roots (possibly repeated) the* **discriminant** *is the following expression*

$$\Delta := {a_N}^{2N-1} \prod_{i<j} (X_i - X_j)^2 \ . \tag{5.11.4}$$

In particular (by definition) the discriminant is zero iff the polynomial has repeated roots.

It can be proved that (example with $N = 4$)

$$\Delta(P) = \det \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ a_1 & 2a_2 & 3a_3 & 4a_4 & 0 & 0 & 0 \\ 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & 0 & 0 \\ 0 & 0 & a_1 & 2a_2 & 3a_3 & 4a_4 & 0 \\ 0 & 0 & 0 & a_1 & 2a_2 & 3a_3 & 4a_4 \end{pmatrix} \tag{5.11.5}$$

from which it is clear that although the definition involves the roots of the polynomial, it is actually a polynomial of the coefficients a_j. So the discriminant of $P(w, z)$ in (5.11.3) is a polynomial in $A_j(z)$ and hence a polynomial in z.

Suppose now $P(w, z)$ is irreducible and z_0 is a point where $A_n(z)\Delta_{(w)}(P)$ is nonzero (and hence nonzero in a neighborhood U_0 of z_0. As a polynomial of w depending parametrically on z, $P(\bullet, z) = 0$ has precisely $n = \deg_w P$ distinct roots $w_k(z)$; furthermore we have

$$P_w(w_k(z), z) \neq 0 \tag{5.11.6}$$

since the roots are distinct (**exercise**). Then Prop. 11.1 guarantees that each $w_k(z)$ is analytic in U_0 and hence we have defined n analytic elements (and germs of analytic functions).

To simplify the discussion we assume from now on that the leading coefficient is a constant (and hence we can assume it to be unity).

$$P(w, z) = w^n + a_1(z)w^{n-1} + \dots \ . \tag{5.11.7}$$

It should be clear that this procedure as described fails in a neighborhood of a point z_0 such that $A_n(z)\Delta(z) = 0$. We denote by $B := \{z_n\}$ the *finite* set of these points (called **ramification points**). Consider a neighborhood U_0 of a non-ramification point as earlier and $f(z) = w_1(z)$ one of the defined germs (or any other solution); the question arises as to what is the largest domain where $f(z)$ admits unrestricted analytic continuation.

The question is addressed by

Theorem 5.11.7 *The analytic elements $(U_0, w_k(z))$ can be analytically continued along any path in the punctured plane $\dot{\mathbb{C}} := \mathbb{C} \setminus B$.*

Proof. Suppose this is not the case: then there exist $a \in \dot{C}$ and a path γ from z_0 to a such that $w_k(z)$ admit continuation to any point $z' \in \gamma$ except a itself. However we know that we can find n germs of analytic functions $\tilde{w}_k(z)$ at a which satisfy the equation (5.11.3) with some common positive radius of convergence $R_{\min}$. Choose a point $c \in \gamma \setminus \{a\}$ inside the disk $D_a(R_{\min})$. Then we have there the n analytic continuations of $w_k(z)$ which we call $[w_k^{(\gamma)}]_c$.Since they must be distinct, they exhaust the solution of $P(w, z) = 0$ and hence must coincide –up to permutation of the indices– with the germs $[\tilde{w}_{k'}]_c$. That means that $\tilde{w}_{k'}$ defines an analytic continuation of w_k along γ to a. Q.E.D.

Since $\dot{\mathbb{C}}$ is not simply connected we do not expect that by continuation of $w_k(z)$ we obtain a single-valued function.

Example 11.1 *Consider* $P(w,z) = w^2 - z$ *and* $z_0 = 1$*. The only branchpoint is* $z = 0$ *so that now* $\dot{\mathbb{C}} = C^\times$*. In a neighborhood of* $z = 1$ *we have two analytic functions* $w_1(z), w_2(z)$ *such that* $w_1(1) = 1 = -w_2(1)$*. Their analytic expression is*

$$w_i(z) = \sqrt{|z|}\mathrm{e}^{i\frac{\arg(z)}{2}} , \qquad \arg z \in (-\pi, \pi), |z| > 0 . \tag{5.11.8}$$

If we analytically continue -say- w_1 *around the origin once counterclockwise and follow the argument of* z *continuously along the chosen path, we obtain*

$$w_1^{(\gamma)} = w_2 = -w_1. \tag{5.11.9}$$

It is unsatisfactory to have to introduce *multivalued functions* (in a certain sense a contradiction in terms): to avoid this awkward situation we resolve the ambiguity by defining the function w_k not as a function of z but as a function on the locus $P(w,z) = 0$ itself! In a certain sense we define the analytic continuations of w_k as functions on its own graph.

11.1 Manifold structure on the locus $P(w,z) = 0$

We continue our consideration of the locus of point $(w,z) \in \mathbb{C}^2$ such that $P(w,z) = 0$. We wish to introduce a manifold structure on it. We restrict to the so–called smooth curves

Definition 11.3 *The set*

$$\mathcal{L} := \{(w,z) \in \mathbb{C}^2 \ : \ P(w,z) = 0\} \tag{5.11.10}$$

is called a **plane algrbraic curve**. *We say that it is* **non-singular** *if the two complex partial derivatives* $P_w(w,z)$ *and* $P_z(w,z)$ *never vanish at the same point* $(w,z) \in \mathcal{L}$.

On non-singular curves we define the local coordinates as follows:

1. In a neighborhood U of a point (w_0, z_0) where $P_w(w_0,z_0) \neq 0$ we know from Prop. 11.1) that there is a unique holomorphic function $w(z)$ on a suitably small disk $D_{z_0}(\epsilon) : \{|z - z_0| < \epsilon\}$ which satisfies identically the equation $P(w(z), z) = 0$. In this disk we use z as coordinate.

2. In a neighborhood U of a point (w_0, z_0) where $P_z(w_0, z_0) \neq 0$ then, by the same arguments as before with interchange of the rôles, we use w as local coordinate.

In a neighborhood of a point where both derivatives P_w, P_z do not vanish, then we can use either w or z as coordinate. The derivatives are computed from the (complex) implicit differentiation theorem

$$\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{P_z(w,z)}{P_w(w,z)} , \qquad \frac{\mathrm{d}z}{\mathrm{d}w} = -\frac{P_w(w,z)}{P_z(w,z)} . \tag{5.11.11}$$

On the plane curve $\mathcal{L}$ we have the two functions $g(w,z) = w$ and $f(w,z) = z$ which are clearly holomorphic. Consider

$$f : \mathcal{L} \to \mathbb{C}(w,z) \mapsto z \tag{5.11.12}$$

Its ramification points are where the point z_j such that

$$P(w,z_j) = 0P_w(w,z_j) = 0\ . \tag{5.11.13}$$

and these are precisely the zeroes of the discriminant of P w.r.t. w.

11.2 Surgery

In this section we consider a non-singular algebraic curve $P(w,z) = 0$. We have established that there are a finite number of branchpoints $B = \{z_k\}$ in a neighborhood of each of which z cannot be used as a local coordinate. The domain $\mathbb{C} \setminus B$ is not simply connected and hence the analytic continuations of the roots $w_k(z)$ do not define holomorphic functions. We now make some "cuts" do $\mathbb{C} \setminus B$ so as to make it simply connected. From each $z_k \in B$ we choose a simple (i.e. non self-intersecting) path $\gamma_j \subset \mathbb{C} \setminus B$ going to infinity and we do so that different paths never intersect. We call $\mathcal{D}$ the domain obtained by removing these paths (the **cuts**) from $\mathbb{C}$. We claim that $\mathcal{D}$ is simply connected (we do not give a formal proof, but a simple drawing should suffice to convince you). The choice of cuts is largely irrelevant and we can choose them to be straight half-lines. Given a $z_0 \in \mathcal{D}$ and the n germs of analytic functions $w_k(z)$, they can be analytically continued to $\mathcal{D}$ to *bona-fide* holomorphic functions.

Consider now n copies of the same domain $\mathcal{D}$: we call them $\mathcal{D}_k$, $k = 1..n$ and choose (arbitrarily but once and for all) to assign to each of them one of the w_k (for instance with the same number).

The boundary of each sheet consists of (copies of) the n cuts: these cuts have a *plus* side and a *minus* side which is defined by their orientation (from the branchpoint to infinity). Imagine that each side of each cut consists of different points (we could be more rigorous but it would be hiding the simplicity of the idea) which we call the **±-edges**.

Exercise 11.1 *Let $\mathcal{D}$ as above: give a rigorous definition of the two edges of a cut.*

We now **glue** the edges of the cuts of different sheets as per the following rule. Consider the +-edge of the j-th cut of the α-sheet $\mathcal{D}_\alpha$: if we analytically continue w_α along a path that crosses the j-th cut in the direction from the $+$ to the $-$ side we obtain a $w_{\alpha'}$ where α' is not necessarily the same as α. We then identify all the points of the $+$edge of the j-th cut of $\mathcal{D}_\alpha$ with the points of the $-$edge of the same j-th cut of $\mathcal{D}'_\alpha$. We continue until each edge of each cut of each sheet has been identified with the other edge of the same cut on a (possibly) different sheet.

The ensuing topological space (actually manifold) is biholomorphic to the punctured plane curve and it can be showed that it is connected (under the hypothesis that $P(w,z)$ is irreducible).

Exercise 11.2 *Consider*

$$P(w,z) = (w - z^2)(w - 2z) \ . \tag{5.11.14}$$

Show that the construction above results in a non-connected topological space (P is not irreducible).

Instead of continuing within this level of abstraction, we take an instructive (and famous) class of plane curves.

11.3 Hyperelliptic curves

By definition they are plane curves of the form

$$w^2 = P(z) \ , \qquad P(z) = c\prod_{j=1}^{n}(z - z_j) \ , c \neq 0 \tag{5.11.15}$$

We immediately assume $c = 1$ by possibly rescaling w.

Exercise 11.3 *This curve is nonsingular iff the z_j's are distinct.*

In this case there are only two sheets

$$w_\pm = \pm\sqrt{P(z)} \tag{5.11.16}$$

where the square root is defined on the simply connected domain obtained as described before. By analytic continuation then we have well–defined analytic functions on the domains $\mathcal{D}_\pm$. We claim (and this can be suitably generalized) that we can choose a different set of cuts in $\dot{\mathbb{C}} := \mathbb{C} \setminus \{z_1, \ldots, z_n\}$ where the new domain $\widetilde{\mathcal{D}}$ is **not simply–connected** but nevertheless the analytic continuation of $w_\pm$ gives bona-fide holomorphic functions.

Choose arcs of curves joining $[z_{2j-1}, z_{2j}]$ and –if n is odd– the last z_n to infinity in such a way that these arcs are simple and mutually avoiding. Take $\widetilde{\mathcal{D}}$ to be the plane $\mathbb{C}$ less these cuts.

We claim that

Theorem 5.11.16 *Given $z_0 \in \widetilde{\mathcal{D}}$ and $w_\pm$ the two germs of analytic functions defined at z_0 by the two square-roots of $P(z)$. Then they can be analytically continued to $\widetilde{\mathcal{D}}$ to holomorphic functions.*

Proof (sketch). Take a closed loop based at z_0 that encircles only one of the cuts say $[z_1, z_2]$; this contour must intersect the original cuts originating from z_1 and z_2. The analytic continuation "changes sign" twice and hence is analytically continued to the same function. Q.E.D.

We define a **compactification** of these plane curves; we distinguish the case of n even or odd and use the model given by the dissection $\widetilde{\mathcal{D}}$.

Even n. In this case no branchcut extends to ∞ in either sheet. We compactify $\mathcal{L}$ by adding two points $\infty_\pm$ to $\mathcal{D}_\pm$ with local coordinates $\zeta = \frac{1}{z}$.

Odd $\mathbf{n = 2k+1}$. There is a branchcut extending to ∞ on both sheet. We compactify $\mathcal{L}$ by adding one points denoted by ∞ with the following local description: define $\eta = \frac{z^k}{w}$ and $\zeta = \frac{1}{z}$, then

$$\eta^2 = \frac{1}{\zeta^{2k} P(\frac{1}{\zeta})} = \frac{\zeta}{\zeta^n P(\frac{1}{\zeta})} = \frac{\zeta}{1 + \mathcal{O}(\zeta)} \tag{5.11.17}$$

The local coordinate can be chosen to be η near $\eta = 0, \zeta = 0$.

We will denote by $\overline{\mathcal{L}}$ the compactifications thus defined.

One has the following result

Theorem 5.11.17 *The meromorphic functions on $\overline{\mathcal{L}}$ are all functions of the form*

$$F = R_0(z) + yR_1(z) \tag{5.11.18}$$

with R_0, R_1 rational functions of z.

[No proof]

We then look at differentials.

Proposition 11.2 The holomorphic differentials are linear combinations of

$$\omega_j := \frac{z^j \mathrm{d}z}{w}, \ , j \leq [(n-1)/2] - 1 \tag{5.11.19}$$

where $[x]$ denotes the integral part of x.

Proof. A holomorphic differential must be such that on the $\pm$ sheets we have

$$g_\pm(z)\mathrm{d}z \tag{5.11.20}$$

where the two functions $g_\pm$ are such that the boundary values on the left of a cut of g_+ is the same as the boundary value on the right (at the same point) for g_- and they are otherwise holomorphic functions of z. Given this property, we can define

$$e_+ := g_+ + g_- \ , \qquad f_+ := g_+ - g_- \tag{5.11.21}$$

Then the analytic continuation of f_+ is $f_- = -f_+$ and $f_\pm(z) \in \mathcal{H}(\mathcal{D}_\pm)$. Similarly for e_+, which however continues to $e_- = e_+$. We first prove that $e \equiv 0$. Indeed, since the analytic continuation of e can be done to the whole $\dot{\mathbb{C}} = \mathbb{C} \setminus \{z_1, \ldots, z_n\}$, then it may have at the punctures at worst some isolated singularities. In the local coordinate near a branchpoint we find that $e(z) = \mathcal{O}(\frac{1}{\sqrt{z - z_i}})$. Since it must admit a Laurent expansion (in integer powers of $z - z_i$) we conclude that $e(z)$ must be holomorphic at z_i's: therefore $e(z)$ is entire. At infinity we find that $e(z) = \mathcal{O}(z^{-2})$ (after requiring holomorphicity) and hence $e(z)$ vanishes identically.

For $f_\pm$ (which actually -as we have just proved- coincides with $g_\pm$), it is then seen that

$$h(z) := f_+(z)w_+(z) = f_-(z)w_-(z) \tag{5.11.22}$$

can be uniquely and analytically continued to the whole $\dot{\mathbb{C}} = \mathbb{C} \setminus \{z_1, \ldots, z_n\}$. At the branchpoints may have still some isolated singularity. In the neighborhood of a branchpoint z_j we have stipulated that the local parameter is w and

$$f_\pm \mathrm{d}z = \frac{f_\pm 2w_\pm}{P'(z)}\mathrm{d}w = \frac{h(z)}{P'(z)}\mathrm{d}w \tag{5.11.23}$$

where in the last expression we should think of z as a function of w, for

$$w^2 = (z - z_j)Q(z) \ , \qquad Q(z_j) \neq 0 \ \Rightarrow z(w) = z_j + w^2 g(w) \tag{5.11.24}$$

Since the curve is non-singular $P'(z_j) \neq 0$ (P has only simple roots) and hence $h(z)$ must have a removable singularity at each branchpoint.

We next analyze the behaviour at ∞. For even n we must have that $f_\pm(z)z^2$ are bounded in a neighborhood of $z = \infty$, namely

$$\frac{h(z)z^2}{w_\pm(z)} = \frac{h(z)z^2}{z^{n/2}(\pm 1 + \mathcal{O}(1/z))} \tag{5.11.25}$$

and therefore $h(z)z^{2-n/2}$ is bounded at infinity so that $h(z)$ must be a polynomial in z of degree at most $n/2 - 2 = [(n-1)/2] - 1$.

For $n = 2k+1$ odd the local parameter is $\eta = \frac{z^k}{w}$ so that

$$\mathrm{d}z = -\frac{1}{\zeta^2}\mathrm{d}\zeta = -\frac{2}{\eta^3}\mathrm{d}\eta(1 + \mathcal{O}(\eta)) \tag{5.11.26}$$

so that the requirement of holomorphicity is that

$$\left|\frac{h(z)}{\eta^3 w_\pm(z)}\right| \sim \left|\frac{h(z)z^{\frac{3}{2}}}{z^{k+\frac{1}{2}}}\right| = \left|h(z)z^{1-k}\right| \tag{5.11.27}$$

is bounded as $|z| \to \infty$. It then follows that $h(z)$ is a polynomial of degree at most $k - 1 = [(n-1)/2] - 1$. Q.E.D

Definition 11.4 *Given a one-dimensional* connected and compact *manifold M the dimension of the vector space of differential of the first kind is called the* **genus** *of the complex curve.*

In view of this definition we have just computed the genus of hyperelliptic curves.

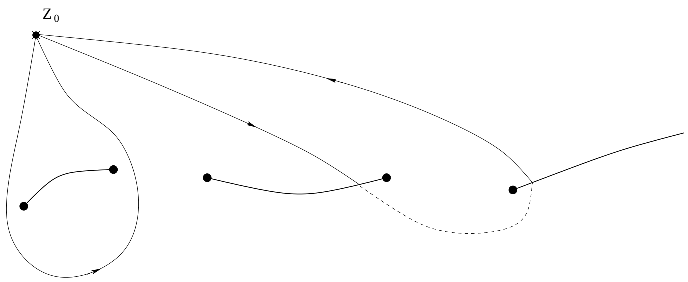

Figure 5.7: The domain $\widetilde{\mathcal{D}}$ and two closed loops on the Riemann surface (the dotted part belongs to the other sheet).

11.4 Abelian Integrals

The name of Abel is for historical reasons (sometimes the 1^{st} kind differentials are called Abelian differentials). They can be defined also for meromorphic differentials but here we limit ourselves to holomorphic (first kind) differentials.

We fix a base-point (w_0, z_0) on our algebraic (compact) surface and in a small schlicht neighborhood of this point we consider the following functions

$$u_j(z) := \int_{z_0}^{z} \frac{\zeta^j \mathrm{d}\zeta}{\sqrt{P(\zeta)}} \ , \qquad j \leq [(n-1)/2] - 1 \ , \tag{5.11.28}$$

where $\sqrt{P(\zeta)}$ is defined in the chosen simply-connected neighborhood by analytic continuation $w_0 = \sqrt{P(z_0)}$. We choose the domain of analyticity of $w_\pm(z) = \pm\sqrt{P(z)}$ to be the cut domain $\tilde{\mathcal{D}}$ (the non simply connected one). The functions $u_j(z)$ are called **Abelian integrals** (of the first kind). The germ of analytic functions defined by them at z_0 can be analytically continued as usual and admits unrestricted domain of continuation the whole punctured plane $\dot{\mathbb{C}} = \mathbb{C} \setminus \{z_1, \ldots, z_n\}$. It is clear that there is a multivaluedness due to the integrand but that is "cured" by putting the appropriate cuts so as to obtain $\widetilde{\mathcal{D}}$. There remains however a further multivaluedness: since $\widetilde{\mathcal{D}}$ is non simply-connected, the functions u_j may fail to extend to an analytic function, and in fact it does/ Note that

$$\frac{\mathrm{d}u_j(z)}{\mathrm{d}z} = \frac{z^j}{w_\pm(z)} \ , \tag{5.11.29}$$

and hence any analytic continuation of u_j must satisfy the (continuation of the) same differential equation. It follows that the analytic continuation $\widetilde{u_j}$ of u_j along a loop in $\widetilde{\mathcal{D}}$ must differ by a **constant** which depends on the homotopy class of γ.

$$\widetilde{u_j}(z) - u_j(z) = C_\gamma \tag{5.11.30}$$

This is valid even if the contour γ is not entirely contained in $\widetilde{\mathcal{D}}$ but $w(z)$ can be analytically continued along it yielding the same analytic element: for example if the contour is a simple closed loop encircling an even number of branchpoints (see Fig. 5.7).

Definition 11.5 *The constants C_γ defined in (5.11.30) are called the* **periods** *of the Abelian integrals.*

We have

Proposition 11.3 Any period C_γ is a linear combination with integer coefficients of the **fundamental periods**

$$\Omega_{k,j} := 2\int_{z_j}^{z_{j+1}} \frac{z^k \mathrm{d}z}{w_+(z)} = \oint_{\gamma_j} \omega_k \ , \quad j = 1 \dots 2[(n-1)/2] - 1, \quad k = 0, \dots, [(n-1)/2] - 1 \tag{5.11.31}$$

where γ_j is a simple closed loop encircling the two branchpoints z_j, z_{j+1} (note that te number of such **cycles** is exactly twice the genus of the surfave).

The rectangular matrix $\Omega \in Mat(g \times 2g, \mathbb{C})$ is called the (unnormalized) **matrix of periods** and plays an essential rôle in the theory of **Theta functions** (topic for another course!).

The case of **elliptic** curve is a special case of these hyper-elliptic ones.

11.5 Symplectic basis in the homology

The definition of homology is beyond the scope of this course: for us here it is sufficient to say that the (first) homology group is the abelian part of the homotopy group. It is a free abelian group on $2g$ (g is the genus) generators and it is usually denoted additively.

The rationale is the following: if ω is a holomorphic differential, then the map

$$I_\omega : \pi_1 \to \mathbb{C} \tag{5.11.32}$$

$$[\gamma] \to I([\gamma]) := \int_\gamma \omega \tag{5.11.33}$$

is a group homomorphism between π_1 and $\mathbb{C}$ (with addition). Namely

$$I([\gamma] \cdot [\gamma']) = I[\gamma] + I[\gamma'] \tag{5.11.34}$$

Clearly it vanishes on $[\pi_1, \pi_1]$ since

$$I([\gamma_1] \cdot [\gamma_2] \cdot [\gamma_1]^{-1} \cdot [\gamma_2]^{-1}) = I([\gamma_1]) + I([\gamma_2]) - I([\gamma_1]) - I([\gamma_2]) = 0. \tag{5.11.35}$$

Therefore it can be convenienetly thought of as a gruoup homomorphism from $H_1 := \pi_1/[\pi_1, \pi_1]$ to $\mathbb{C}$. Elements of H_1 are referred to as "cycles".

The group H_1 is -as well as π_1– generated by $2g$ generators. A convenient choice of generators is to choose g "a"-cycles and g "b"-cycles in such a way as to make the representation of the intersection number symplectic in this basis

$$a_i \sharp a_j = 0 = b_i \sharp b_j \ , \qquad a_i \sharp b_j = \delta_{ij} = -b_j \sharp a_i \tag{5.11.36}$$

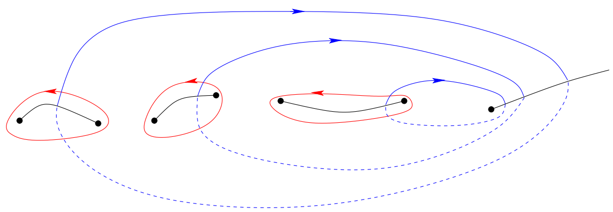

Figure 5.8: A canonical choice of symplectic basis in the homology of a hyperelliptic curve: the red cycles are the a-cycles, the blue are the b-cycles. In this picture the hyperelliptic curve has equation $w^2 = P_7(z)$.

The matrix of periods for some choice of g linearly independent holomorphic differentials $du_1, \ldots, du_g$ then splits in two $g \times g$ blocks

$$\mathbb{A}_{ij} := \oint_{a_i} du_j \ , \qquad \mathbb{B}_{ij} := \oint_{b_i} du_j \tag{5.11.37}$$

For a hyperelliptic curve the usual choice is as in figure 5.8. We quote some facts

Fact 1

Both matrices $\mathbb{A}$ and $\mathbb{B}$ are invertible.

Fact 2a

If $\{a_j, b_j\}$ is a set of generators chosen as described above then the matrix $\boldsymbol{\tau} := \mathbb{A}^{-1}\mathbb{B}$ has positive definite imaginary part, $\Im(\boldsymbol{\tau}) > 0$.

Fact 2b

If we choose an equivalent basis for holomorphic differentials $\omega_i := \sum_{j=1}^{g} \mathbb{A}^{-1}_{ij} du_j$ then we have

$$\oint_{a_j} \omega_k = \delta_{jk} \ , \qquad \oint_{b_j} \omega_k = \tau_{jk}. \tag{5.11.38}$$

These holomorphic differentials are called **normalized Abelian differentials**. The normalization depends on the choice of the symplectic basis of cycles a_j, b_j.

Chapter 6

Harmonic functions

1 Harmonic conjugate

Definition 1.1 *Let $u(x,y) : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^2(\mathcal{D})$ function (the second partial derivatives exist and are continuous). The function is called* **harmonic** *if it satisfies* **Laplace's equation**

$$\nabla^2 u := \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0 \ . \tag{6.1.1}$$

By identification $z = x+iy$ we will write $u(z)$ although (clearly) f cannot be holomorphic; more properly we will write $u(z,\overline{z})$ if required for clarity. Recalling the definition of the differential operators $\partial_z, \partial_{\overline{z}}$ we can rewrite Laplace's equation as

$$\partial_z \partial_{\overline{z}} u(z,\overline{z}) = 0 \tag{6.1.2}$$

Consider now a harmonic function $u(z)$ and consider the one-form

$$\omega := u_y \mathrm{d}x - u_x \mathrm{d}y \ . \tag{6.1.3}$$

This differential is **closed** because of harmonicity of $u(x,y)$

$$\partial_y(u_y) = -\partial_x(u_x) \ \Leftrightarrow \ \nabla^2 u = 0 \tag{6.1.4}$$

Therefore in any simply-connected subregion of $\mathcal{D}$ we can construct a $\mathcal{C}^2$ function v such that $\mathrm{d}v = \omega$: explicitly this means that there is a $\mathcal{C}^2$ function $v(x,y)$ such that

$$v_x = u_y \ , \qquad v_y = -u_x \tag{6.1.5}$$

Definition 1.2 *The function v is called the* **harmonic conjugate** *of the harmonic function u and it is also harmonic where defined.*

Explicitly, the formula for v is

$$v(z) = \int_{z_0}^{z} \left(u_y \mathrm{d}x - u_x \mathrm{d}y \right) \tag{6.1.6}$$

where the integral is independent of the path in the same homotopy class. If z is in a simply connected neighborhood of z_0 then the definition is unambiguous.

Suppose now that $\mathcal{D}$ itself is simply connected. Then v is defined and harmonic throughout $\mathcal{D}$. The equations defining v are precisely the Cauchy-Riemann equations for the complex–valued function

$$f(z) = u(x, y) + iv(x, y) \ , \tag{6.1.7}$$

which is then holomorphic. In particular this implies that a harmonic function is in fact infinitely differentiable (since so is $f(z)$). Viceversa, given a holomorphic function $f \in \mathcal{H}(\mathcal{D})$ (whether $\mathcal{D}$ is simply connected or not), then its real and imaginary parts define harmonic functions for the CR equations give

$$f = u + iv \tag{6.1.8}$$

$$\begin{cases} u_x = -v_y \\ u_y = v_x \end{cases} \quad \Rightarrow \quad \begin{cases} u_{xx} = -v_{xy} = -u_{yy} \\ v_{xx} = u_{yx} = -v_{yy} \end{cases} \tag{6.1.9}$$

and v is the harmonic conjugate of u. We have proved

Proposition 1.1 Given a harmonic function u on a domain $\mathcal{D}$ then for each simply connected subdomain $\mathcal{D}_{sc} \subset \mathcal{D}$ there is a holomorphic function f whose real part coincides with u. Such holomorphic function is uniquely defined up to addition of a purely imaginary constant.

Put it differently, any harmonic function defines at any point a germ of analytic function (modulo additive imaginary constants) which admits **unrestricted analytic continuation** to $\mathcal{D}$. The obstruction to the existence of a harmonic conjugate is the same as the obstruction to the analytic continuation being a holomorphic (single-valued) function.

2 Mean and maximum value theorems

Harmonic functions share many properties of holomorphic functions (clearly!)

Theorem 6.2.0 *Let $u : \mathcal{D} \to \mathbb{R}$ be harmonic and $\overline{D_a(r)} \subset \mathcal{D}$. Then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + r\mathrm{e}^{i\theta}\right) \mathrm{d}\theta \tag{6.2.1}$$

Proof. Choose $\epsilon > 0$ such that $D_a(r + \epsilon) \subset \mathcal{D}$; this is a simply connected domain, hence we can define the harmonic conjugate v so that $u = \Re f$. Then

$$f(a) = \frac{1}{2i\pi} \oint_{|z-a|=r} f(z) \frac{\mathrm{d}z}{z - a} \ . \tag{6.2.2}$$

If we parametrize the boundary $z = a + re^{i\theta}$ then

$$\frac{\mathrm{d}z}{z-a} = i\mathrm{d}\theta \ , \tag{6.2.3}$$

and the formula follows taking the real part. Q.E.D.

Definition 2.1 (Mean value property for real functions) *A function $u : \mathcal{D} \to \mathbb{R}$ is said to satisfy the* **mean value property** *(MVP) if eq. (6.2.1) holds for all $a \in \mathcal{D}$ and radii r such that $\overline{D_a(r)} \subset \mathcal{D}$.*

Quite clearly this is just a rephrasing of Def. 5.2.

Proposition 2.1 [Maximum principle] If $u : \mathcal{D} \to \mathbb{R}$ is a continuous function satisfying the MVP, and $a \in \mathcal{D}$ is a maximum

$$u(a) \geq u(z) \ , \tag{6.2.4}$$

then $u(z)$ is identically constant.

Proof. Consider the set of maxima

$$M := \{z \in \mathcal{D} : \ u(z) = u(a)\} \ . \tag{6.2.5}$$

Since u is continuous this set is closed. We prove that is also open, and then –since $\mathcal{D}$ is connected– $M = \mathcal{D}$. Indeed let $a_0 \in M$: we can find a small circle of radius r centered there and contained in $\mathcal{D}$ where to apply the MVP. From the continuity it follows easily that

$$u(a_0) = \frac{1}{2\pi}\int u(z(\theta))\mathrm{d}\theta \leq\leq \max_{|z-a|=r} u(z) \tag{6.2.6}$$

with the equaility iff $u(z)$ is constant on the circle. Since r is arbitrary, it follows that u is constant in a small disk centered at a_0, hence M is open. Q.E.D.

Remark 2.1 If u is harmonic then so is $-u$, hence the maximum principle is also a minimum principle.

We now prove the equivalent of Cauchy's integral formula for harmonic functions; we need to find the analogue of $\frac{\mathrm{d}\zeta}{\zeta - z}$. This analogoue is called the Poisson kernel.

Consider a disk $D_0(\rho)$. and define the following

$$P(\zeta, z) := \frac{|\zeta|^2 - |z^2|}{|z-\zeta|^2} \tag{6.2.7}$$

Then we have

Theorem 6.2.7 (Integral formula for harmonic functions on the disk) *Let $u(z)$ be harmonic on a domain containing $|z| \leq \rho$; then, for all $|z| < \rho$ we have*

$$u(z) = \frac{1}{2\pi}\oint_{|\zeta|=\rho} u(\zeta)P(\zeta, z)\frac{\mathrm{d}\zeta}{i\zeta} \tag{6.2.8}$$

Proof. Let f be the holomorphic function defined on $D_0(\rho)$ whose real part is u. We want to "massage" Cauchy integral formula into the above

$$f(z) = \frac{1}{2\pi} \oint_{|\zeta|=\rho} f(\zeta) \frac{\zeta}{\zeta - z} \frac{d\zeta}{i\zeta} \tag{6.2.9}$$

We add to this integral another one which is identically zero: namely let $z^\star$ be the reflection of z on the circle

$$\zeta^\star := \frac{\rho^2}{\overline{z}} , \qquad |z^\star| > \rho . \tag{6.2.10}$$

Therefore

$$0 \equiv \frac{1}{2\pi} \oint_{|\zeta|=\rho} f(\zeta) \frac{\zeta}{\zeta - z^\star} \frac{d\zeta}{i\zeta} \tag{6.2.11}$$

Adding the two we have (note that $\zeta\overline{\zeta} = \rho^2 = z\, z^\star$)

$$\frac{\zeta}{\zeta - z} + \frac{\zeta}{\zeta - z^\star} = P(\zeta, z) . \tag{6.2.12}$$

which is **real** valued. The proof now follows by taking the real part. Q.E.D

More generally one can prove

Theorem 6.2.12 *Let $\phi(z)$ continuous and defined on $|z| = \rho$. Then*

$$u(z) := \frac{1}{2\pi} \oint_{|\zeta|=\rho} P(\zeta, z)\phi(\zeta) \frac{d\zeta}{i\zeta} , \tag{6.2.13}$$

is (the unique) harmonic function with continuous extension to the closed disk and boundary values given by $\phi(z)$.

The fact that this function is harmonic is simple since $P(z, \zeta)$ is harmonic w.r.t. z. The slightly complicated part is to prove that the boundary value of u is ϕ (we omit this part).

Theorem 6.2.13 *Let $u(z) : \mathcal{D} \to \mathbb{R}$ satisfy the MVP. Then u is harmonic.*

Proof. It is enough to show that u is harmonic in each disk. Choose a closed disk contained in $\mathcal{D}$: define $w : \mathbf{D} \to \mathbb{R}$ as the unique harmonic function whose boundary value on $\partial\mathcal{D}$ coincides with u. Since $u - w$ satisfies the MVP on the disk, then it also satisfies the maximum/minimum principle and since $u - w \equiv 0$ on the boundary, it follows that $u \equiv w$ on the disk, and hence u is harmonic. Q.E.D.

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Chapter 7

Riemann mapping theorem

1 Statement

We are now going to prove the following result, going under the name of Riemann's mapping theorem.

Theorem 7.1.0 (Riemann's mapping theorem) *Every simply connected domain $\mathcal{D} \subset \mathbb{C}$, $\mathcal{D} \neq \mathbb{C}$ is biholomorphic to the unit disk* $\mathbf{D} := \{|z| < 1\}$

The proof requirese quite some preparations.

Lemma 7.1.0 Given a simply connected domain $\mathcal{D} \neq \mathbb{C}$ then there is a biholomorphic mapping to a *bounded* domain B.

Proof. Let $a \notin \mathcal{D}$ and choose arbitrarily and fix $z_0 \in \mathcal{D}$. Then

$$g(z) := \int_{z_0}^{z} \frac{\mathrm{d}\zeta}{\zeta - a} = \ln(z - a) - \ln(z_0 - a) \tag{7.1.1}$$

is well defined globally on $\mathcal{D}$ because of the simple-connectivity. By analytic continuation of the identity

$$\mathrm{e}^{g(z)} = \frac{z - a}{z_0 - a} \ , \tag{7.1.2}$$

valid in a neighborhood of z_0, we see that $g(z)$ is univalent and hence establishes a biholomorphic mapping between $\mathcal{D}$ and $g(\mathcal{D})$. We claim that

$$T := g(\mathcal{D}) \cap (g(\mathcal{D}) + 2i\pi) = \emptyset \tag{7.1.3}$$

Indeed suppose that $w \in T$: then there are two **distinct** points z_1, z_2 with $g(z_1) = g(z_2) + 2i\pi$; this however means that $|z_1 - a| = |z_2 - a|$ and $\arg(z_1 - a) = \arg(z_2 - a) + 2\pi$, which implies that $z_1 = z_2$, a contradiction. We also claim that there is a positive $\epsilon > 0$ such that $|g(z) - 2i\pi| > \epsilon$: indeed suppose that there is a sequence $z_n \in \mathcal{D}$ such that $g(z_n) \to 2i\pi$. This sequence would necessarily converge to z_0 for

$$\frac{z_n - a}{z_0 - a} = \mathrm{e}^{g(z_n)} \to 1 \ . \tag{7.1.4}$$

But this is impossible since $g(z)$ is a biholomorphism in a neighborhood of z_0. Consider then the function

$$G(z) := \frac{1}{g(z) - 2i\pi} \ . \tag{7.1.5}$$

Then we have $|G(z)| < \frac{1}{\epsilon}$ and it is still a biholomorphism between $\mathcal{D}$ and $G(\mathcal{D})$, which is bounded. Q.E.D.

This lemma shows that without loss of generality we may assume (up to a dilation) that $\mathcal{D} \subset \mathbf{D}$. By possibly a small translation we can also assume that $0 \in \mathcal{D}$. The crucial point is that a biholomorphic equivalence between such a $\mathcal{D}$ and $\mathbf{D}$ has a certain extremal property described in the following theorem.

Theorem 7.1.5 *Let $\mathcal{M}$ be the set of univalent holomorphic maps $f : \mathcal{D} \to \mathbf{D}$, where $\mathcal{D} \subset \mathbf{D}$ is a simply connected domain containing $z = 0$ and such that $f(0) = 0$. A map $\varphi : \mathcal{D} \to \mathbf{D}$ with $\varphi(0) = 0$ is a biholomorphic equivalence if and only if*

$$\forall f \in \mathcal{M} \ , \qquad |f'(0)| \le |\varphi'(0)| \ . \tag{7.1.6}$$

Proof. The set $\mathcal{M}$ is nonempty since the identity map belongs to it. Suppose φ is a biholomorphic equivalence with the stated properties and $f : \mathcal{D} \to \mathbf{D}$ a univalent map. Let

$$h := f \circ \varphi^{-1} : \mathbf{D} \to \tilde{\mathcal{D}} \subset \mathbf{D} \ . \tag{7.1.7}$$

Note that $h(0) = 0$ We claim that $|h'(0)| \le 1$, indeed $\forall \epsilon > 0$

$$|h'(0)| = \left| \frac{1}{2i\pi} \oint_{|z|=1-\epsilon} \frac{h(z)}{z^2} \mathrm{d}z \right| \le \frac{\sup_{z\in\mathbf{D}} |h(z)|}{1-\epsilon} \le \frac{1}{1-\epsilon} \tag{7.1.8}$$

Taking the inf w.r.t. ϵ yields the result. From the chain rule we have then

$$\left| \frac{f'(0)}{\varphi'(0)} \right| \le 1 \tag{7.1.9}$$

which proves the necessity.

In order to prove the sufficiency we must prove that if $\varphi : \mathcal{D} \to \mathbf{D}$ is univalent with $\varphi(0) = 0$ and has the claimed extremal property then it is onto. So let φ such a function and suppose it is not onto. Let $a \in \mathbf{D}$ such that $a \notin \varphi(\mathcal{D})$. We then construct a function $g \in \mathcal{M}$ with $|g'(0)| > |\varphi'(0)|$.

Construction of g(z). We define

$$\psi(z) = \frac{\varphi(z) - a}{1 - \overline{a}\varphi(z)} : \mathcal{D} \to \mathbf{D} \tag{7.1.10}$$

since this linear/fractional transformation is a map of $\mathbf{D}$ into itself. So now we have a $\psi(z) : \mathcal{D} \to \mathbf{D}$ which never vanishes. We can define unambiguously

$$H(z) := \ln \psi(z) : \mathcal{D} \to \mathbf{D} \tag{7.1.11}$$

since $\mathcal{D}$ is simply connected and $\varphi(z)$ is never zero. Note that $\Re H(z) < 0$ since $|\psi(z)| < 1$. We set

$$g(z) = \frac{H(z) - H(0)}{H(z) + \overline{H(0)}} \tag{7.1.12}$$

and claim that this function yields a contradiction with the hypothesis of extremality. First of all we must check that $g \in \mathcal{M}$: indeed $g(0) = 0$ and

$$|g(z)| = \left| \frac{H - H_0}{H + \overline{H_0}} \right| < 1 \tag{7.1.13}$$

since the distance of H_0 from H is less than the distance of $-\overline{H_0}$ (which has same imaginary part as H_0). We finally compute

$$g'(0) = \frac{H'(0)}{H(0) + \overline{H(0)}} \ , \qquad H'(0) = \left(\overline{a} - \frac{1}{a}\right)\varphi'(0) \ , \qquad H(0) = \ln(-a) \tag{7.1.14}$$

$$g'(0) = \left(\overline{a} - \frac{1}{a}\right)\frac{1}{\ln(|a|^2)}\varphi'(0) \tag{7.1.15}$$

hence

$$\left| \frac{g'(0)}{\varphi'(0)} \right| = \frac{|a|^2 - 1}{|a| \ln(|a|^2)} \tag{7.1.16}$$

It is then an exercise to see that the function of $|a|$ on the RHS is strictly greater than 1 on the interval $0 < |a| < 1$. Q.E.D.

The only missing step now is to show existence of an extremal map φ; the idea is that of endowing the set $\mathcal{M}$ with a topology such that it is **(a)** compact and **(b)** the function

$$\delta_0' : \mathcal{M} \to \mathbb{R} \ , \delta_0'(f) := |f'(0)| \tag{7.1.17}$$

is continuous and bounded. Then it must have a maximum on this set. This requires the study of the topology of $\mathcal{H}(\mathcal{D})$.

2 Topology of $\mathcal{H}(\mathcal{D})$

The vector space $\mathcal{H}(\mathcal{D})$ is a subspace of the continuous maps, for which we have the sup norm. The small problem is that $\mathcal{D}$ is not compact and hence we must introduce a work-around.

Recall that if (X, d) is a compact metric space, the vector space of continuous functions

$$\mathcal{C}(X) := \{f : X \to \mathbb{C} \ , \quad f \text{ continous } \} \tag{7.2.1}$$

is endowed with the **distance**

$$\delta(f, g) := \sup_{x\in X} |f(x) - g(x)| = \max_{x\in X} |f(x) - g(x)| \ . \tag{7.2.2}$$

The metric space $(\mathcal{C}(X), \delta)$ is then a **complete** metric space (i.e. any Cauchy sequence converges to a continuous function).

Let now $\mathcal{D} \subset \mathbb{C}$ be an **open** set.

Lemma 7.2.2 There is a collection $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of $\mathcal{D}$ such that

1. $\displaystyle\bigcup_{n\in\mathbb{N}} K_n = \mathcal{D}$
2. $K_n \subset \overset{\circ}{K}_{n+1}$
3. $\forall C \subset \mathcal{D}$, C compact there is N such that $C \subset K_n$.

Such a sequence of increasing compact sets is called a **saturating** sequence.

Proof. We define

$$K_n := \{|z| \le n\} \cap \left\{ z \in \mathcal{D} \text{ s.t. } d(z, \mathbb{C} \setminus \mathcal{D}) \ge \frac{1}{n} \right\} \tag{7.2.3}$$

Note that $d(z, \mathbb{C} \setminus \mathcal{D})$ for $z \in \mathcal{D}$ is defined as

$$d(z, \mathcal{D}^c) := \inf_{w\notin\mathcal{D}} |z - w| \tag{7.2.4}$$

and the sets $d(z, \mathcal{D}^c) \ge x$ are all **closed** sets. Indeed if $\{z_n\} \subset \mathcal{D}$ is a Cauchy sequence such that $d(z_n, \mathcal{D}^c) \ge x > 0$ then the limit point z_∞ also has the property $d(z_\infty, \mathcal{D}^c) \ge x$ and hence belongs to $\mathcal{D}$ (the disk $D_{z_\infty}(x/2)$ has no intersection with $\mathcal{D}^c$ and hence belongs to $\mathcal{D}$).

The sets K_n are therefore closed and bounded in $\mathbb{C}$ and hence compacts. The rest of the proof is left as **exercise**. Q.E.D.

Let now $\{K_n\}$ be such a saturating sequence of compacts in $\mathcal{D}$. Define

$$\rho_n(f, g) := \max_{z\in K_n} |f(z) - g(z)| \ . \tag{7.2.5}$$

and then

$$\rho(f, g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \ . \tag{7.2.6}$$

We claim that ρ is a metric on $\mathcal{C}(\mathcal{D})$, indeed

1. If $\rho(f,g)=0$ then all $\rho_n(f,g)=0$ and hence $f\equiv g$ (and viceversa obviously).

2. Trivially $\rho(f,g)=\rho(g,f)$.

3. $\rho(f,g)\leq\rho(f,h)+\rho(h,g)$ because the same is true for all ρ_n's.

The convergence in the metric ρ is such that a sequence f_n converges to f iff the sequence converges **uniformly on all compact sets** $K\subset\mathcal{D}$. Since uniform convergence on a compact set is sufficient to guarantee continuity, the space $(\mathcal{C}(\mathcal{D}),\rho)$ is **complete**. In the metric ρ the space $\mathcal{C}(\mathcal{D})$ is of finite diameter, therefore the notion of bounded set is slightly different

Definition 2.1 *A set $\mathcal{B}\subset\mathcal{C}(\mathcal{D})$ is called* **bounded** *if all elements of $\mathcal{B}$ are uniformly bounded on any compact subset $K\subset\mathcal{D}$; Being* **bounded** *in ρ means that the functions in $\mathcal{M}$ are* **uniformly** *bounded on any compact set*

$$\forall K\subset\mathcal{D}, K\ compact\ ,\ \exists M\ s.t.\quad \forall f\in\mathcal{B}\ \ \sup_{z\in K}|f(z)|<M\ . \tag{7.2.7}$$

Theorem 7.2.7 *The subspace $\mathcal{H}(\mathcal{D})\subset\mathcal{C}(\mathcal{D})$ is* **closed** *(hence complete) in the metric ρ of uniform convergence on compact sets.*

Proof. Let f_n be a Cauchy sequence of holomorphic functions: we know that f_∞ is at least continuous. To prove that it is holomorphic at all points choose a small circle γ contained in $\mathcal{D}$. Note that γ is a compact set and hence we have uniform convergence on the points of γ. Then for all z inside this disk

$$f_n(z)=\frac{1}{2i\pi}\oint_\gamma\frac{f_n(\zeta)\mathrm{d}\zeta}{\zeta-z} \tag{7.2.8}$$

Passing to the limit on the RHS and LHS we have a similar Cauchy integral representation for f_∞ which is therefore holomorphic in the (arbitrary) disk. Q.E.D.

Theorem 7.2.8 *The map*

$$D:\mathcal{H}(\mathcal{D})\to\mathcal{H}(\mathcal{D})\ ,D(f)(z):=f'(z)\ , \tag{7.2.9}$$

is continuous. Moreover it maps any bounded set $\mathcal{M}\subset\mathcal{H}(\mathcal{D})$ into a bounded set.

Proof. We need to prove that if $f_n\to f$ in the metric ρ then $f_n'\to f'$ in the metric ρ or, equivalently that we have uniform convergence on any compact $K\subset\mathcal{D}$ of f_n'.

Note that –since any compact is covered by a finite number of disks contained in $\mathcal{D}$, it is sufficient to verify the uniform convergence on an arbitrary disk $\overline{D_a(R)}\subset\mathcal{D}$. To do this we need to take a slightly larger disk such that its closure is still in $\mathcal{D}$, namely we choose $\epsilon>0$ such that $D_a(R+\epsilon)\subset\mathcal{D}$. Then for $z\in\overline{D_a(R)}$

$$|f_n'(z)-f'(z)|=\frac{1}{2\pi}\left|\oint_{|w-a|=R+\epsilon}\frac{(f_n(w)-f(w))\mathrm{d}w}{(w-z)^2}\right|\leq\frac{1}{2\pi}\oint_{\partial D_a(R+\epsilon)}\frac{|f_n(w)-f(w)|}{|z-w|^2}\mathrm{d}s\leq \tag{7.2.10}$$

$$\leq\sup_{z\in\overline{D_a(R+\epsilon)}}|f(z)-f_n(z)|\frac{R+\epsilon}{\epsilon^2}\underset{n\to\infty}{\longrightarrow}0 \tag{7.2.11}$$

This proves the continuity. For the boundedness let $\mathcal{M}$ be a bounded set of holomorphic functions on $\mathcal{D}$. Once more it is sufficient to check the assertion for disks. Using the same disks as before and letting M be the bounding constant on the set $\overline{D_a(R+\epsilon)}$, we have easily that $D(\mathcal{M})$ is bounded on the disk $\overline{D_a(R)}$,

$$|f'(z)| \leq M\frac{R}{\epsilon^2}. \tag{7.2.12}$$

Q.E.D.

Proposition 2.1 The evaluation map $ev_a : \mathcal{H}(\mathcal{D}) \to \mathbb{C}$, $ev_a(f) = f(a)$ is continuous.

Proof. Uniform convergence implies pointwise convergence. Q.E.D.

Composing the differentiation with the evaluation we have that the map $D_a(f) = f'(a)$ is a continuous map, and so is $|f'(a)|$.

In particular we have

Corollary 7.2.12 *For a uniformly convergent sequence $f_n \in \mathcal{H}(\mathcal{D})$ all derivatives at any point of $\mathcal{D}$ are convergent sequences. Viceversa a sequence $f_n \in \mathcal{H}(\mathcal{D})$ is uniformly convergent on any compact $K \subset \mathcal{D}$ if all the derivatives at all points are convergent and the sequence is* **bounded** *(i.e. uniformly bounded on compact sets).*

Proof. Only the "viceversa" part is not already proved. Once more it is sufficient to verify the statement for an arbitrary disk $\overline{D_a(r)} \subset \mathcal{D}$. So let f_n be such a sequence. Fix $a \in \mathcal{D}$ and let $r < R$ such that $\overline{D_a(R)} \subset \mathcal{D}$. Let M be the bounding constant for the sequence on this compact. We have

$$|f_n^{(k)}(a)| \leq \frac{k!}{2\pi}\oint_{|z-a|=R}\left|\frac{f_n(z)\mathrm{d}z}{(z-a)^{k+1}}\right| \leq \frac{k!M}{R^k} \tag{7.2.13}$$

Let us estimate the distance on the disk of radius r, $|z-a| \leq r$

$$|f_n(z) - f_m(z)| \leq \sum_{k=0}^{N}\frac{|f_n^{(k)} - f_m^{(k)}|}{k!}|z-a|^k + \sum_{k>N}\frac{|f_n^{(k)} - f_m^{(k)}|}{k!}r^k \leq \tag{7.2.14}$$

$$\leq \sum_{k=0}^{S}\frac{|f_n^{(k)} - f_m^{(k)}|}{k!}r^k + 2M\sum_{k>S}\left(\frac{r}{R}\right)^k \tag{7.2.15}$$

Fix $\epsilon > 0$: since the last geometric series is convergent there is S_0 such that $S > S_0$ implies that the last term is less than $\epsilon/2$. Since the first term contains only a finite number of convergent sequences (and S_0 terms), then there is a $N = N(S_0)$ such that for $n, m > N$ the first sum is less than $\epsilon/2$. Hence we have the uniform convergence. Q.E.D.

In order to be able to complete the proof of Riemann's mapping theorem we need to find a **compact** set in our $\mathcal{M}$. We have

Theorem 7.2.15 *A set $\mathcal{M} \subset \mathcal{H}(\mathcal{D})$ is compact iff it is closed and bounded.*

Proof. Let $\mathcal{M}$ be compact. Then it is obviously closed. The boundedness is almost obvious: let $K \subset \mathcal{D}$ be a compact set. The function

$$\|f\|_K := \max_{z\in K} |f(z)| \tag{7.2.16}$$

is continuous and hence has a maximum for f_0 in $\mathcal{M}$. This maximum cannot be infinite and hence $\mathcal{M}$ is bounded.

The nontrivial direction is the proof that boundedness+closedness implies compactness. Let then $\mathcal{M}$ be closed and bounded and let f_n a sequence in $\mathcal{M}$: we need to find a convergent subsequence, namely a subsequence that converges uniformly on all compacts.

We first find a uniformly convergent subsequence on a arbitrary but fixed compact set $K \subset \mathcal{D}$. By compactness, this set can be covered by a **finite** number of disks whose closure still is contained in $\mathcal{D}$. Let a be the center of one of these finite disks and R its radius. For $k \in N$ fixed, each sequence $\{f_n^{(k)}(a)\}_{n\in\mathbb{N}}$ is bounded and hence admits a convergent subsequence. Let $\mathbb{N}_0$ be the subsequence for $k = 0$. From this one we can extract another subsequence $\mathbb{N}_1 \subset \mathbb{N}_0$ for which $f_n'(a)$ converges. From this one we can extract another subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ for which $f_n''(a)$ converges. And so on and so forth.

We then extract the "diagonal" subsequence, namely n_ℓ is the ℓ-th element of $\mathbb{N}_\ell$: since the subsequences are nested, n_ℓ is a subsequence of N_{ℓ_0} for $\ell > \ell_0$. In particular this implies that for any $\ell_0 \in \mathbb{N}$ the sequence $f_{n_\ell}^{(\ell_0)}(a)$ converges as $\ell \to \infty$. By Corollary 7.2.12 this subsequence converges uniformly on the disk $\overline{D_a(r)}$, for any $r < R$. We repeat this procedure for the finite number of disks that cover K and note that if the finite disks $D_{a_i}(R_i)$ cover K so do the $D_{a_i}(R_i - \epsilon)$ for some small enough $\epsilon > 0$. Therefore we have uniform convergence on K.

To complete the proof we take a saturating sequence of compact sets K_ν, $\nu \in \mathbb{N}$; for each of them there is a convergent subsequence f_{n_ℓ} as before. We repeat the diagonal argument by constructing nested subsequences and reading them on the diagonal, and thus complete the proof. Q.E.D.

We will need also a result of Hurwitz

Proposition 2.2 [Hurwitz's Theorem] Let $\{f_n\} \subset \mathcal{H}(\mathcal{D})$ be a sequence uniformly convergent to $f \in \mathcal{H}(\mathcal{D})$ on compact sets. Let $\overline{D_a(R)} \subset \mathcal{D}$ be such that $f(a) \neq 0$: then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$ the functions f_n have the same number of zeroes in $\overline{D_a(R)}$ as f does.

Proof. The number of zeroes of f_n in the closed disk is

$$\frac{1}{2i\pi} \oint_{|z-a|=R} \frac{f_n'(z)\mathrm{d}z}{f_n(z)} \tag{7.2.17}$$

and a similar expression for f. If a zero of f happens to be exactly on the boundary of the chosen closed disk, we slightly enlarge it $R' = R + \epsilon$ in such a way as to **not include** any further zero (this is possible since the zeroes of a holomorphic function is discrete in its domain). We will therefore assume that no zero of f falls exactly on the boundary. We know that both f_n and f_n' are uniformly convergent on any compact set to f and f': since $f(z) \neq 0$ on the circle $|z-a| = R$ then also

$$\min_{|z-a|=R} |f(z)| = \delta > 0 \ , \tag{7.2.18}$$

and there is $N' \in \mathbb{N}$ such that $|f_n(z)| \geq \delta/2$ on $|z-a| = R$ for $n \geq N'$. Moreover we have for $|z-a| = R$

$$\left|\frac{f_n'}{f_n} - \frac{f'}{f}\right| = \left|\frac{f_n' f - f' f_n}{f f_n}\right| = \left|\frac{f_n' f - f' f + f' f - f' f_n}{f f_n}\right| \leq \left|\frac{f_n' - f'}{f_n}\right| + \left|\frac{f'(f - f_n)}{f f_n}\right| \tag{7.2.19}$$

Since the numerators are bounded away from zero and the numerators tend to zero we have N such that $n > N$ implies

$$\left|\frac{f_n'}{f_n} - \frac{f'}{f}\right| \leq \frac{1}{2R}. \tag{7.2.20}$$

Then the difference of the number of zeroes is less than $\frac{1}{2}$ and hence they must be the same (since they must be integers). Q.E.D.

Corollary 7.2.20 *If $f_n : \mathcal{D} \to \mathbb{C}$ is a sequence of univalent maps uniformly convergent on compact sets $K \subset \mathcal{D}$ to f, then either f is a constant or else f is univalent as well.*

Proof. Suppose f not constant and suppose it is not univalent. Let $z_1, z_2 \in \mathcal{D}$ such that $f(z_1) = f(z_2) = a$. Then $f_n(z) - a$ must also have zeroes in suitable neighborhoods of z_1, z_2 for n large enough, contradicting the univalency. Q.E.D.

3 Proof

Proof of Thm. 7.1.0. Recall that $\mathcal{D} \subset \mathbf{D}$ is a simply-connected domain and $\mathcal{M}$ is the set of holomorphic univalent maps $f : \mathcal{D} \to \mathbf{D}$ fixing the origin $f(0) = 0$. We have established that the map achieving the equivalence is the one for which $|f'(0)|$ is maximal in $\mathcal{M}$.

The set $\mathcal{M}$ is clearly bounded. We also know from Corollary 7.2.20 that a Cauchy sequence in $\mathcal{M}$ either converges in $\mathcal{M}$ or converges to a constant (which actually must be 0) . We must exclude this possibility. Consider the subsets

$$\mathcal{M}_r := \{f \in \mathcal{M} : \ \ |f'(0)| \geq r\} \ . \tag{7.3.1}$$

These sets are still bounded and moreover closed because the function $|f'(0)|$ is continuous. Since $f(z) = z$ belongs to $\mathcal{M}$ then $\mathcal{M}_r$ are certainly nonempty for $0 < r \leq 1$. A convergent sequence in $\mathcal{M}_r$ cannot converge to a constant and it is easy to see (from the continuity of $|f'(0)|$) that converges within $\mathcal{M}_r$.

In view of Thm. 7.2.15 then $\mathcal{M}_r$ is compact; the continuous functional

$$\delta' : \mathcal{M}_r \to \mathbb{R} \ , \delta'(f) := |f'(0)| \tag{7.3.2}$$

must have a point $\varphi \in \mathcal{M}_r$ of maximum. This map is the required equivalence. Q.E.D.

4 Extensions of the theorem

Riemann's theorem can be significantly extended to the following situation: let $\mathcal{L}$ be a compact one-dimensional complex manifold. It can be shown that it is (equivalent) to (the compactification of) an algebraic curve.

Let $\widetilde{\mathcal{L}}$ be its universal covering: unless $\mathcal{L}$ is simply connected to begin with, $\widetilde{\mathcal{L}}$ is not compact. Let g be the **genus** of this curve; we recall that we have defined it to be the dimension of the space of first-kind differentials (holomorphic one-forms). There are more "geometric" but equivalent definitions which we omit. We have

Proposition 4.1 Let $g \geq 1$ and fix $p_0 \in \mathcal{L}$. There are $2g$ closed simple curves γ_j whose homotopy classes generate the fundamental group $\pi_1(\mathcal{L}, p_0)$ and such that $\mathcal{D} := \mathcal{L} \setminus \bigcup_{j=1}^{2g} \gamma_j$ is a simply-connected domain. Furthermore these generators satisfy a single group-relation in the fundamental group $\pi_1(\mathcal{L})$

$$[\gamma_1] \cdot [\gamma_2] \cdots [\gamma_{2g}] = \mathbf{1} \tag{7.4.1}$$

The extension of Riemann's theorem is the following

Theorem 7.4.1 *Let $\widetilde{\mathcal{L}}$ be the universal covering of an algebraic compact curve $\mathcal{L}$ of genus g. Then there is a univalent holomorphic map φ which establishes a biholomorphic equivalence of $\widetilde{\mathcal{L}}$ with the following domains*

1. *The Riemann sphere $\overline{\mathbb{C}}$ if $g = 0$.*

2. *The complex plane $\mathbb{C}$ is $g = 1$.*

3. *The upper half plane $\mathcal{H}_+$ if $g \geq 2$.*

These three cases are called (confusingly), **elliptic** *($g = 0$),* **parabolic** *($g = 1$) or* **hyperbolic** *($g \geq 2$).*

Since the deck transformations of $\widetilde{\mathcal{L}}$ act properly-discontinuously on $\widetilde{\mathcal{L}}$, then their composition with the biholomorphic equivalence φ also has the same property. Let Γ be the group of such deck transformations. Then we know that $\widetilde{\mathcal{L}}/\Gamma \simeq \mathcal{L}$ (basically by definition of universal covering). Therefore

Theorem 7.4.1 *Any one-dimensional compact (connected) complex manifold is biholomorphically equivalent to the quotient of $\mathbb{C}$ or $\mathcal{H}_+$ for the cases $g = 1$, $g \geq 2$ respectively, by a group of automorphisms of $\mathbb{C}$ or $\mathcal{H}_+$ respectively which acts properly-discontinuously.*

The proof (cf. Siegel) is quite involved and goes along these points.

Let $\mathfrak{U}$ be a connected, simply connected one dimensional manifold (usually the universal cover of a compact curve). Prove the existence of a meromorphic functiion $f(\zeta)$ which

1. $f(\zeta)$ has only one simple pole with residue 1 at some point ζ_0.

2. $f(\zeta)$ minimizes a Dirichlet integral.

Using the minimum property one is able to show that $f(\zeta)$ is univalent and performs the requested equivalence.

4.1 Some conformal mappings

There is an endless list of type of domains for which the explicit conformal mapping has been worked out, the reason being their interest in problems of 2-dimensional, irrotational stationary fluid dynamics. Just one example is the following.

Schwarz–Christoffel transformations

These give the conformal map of an arbitrary polygon to the upper half plane (or rather the viceversa). Let $x_1 < x_2, \ldots, x_n \in \mathbb{R}$ and $a_1, a_2, \ldots, a_n \in (0, 2\pi)$. Let

$$\frac{\mathrm{d}w}{\mathrm{d}z} = A \prod_{j=1}^{n} (z - x_j)^{\frac{a_j}{\pi} - 1} \ , \quad \Im z > 0, \ A \neq 0 \ . \tag{7.4.2}$$

Then $w(z)$ establishes a biholomorphic equivalence between the upper half z–plane and the interior of a polygon with interior angle a_j at the w_j vertex. Note that the polygon may be degenerate if we allow $a_j = 0, 2\pi$ and it may also be non closed.

5 Exercises

In this section we mean by "equivalence" a biholomorphic mapping. The unit disk will be denoted by

$$\mathbf{D} := \{|z| < 1\} \ , \tag{7.5.1}$$

and the upper half plane by

$$\mathcal{H}_+ := \{z : \ \Im(z) > 0\} \ . \tag{7.5.2}$$

Exercise 5.1 *Let $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ be simply-connected domains in $\mathbb{C}$ with $\widetilde{\mathcal{D}}$ bounded. Let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic on $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$ (the closure) and such that $f : \partial\mathcal{D} \to \partial\widetilde{\mathcal{D}}$ gives a piecewise smooth correspondence between the oriented boundaries. Prove that f is* **onto** *(surjective).*

Exercise 5.2 *Find an equivalence of the following domain with the unit disk* $\mathbf{D}$*.*

$$\mathcal{D} := \left\{ z = x + iy : \ y^2 > 4(x+1) \right\} \tag{7.5.3}$$

Exercise 5.3 *Construct an equivalence of the upper half plane with a cut*

$$\mathcal{D} := \{z : \ \Im(z) > 0\} \setminus [a, a+ih] \ , \qquad a, h \in \mathbb{R} \ , \ h > 0 \ , \tag{7.5.4}$$

(where $[a, a+ih]$ denotes a straight segment) with the upper half plane $\mathcal{H}_+$ **without** *the cut.*

Chapter 8

Picard's theorems

1 Picard's Little Theorem

We have

Theorem 8.1.0 (Picard's little theorem) *Let f be an entire function such that it omits two values. Then f is a constant.*

Consider an elliptic curve

$$Y^2 = 4X^3 - g_2 X - g_3 \ , \qquad {g_2}^3 - 27{g_3}^2 \neq 0 \ , \tag{8.1.1}$$

and denote by e_1, e_2, e_3 the roots (which are distinct!). We already know that the compactification of this curve if equivalent to the manifold $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ where the periods ω_i are defined by certain elliptic integrals. We also know that these lattices are equivalent if the modular parameters $\tau = \frac{\omega_1}{\omega_2}$ are related by the transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ . \tag{8.1.2}$$

In fact we have also proved that it suffices to look at the modular invariant

$$J(\tau) := \frac{{g_2}^3}{{g_2}^3 - 27{g_3}^2} \tag{8.1.3}$$

and this unique number characterizes the classes of elliptic curves.

Note that the algebraic curve

$$Y^2 = 4X^3 - \varkappa^4 g_2 X - \varkappa^6 g_3 \tag{8.1.4}$$

has the same modular invariant and hence is equivalent (it is sufficient to rescale $Y \mapsto \varkappa^3 Y$ and $X \mapsto \varkappa^2 X$). The roots are now $e_i' = \varkappa^{-2} e_i$. We define

$$\lambda := \frac{e_1 - e_2}{e_3 - e_2} \ \neq \ 0, 1 \tag{8.1.5}$$

Under a modular transformation

$$\omega_1' = a\omega_1 + b\omega_2 \tag{8.1.6}$$

$$\omega_2' = d\omega_1 + c\omega_2 \tag{8.1.7}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{8.1.8}$$

the half periods σ_i $(i = 1, 2, 3)$ in general are permuted

$$\sigma_1' = \frac{a}{2}\omega_1 + \frac{b}{2}\omega_2 \tag{8.1.9}$$

$$\sigma_2' = \frac{c}{2}\omega_1 + \frac{d}{2}\omega_2 \ , \tag{8.1.10}$$

depending on the class of a, b, c, d mod 2. They are not permuted iff

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \tag{8.1.11}$$

because in this case the new half periods are equivalent to the old ones by a shift in the lattice. Transformations of these type constitute a subgroup of $SL_2(\mathbb{Z})$ denoted

$$\Gamma_2 := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ g = \mathbf{1} \bmod 2 \right\} \ . \tag{8.1.12}$$

Proposition 1.1 The group Γ_2 is generated by the transformations

$$\pm\,1,\ \alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = T^2 \ , \ \beta = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = -ST^2S\alpha(\tau) = \tau + 2 \ , \qquad \beta(\tau) = \frac{\tau}{2\tau+1} \tag{8.1.13}$$

and has fundamental domain depicted in Fig. 8.1.

Proof. Consider the group

$$G := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{Z}_2 : ab - cd = 1 \right\} = SL_2(\mathbb{Z}_2) \ . \tag{8.1.14}$$

(That this is a group is left as exercise). This is a finite group of $Mat(2 \times 2, \mathbb{Z}_2)$ (which has 16 elements), with elements

$$\mathbf{1},\ e = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ f = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ \ell = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.1.15}$$

with the following multiplication table:

	$\mathbf{1}$	e	f	g	h	ℓ
$\mathbf{1}$	$\mathbf{1}$	e	f	g	h	ℓ
e	e	$\mathbf{1}$	h	ℓ	f	g
f	f	ℓ	$\mathbf{1}$	h	g	e
g	g	h	ℓ	$\mathbf{1}$	e	f
h	h	g	e	f	ℓ	$\mathbf{1}$
ℓ	ℓ	f	g	e	$\mathbf{1}$	h

(8.1.16)

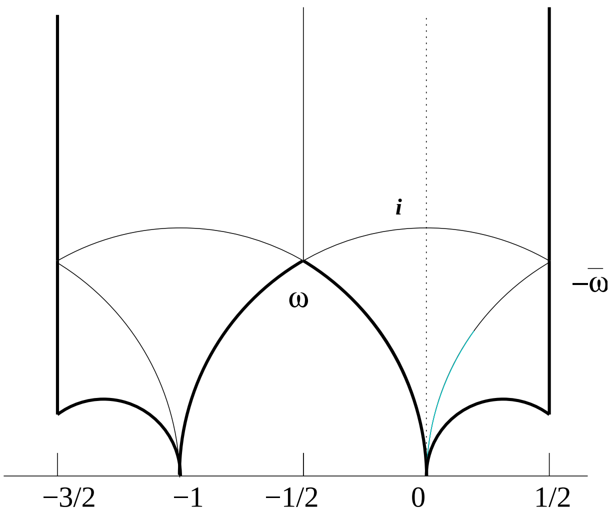

Figure 8.1: The fundamental domain of Γ_2.

(In particular this is the table of S_3, the permutation group of three elements, with $e \sim (1,2)$, $f \sim (2,3)$ and $g \sim (13)$.) The natural map $\pi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_2)$ is a group homomorphism and Γ_2 is the kernel of this map, hence it is a normal subgroup. Hence $SL_2(\mathbb{Z}_2) \simeq SL_2(\mathbb{Z})/\Gamma_2$. Since $\Gamma_2 \triangleleft SL_2(\mathbb{Z})$ (in particular it is contained), the fundamental domain of Γ_2 is obtained by union of the images of the fundamental domain D_0 of $SL_2(\mathbb{Z})$ under some choice of representatives of the coset group. We choose as representatives

$$\alpha_1 = \mathbf{1}, \alpha_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ \alpha_3 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \ \alpha_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \alpha_5 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \ \alpha_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{8.1.17}$$

and we take $D_2 = \bigcup_{j=1}^{6} \alpha_j \cdot D_0$. The result is in Fig. 8.1.

That this is a fundamental domain for Γ_2 follows from the fact that $\forall \tau \in \mathcal{H}_+$ there is a $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma \cdot \tau \in D_0$; since we can write it as $\gamma = \alpha_j^{-1}\gamma'$ with $\gamma \in \Gamma_2$ and for some $j = 1 \dots 6$ then $\gamma'\tau \in \alpha_j D_0$. Moreover this decomposition is unique: indeed if $\tau' = g\tau$ then g is uniquely defined up to left multiplication by the stabilizer of τ' and right multiplication by the stabilizer of τ. In any case the different choices of g represent the same class in $SL_2(\mathbb{Z})/\Gamma_2$ (check it!).

In order to prove that Γ_2 is generated by $\pm\mathbf{1}$, $\alpha = T^2$ and $\beta = -ST^2S$ we remark that the group generated by these elements is **(a)** normal (you need to check that it is invariant by conjugation by S and T, which is easy remembering that $(ST)^3 = (TS)^3 = \mathbf{1}$) and (**b**) contained in Γ_2 (obviously). Since all elements of $SL_2(\mathbb{Z})$ can be written as

$$\gamma = T^{j_1} S T^{j_2} \cdots S T^{j_k} \ , \tag{8.1.18}$$

we can pull to the right the even powers of $T^{2k} = \alpha^k$ and of $ST^{2k}S = \pm\beta^k$, leaving on the left one of the following words

$$\mathbf{1},\ T^{-1}, S^{-1}, (TS)^{-1}, (ST)^{-1}, TS^{-1}T^{-1}\ , \tag{8.1.19}$$

which are the α_j. Hence the two quotients have the same classes and hence the group generated by $\pm 1, \alpha, \beta$ coincides with Γ_2. Note also that the action of Γ_2 has no points with nontrivial stabilizer (i.e. bigger than $\pm\mathbf{1}$ in this case). Q.E.D.

Given that $e_i = \wp(\omega_i/2)$ it is easy to see that $\lambda(\omega_1, \omega_2)$ depends in fact only on their ratio τ. From the previous discussion it should be clear also that $\lambda(\tau)$ is invariant under Γ_2:

$$\lambda(\tau + 2) = \lambda(\tau) \qquad \lambda\left(\frac{\tau}{2\tau + 1}\right) = \lambda(\tau)\ . \tag{8.1.20}$$

and cannot take the values $0, 1$. It can take any other value for in this case we can set e.g.

$$e_1 = \varkappa(1 - 2\lambda)\ , \qquad e_2 = \varkappa(1 + \lambda), \qquad e_3 = -e_1 - e_2 \tag{8.1.21}$$

and these are all the numbers with the prescribed λ. It is also easy to see that if two algebraic curves have the same λ (up to a permuation of the roots) then they have the same J: in fact

Exercise 1.1 (Check!) *Prove that the modular invariant J is related to λ by*

$$J = \frac{4(1 - \lambda + \lambda^2)^3}{27\lambda^2(1 - \lambda)^2} \tag{8.1.22}$$

Prove that this expression is invariant under the transformations

$$\lambda \mapsto \frac{\lambda}{\lambda - 1}\ , \qquad \lambda \mapsto 1 - \lambda \tag{8.1.23}$$

which corresponds to permutations of the three roots.

It also follows that if $\lambda(\tau) = \lambda(\tau')$ then τ and τ' must be related by a transformation in Γ_2 because the two elliptic curves have the same half periods, (and clearly the viceversa is true as well). Since the group Γ_2 acts properly and discontinuously on $\mathcal{H}_+$ with trivial stabilizer, then $\lambda(\tau)$ is locally a univalent map and hence

Proposition 1.2 The function $\lambda(\tau) : \mathcal{H}_+ \to \mathbb{C} \setminus \{0, 1\}$ is **unramified** and onto.

2 Proof of Thm. 8.1.0

The theorem can be rephrased as

Theorem 8.2.0 *Given an entire non-constant function $f(z)$, then $f(z)$ takes on all values except possibly one.*

Proof. Suppose f does not take on the values a, b. By the transformation

$$\tilde{f}(z) = \frac{f(z) - a}{b - a} \ , \tag{8.2.1}$$

we can always assume that $a = 0,\ b = 1$. Consider then the Riemann surface

$$\lambda(\tau) = f(z) \ , \qquad \tau \in \lambda^{-1}(f(z)) \tag{8.2.2}$$

This defines a covering of $\mathcal{H}_+$ onto $\mathbb{C} \setminus \{0, 1\}$: the number of sheets is infinite since all τ related by a transformation of Γ_2 have the same λ value.

Take z_0 and $\tau_0 \in \lambda^{-1}(f(z_0))$; we can define a germ of analytic function $\tau(z)$ such that $\tau(z_0) = \tau_0$. Since λ is a local equivalence, this germ admits unrestricted analytic continuation to the whole z plane and since this is simply connected then this analytic continuation defines an analytic function (in fact entire).

Consider finally

$$G(z) := \frac{\tau(z) - i}{\tau(z) + i} \tag{8.2.3}$$

Since $\Im(\tau(z)) > 0$ then $|G(z)| < 1$ and hence it is an entire, bounded function. Therefore $G(z)$ is constant, which can be iff $f(z)$ is constant, a contradiction. Q.E.D.

3 Picard's great theorem

Here we cheat a bit: we state without proof the following theorem.

Theorem 8.3.0 (Montel-Charateodory Theorem) *Let $\mathcal{D}$ be a domain and $\mathfrak{F}$ be the family of functions that never assume the values $0, 1$: then $\mathfrak{F}$ is relatively compact in $\mathcal{C}(\mathcal{D}, \overline{\mathbb{C}})$, namely any sequence of functions admits a subsequence which is either uniformly convergent on all compact subsets of $\mathcal{D}$ or else uniformly tends to ∞ on any compact.*

This is all we need to prove

Theorem 8.3.0 (Picard's great theorem) *Let $f(z)$ have an essential singularity at $z = a$. In each neighborhood of a $f(z)$ assumes all complex values -except possibly one- infinitely many times.*

Proof. Up to translation we can assume that the essential singularity is at $z = 0$. Suppose there is $R > 0$ and two numbers a, b (which we assume to be $0, 1$ up to a shift/rescaling of f) which are never assumed in the punctured disk of radius R, $\mathcal{D} = D_0(R) \setminus \{0\}$. Construct the sequence

$$f_n(z) = f\left(\frac{z}{n}\right) \ , \qquad n \geq 1 \tag{8.3.1}$$

This family is never assumes the values $0, 1$ and hence admits a subsequence f_{n_k} which either converges to an analytic function φ or converges uniformly to ∞ on all compact subsets of $\mathcal{D}$.

If we are in the first case we set

$$M := \max_{|z|=R/2} |\varphi(z)| \tag{8.3.2}$$

From the uniform convergence it follows that on $|z| = R/2$

$$\left| f\left(\frac{z}{n_k}\right) \right| \le \left| f\left(\frac{z}{n_k}\right) - \varphi(z) \right| + |\varphi(z)| < M + 1 \tag{8.3.3}$$

for n_k sufficiently large. Therefore $f(z)$ is uniformly bounded on concentric circles of radii $R/(2n_k)$ and by the maximum modulus theorem it must be bounded on $\mathcal{D}$. Hence $z = 0$ is a removable singularity, a contradiction.

If $\varphi \equiv \infty$ then it is left as exercise to show that $f(z)$ has a pole. Q.E.D.